SOLUTIONS FOR EXTRA ADMISSIONS TEST IN MATHEMATICS, COMPUTER SCIENCE AND JOINT SCHOOLS NOVEMBER 2023

A Each of the given polynomials is of the form $y = x^3 - 3x^2 + kx$. The first derivative is $3x^2 - 6x + k$, and the second derivative is $6x - 6$. So if there is a point with zero derivative, then it will either be a local maximum or a local minimum, unless $x = 1$. At $x = 1$ the first derivative is $k - 3$, which is only zero if $k = 3$.
The answer is (c)

B Note that $\sqrt[10]{10^{11}}$ is $10^{11/10}$, so $\log_{10}\left(\sqrt[10]{10^{11}}\right) = \frac{11}{10}$. This is less than $\frac{11}{9}$.

To compare $\sqrt{\frac{3}{2}}$ with $\frac{11}{10}$, compare their squares; compare $\frac{3}{2}$ with $\frac{121}{100}$. The former is $\frac{150}{100}$ which is larger than $\frac{121}{100}$.

Now note that $\sqrt{3}\cos(44°)$ is slightly larger than $\sqrt{3}\cos(45°)$ which is $\sqrt{\frac{3}{2}}$.

Finally note that $\pi > 3$ so $\frac{\pi}{2} > \frac{3}{2} > \frac{11}{10}$.

So the smallest of the numbers is $\frac{11}{10}$.
The answer is (a)

C The required sum is the sum of all cubes up to 20^3 minus the sum of the even cubes up to 20^3.

$$\begin{aligned}
1^3 + 3^3 + 5^3 + 7^3 + \cdots + 19^3 &= \left(1^3 + 2^3 + 3^3 + 4^3 + \cdots + 20^3\right) - \left(2^3 + 4^3 + 6^3 + \cdots + 20^3\right) \\
&= \left(1^3 + 2^3 + 3^3 + 4^3 + \cdots + 20^3\right) - 2^3\left(1^3 + 2^3 + 3^3 + \cdots + 10^3\right) \\
&= \frac{20^2 \times 21^2}{4} - 8 \times \frac{10^2 \times 11^2}{4} \\
&= 10^2 \times \left(21^2 - 2 \times 11^2\right) \\
&= 10^2 \times (441 - 2 \times 121) \\
&= 19,900
\end{aligned}$$

The answer is (b)

D The numbers x and y are each either odd or even, so the squares x^2 and y^2 are each either multiples of 4 or one more than a multiple of 4. There are four cases to check, and after checking each, we find that the expression on the left-hand side could be a multiple of 4 (if x and y are both even, or both odd), or could be one more than a multiple of 4 (if x is odd and y is even), or could be three more than a multiple of 4 (if x is even and y is odd). The number on the right-hand side is two more than a multiple of 4. So there are no solutions.
The answer is (a)

E Starting with 1, there are nine one-digit numbers. Then from 11 to 99 there are 90 two-digit numbers, then there are 900 three-digit numbers, and so on. So the required sum is (grouping terms by the number of digits of n) equal to

$$9 \times 20^{-1} + 90 \times 20^{-2} + 900 \times 20^{-3} + \ldots$$

which is a geometric series with first term $\frac{9}{20}$, common ratio $\frac{1}{2}$ and sum to infinity equal to $\frac{9}{10}$.
The answer is (c)

F If $a = c$ and $b = d$ then the expression we're given for $f\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right)$ becomes $\begin{pmatrix} a^2 + b^2 \\ 2ab + b^2 \end{pmatrix}$.

We're looking for whole numbers (positive or negative or zero) for a and b such that $a^2 + b^2 = 2$ and $2ab + 2b^2 = 0$. From the second equation we conclude that $b = 0$ or $a + b = 0$. If $b = 0$ then $a^2 = 2$ but there are no whole numbers that square to 2. If $a + b = 0$ then we can eliminate b from the first equation and conclude that $a = \pm 1$. We should check that $\binom{1}{-1}$ and $\binom{-1}{1}$ both work.

The answer is (c)

G The triangular numbers described in the question as the sum of the first k positive integers for some $k \geq 1$ have the form $\frac{1}{2}k(k + 1)$, if we sum the arithmetic series.

Let's write $N = x + y$. Then $0 < y \leq N$ and $f = \frac{1}{2}N(N + 1) + y$. In words, first we choose a triangular number and then we add some positive quantity y. The number y can never be quite large enough to make the "next" triangular number because the difference between $\frac{1}{2}N(N+1)$ and $\frac{1}{2}(N + 1)(N + 2)$ is $N + 1$. But f can take all other values.

The answer is (e)

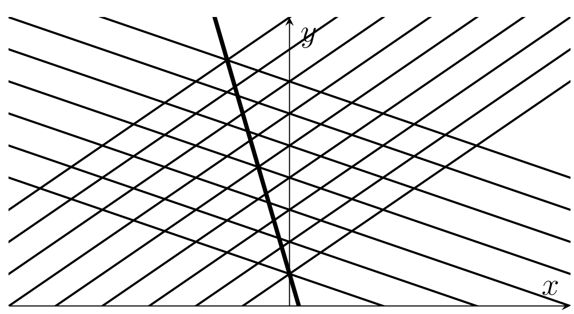
H We're told that $2x^4 - 3x^3 - 5x^2 + 2x + 2 = mx$ has exactly four real solutions x_1, x_2, x_3, x_4. Moving the mx to the other side of the equation, the Factor Theorem implies that the resulting polynomial factorises like this;

$$2x^4 - 3x^3 - 5x^2 + (2 - m)x + 2 = 2(x - x_1)(x - x_2)(x - x_3)(x - x_4),$$

where we have been careful to include the factor of 2 in order to match the leading coefficient of x. Now subsitute $x = 0$ in both sides of the equation for the result that $2 = 2x_1x_2x_3x_4$.

The answer is (b)

I The equations describe grid of parallel lines, with the line $y = 1 - 10x$ crossing opposite corners and the midpoint of the grid, but no other grid points.



There would normally be 16 points where the line $y = 1 - 10x$ crosses 16 given lines, but at three grid points the line $y = 1 - 10x$ crosses two of the given lines "simultaneously". There are only 13 distinct points where the line $y = 1 - 10x$ crosses one or more of the other lines.

The answer is (b)

J The equation rearranges to

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = x^2$$

If we recognise the left-hand side as $(x + y)^4$ then we can simplify this down to $(x + y)^4 = x^2$. Now if x is positive then $(x + y)^2 = x$ and so $y = -x \pm \sqrt{x}$. Since x grows faster than $\sqrt{x}$, the value of y is eventually negative for both of these solutions (once $x > 1$). Only one of the graphs behaves like this.

The answer is (d)