SOLUTIONS FOR EXTRA ADMISSIONS TEST IN
MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS
DECEMBER 2022

A Note that the probability that the coin does not land on heads is $1 - \cos^2\alpha = \sin^2\alpha$. The probability of exactly two heads is $\binom{3}{2}\cos^4\alpha\sin^2\alpha$ and the probability of exactly three heads is $\binom{3}{1}\cos^6\alpha$. Simplifying gives

$$3\cos^4\alpha\sin^2\alpha + \cos^6\alpha = 3(1-\sin^2\alpha)^2\sin^2\alpha + (1-\sin^2\alpha)^3 = 1 - 3\sin^4\alpha + 2\sin^6\alpha$$

The answer is (d)

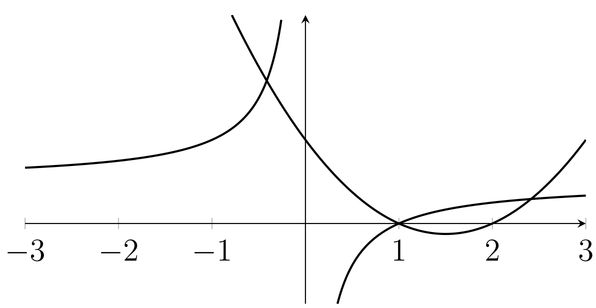
B If we write $u = e^{-x/2}$ then the given expression is $u - u^2$. This quadratic has roots at 0 and 1, and is positive in between. Now as x varies from 0 to ∞, u will decay exponentially from 1 to 0. Along the way, $u - u^2$ will start at 0, then vary through positive numbers, first increasing for $u > 1/2$ then decreasing for $u < 1/2$ and eventually tending towards 0.
The answer is (a)

C The left-hand side factorises to give

$$(x-1)(x-2) < \frac{(x-1)}{x}$$

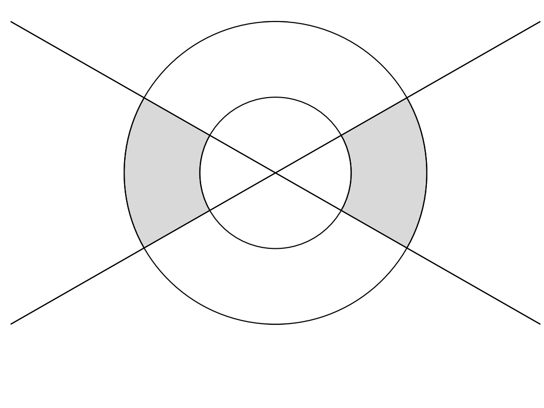
When are the two sides equal? That happens when $x = 1$ or when $x(x-2) = 1$, that is when $x = 1 \pm \sqrt{2}$. Note that one of these roots is negative and the other is positive.

That helps us to draw a graph (it's perhaps useful to note that the right-hand side is $1 - 1/x$).



From the graph, we see that the quadratic is below the reciprocal graph for $1 - \sqrt{}<x < 0$ and for $1 < x < 1 + \sqrt{2}$.
The answer is (b)

D The first inequality describes a region between two concentric circles. The second inequality is true if either (case 1) $x \geq 0$ and $-x \leq \sqrt{3}y \leq x$, or (case 2) $x \leq 0$ and $x \leq \sqrt{3}y \leq -x$.



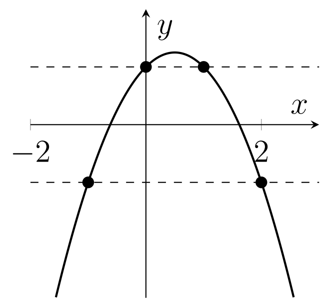
The intersection points between the inner circle and those lines are $(\sqrt{3}/2, 1/2)$ and reflections, so the acute angle subtended at the origin by the two lines is $60°$. We would therefore like one-third of the area between the two circles, which is $\frac{1}{3}\pi(2^2 - 1^2) = \pi$.
The answer is (b)

E The centre of the circle is $\left(\frac{p}{2}, \frac{q+1}{2}\right)$ and the radius is $\sqrt{\frac{p^2}{4} + \frac{(q-1)^2}{4}}$, so the equation of the circle is

$$\left(x - \frac{p}{2}\right)^2 + \left(y - \frac{q+1}{2}\right)^2 = \frac{p^2}{4} + \frac{(q-1)^2}{4}$$

On the line $y = 0$, this a quadratic for x. We can set the discriminant equal to zero for a repeated solution. The quadratic is $x^2 - px + q = 0$ and the discriminant is $p^2 - 4q$.
The answer is (c)

F We need $-1 < 1 + x - x^2 < 1$. Let's find the points where $1 + x - x^2 = 1$ (that's 0 and 1) and find the points where $1 + x - x^2 = -1$ (that's -1 and 2). Consider the graph.



We need either $-1 < x < 0$ or $1 < x < 2$.
The answer is (c)

G All of the options are of the form $y = kf(x)$ with k constant so let's try that in the equation. We get $k\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ on the left and $2(kf(x))^{1/4}$ on the right. Now $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = (f(x))^{1/4}$ so we just need $k = 2k^{1/4}$. We can rearrange this to get $k = 2^{4/3}$.
The answer is (c)

H Clearly f is always a positive whole integer from the recursion relations.

First find n with $f(n) = 1$. Because $f(2n) = f(n)$ and $f(1) = 1$, it must be the case that for all powers of 2, $f(n) = 1$. On the other hand, $f(2n+1) = f(n) + f(n+1)$ is definitely at least two, so $f(n)$ is only equal to 1 on the powers of 2.

Next consider values of n for which $f(n) = 2$. The first equation shows us that if we have n even and $f(n) = 2$ then $f(n/2) = 2$. Let's suppose instead that n is odd and write $n = 2m+1$. Then we would have $f(n) = f(m) + f(m+1)$. The only way for this to be 2 is if the right-hand side is $1 + 1$, which only happens if both the consecutive numbers m, $m+1$ are powers of 2, which only happens for 1, 2. Check that $f(3) = f(1) + f(2) = 2$. So the points where $f(n) = 2$ are precisely those where n is three times a power of 2 (they're of the form 3×2^k).

Now finally consider values of n for which $f(n) = 3$. The first equation again tells us that we can multiply any such n by 2. Let's look for odd solutions. They must come from $1 + 2$ on the right-hand side of the second equation, which only happens if a power of 2 (2^l say) is one more or one less than a number of the form 3×2^k. That's a bit unusual though, because both 2^l

and 3×2^k are even, unless one of the powers of 2 is equal to 1. So those consecutive numbers must be either 2 and 3 or they could be 3 and 4. This gives $f(5) = 3$ and $f(7) = 3$, but that's it for odd numbers. The solutions to $f(n) = 3$ are therefore 5×2^k or 7×2^k. None of these are multiples of 35.
The answer is (a)

I Take two to the power of each side for $x = (x + a)2^b$

The right-hand side is a line with gradient 2^b and y-intercept $a2^b$. We're looking for positive solutions, and there are two ways this could happen; either $a2^b$ is larger than zero, but the line grows slower than x, or $a2^b$ is negative but the line grows faster than x. This corresponds to (in the first case) $a > 0$ and $b < 0$, or (in the second case) $a < 0$ and $b > 0$. Either way, $ab < 0$. There's only one intersection between two straight lines, if any.
The answer is (c)

J Consider adding together these integrals to get

$$\int_1^2 \frac{x^2 + x^{-2}}{1 + x^4}\,\mathrm{d}x.$$

Now note that the numerator is x^{-2} multiplied by the denominator, so this simplifies to

$$\int_1^2 x^{-2}\,\mathrm{d}x = \left[-x^{-1}\right]_1^2 = \frac{1}{2}$$

The integral we're asked for is therefore $\frac{1}{2} - A$.
The answer is (e)