Categorical Post-Quantum Theories

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I could have been a signpost

Could have been a clock

As simple as a kettle

Steady as a rock

-Nick Drake, *One of These Things First*

Acknowledgements

This thesis is the culmination of the wonderful three and a half years I spent in Oxford, a city I found to be rich in both scholarly and culinary delights. Let the spicy potato and falafel wraps not be forgotten, especially for their importance during the writing of this thesis.

Of course, nobody completes such an endeavour without the assistance of others. Firstly I owe my thanks to my supervisors Aleks and Stefano. The work I did with Stefano forms Part I of this thesis and the work with Aleks forms Part II. Thank you both for the insights I gained from you, Stefano for helping to mould me as a junior researcher and Aleks for helping to cement this.

The last chapters of Part II are also indebted to collaborations with Mario and Matt. A particular thank you to Mario for his never-ending enthusiasm and his encouragement to finally write down some of the ideas that form the end of this thesis.

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A very special mention goes to Matt: for all the weird times at Slug House. Were it not for him, I likely would never have discovered this research area, nor the group at Oxford. You have him to blame for the chaos that is about to ensue over the next ~ 200 pages.

Alongside Matt, there are the other students of the UCL CDT: thank you all for a tremendous year. To Amy, Dan and Ed our lockdown virtual pub nights helped to keep me sane and I hope for more shenanigans soon.

Finally, I would like to thank my family, my parents for their everlasting encouragement; and my sister for always making me laugh. I am deeply grateful for all the support they have given me.

Abstract

In this thesis of two Parts, we investigate the application of categorical methods to modelling post-quantum theories.

In Part I we study hyper-decoherence between quantum-like theories. Chapter 1 serves as an introduction to Categorical Probabilistic Theories which combine elements of Categorical Quantum Mechanics and Operational Probabilistic Theories, and to CPM categories which generalise the CPM construction of Selinger to allow for richer group symmetries. In Chapter 2 we study the theory of density hypercubes which exhibits a hyper-decoherence mechanism witnessing quantum theory as an effectful subtheory. We show that this hyper-decoherence process is probabilistic within the theory of density hypercubes and discuss some plausible operational interpretations of this. As a result, we side-step a no-go result regarding the existence of deterministic hyper-decoherence maps, showing that it is nevertheless possible for a post-quantum theory to possess probabilistic hyper-decoherence maps. In Chapter 3 we focus on a particular case of the CPM construction, where the symmetries are generated by the Galois group of a finite field extension. We discuss how to construct probabilistic theories which form towers of decoherence in bijection with the subfields of a Galois extension. These towers generalise the decoherence process of standard quantum theory.

In Part II we study profunctorial methods and their application to spacetime and quantum supermaps. Chapter 4 serves as an introduction to profunctors, promonoidal categories and premonoidal categories, including the enriched version of the latter. Chapter 5 introduces some toy categories of causal curves in spacetime and discusses how we might upgrade the partial monoidal structure of such categories to a total tensor using both pre- and promonoidal categories. Chapter 6 makes this combination of pre- and promonoidal categories more formal, introducing the notion of a pro-effectful category. In the final Chapter 7 we describe how we can use the category of coend optics as a model of quantum combs. We describe the promonoidal structures on this category and their interpretation as horizontal and vertical composition of holes in monoidal categories. We also generalise coend optics to allow for a premonoidal base category, and point towards how the methods of this Chapter might be extended to include arbitrary quantum supermaps.

Contributions

Alongside some small changes in presentation and the inclusion of more background material, this thesis is largely based on work from the papers [96, 95, 97, 94, 72, 98]. Part I focuses on hyper-decoherence between probabilistic theories and is taken from the following two works co-authored with Stefano Gogioso.

[96] James Hefford and Stefano Gogioso. Hyper-decoherence in Density Hypercubes. *EPTCS*, 340:141–159, 2021. DOI: `10.4204/eptcs.340.7`. In Proceedings QPL 2020.

[95] James Hefford and Stefano Gogioso. CPM Categories for Galois Extensions. *EPTCS*, 343:165–192, 2021. DOI: `10.4204/eptcs.343.9`. In Proceedings QPL 2021.

I was the lead author of both of these works, and carried out the majority of the research contained therein. Stefano suggested the original idea of studying the phase groups of density hypercubes and trying to find a causal completion of its hyper-decoherence map, but the results are my own. Similarly, Stefano suggested studying CPM categories that were generated by Galois extensions, but the intricacies of the resulting decoherence towers and the tools required to study these were developed by myself.

Chapter 2 is based upon [96] and Chapter 3 is based upon [95]. Chapter 1 is contains a mixture of background material and novel contributions. Section 1.4 is a generalisation of the construction of [87], and Section 1.5 is novel - both are taken from [95]. The definitions given in 1.2 of hyper-decoherence within the framework of CPTs are novel, building upon those given in [117, 89, 96].

Part II focuses on the application of profunctorial methods to models of spacetime and quantum supermaps, with a particular focus on the monoidal-like structure of these models. It is taken from the following three works co-authored variously with Aleks Kissinger, Cole Comfort and Mario Román.

[97] James Hefford and Aleks Kissinger. On the Pre- and Promonoidal Structure of Spacetime. *EPTCS*, 380:284–306, 2023. DOI: `10.4204/EPTCS.380.17`. In Proceedings ACT 2022.

[94] James Hefford and Cole Comfort. Coend Optics for Quantum Combs. *EPTCS*, 380:63–76, 2023. DOI: `10.4204/EPTCS.380.4`. In Proceedings ACT 2022.

[98] James Hefford and Mario Román. Optics for Premonoidal Categories, 2023. DOI: `10.48550/arXiv.2305.02906`. arXiv: `2305.02906`. To appear in Proceedings ACT 2023.

I was the lead author of [97] and [98], while [94] was an even contribution.

Aleks and I had been independently thinking about decompositional models of spacetime, their partial monoidality and the possibility of using promonoidal categories to model joint systems. The category we describe in [97] is based upon one Aleks had written down many years ago, and the figures demonstrating the morphisms and composition in Slice are due to Aleks. All the proofs, including those of the promonoidality of Slice, and the key result connecting representability with spacelike separation are my own. The discussion of a possible logical interpretation of Slice is due to Aleks, and not included in this thesis.

The combination of promonoidal and premonoidal structure suggested by the investigations into Slice led to the work contained in [98], of which all the proofs and results are my own. Mario helped to give a better overall view to the work; understand its connections to his own work on effectful categories [138]; and understand better the connections to the literature on effectful optics. Futhermore, our conversations during his visit in October 2022 were very helpful for encouraging me in the development of this work.

Cole and I worked together on [94] and most results can be considered a joint effort: in particular the definition of the category of combs; the equivalence with optics in the Propositions 47 and 48; and the polycategory of combs. The equivalence of Proposition 49 is wholly mine.

Chapter 4 is almost entirely background material. Section 4.5.1 is novel and does not seem to have been explicitly written down elsewhere, though it could be considered to be folklaw and follows from some considerations in [25]. The discussion of enriched premonoidal categories in Section 4.6 appears to be novel.

Chapter 5 is based upon [97] and is therefore completely novel. Chapter 6 is based upon [98] - it contains a mixture of background material on effectful categories, and novel material. In particular, the results on effectful categories as pseudomonoids (in Section 6.1) is novel and the Sections 6.3 and 6.4 about tight $\mathcal{V}^2$-profunctors and characterising pro-effectful categories as pseudomonoids are also novel.

Chapter 7 is based upon [94] and [98]. Sections 7.2 and some of 7.3 come from [94]. The remainder of Section 7.3 is background material on Tambara modules, the produoidal structure of which comes originally from [83] and is also discussed in the following article co-authored with Mario Román and Matt Earnshaw:

[72] Matt Earnshaw, James Hefford, and Mario Román. The Produoidal Algebra of Process Decomposition, 2023. DOI: `10.48550/arXiv.2301.11867`. arXiv: `2301.11867`.

No other results from [72] are used in this thesis. Section 7.4 comes from [98]. The suggestions of how to connect some current investigations into supermaps [161, 160] with the optics literature and the work contained here are novel.

Contents

Introduction

One of the current drivers of theoretical physics is the effort to reconcile quantum theory with gravity, as it is widely understood that the two most successful theories of modern physics – Quantum Field Theory and General Relativity – are fundamentally incompatible. There are many ongoing endeavours to a develop a unifying theory, including string theory and quantum loop gravity, although none have yet to be widely accepted as the "true" theory.

Alongside this, there are the endeavours of Topological Quantum Field Theory [7, 9] and Algebraic Quantum Field Theory [93] to place Quantum Field Theory on a firmer mathematical foundation. Both of these frameworks are inherently categorical - they can be seen very roughly as the studies of certain classes of functors from categories of "spacetimes" to categories of "processes" or "algebras of observables". It is hoped that Topological and Algebraic Quantum Field Theory might also provide a novel mathematical setting for unifying Quantum Field Theory and General Relativity, perhaps by providing new perspectives on currently existing reconciliations or by providing an altogether new theory.

Stepping beyond the quantum into a truly post-quantum theory is also of interest in the field of Quantum Foundations. Categorical Quantum Mechanics (CQM) [3, 54] and Operational Probabilistic Theories [42] provide frameworks for quantum-like theories, both taking *compositionality* as their fundamental paradigm over the more traditional dynamical view of physics. CQM stresses the importance of studying how systems and processes compose to produce emergent effects in larger systems, with category theory being the natural mathematical tool for capturing this.

It is the paradigm of compositionality in which this thesis sits, with a focus on the post-quantum. In Part I, we consider what it might mean from a process theoretic perspective for a post-quantum theory to "contain" quantum theory. Much as quantum theory contains classical theory as an effective subtheory accessed by decoherence, it is plausible that a post-quantum theory ought to possess a hyper-decoherence process witnessing quantum theory as an effective subtheory. Taking this view to its extreme we could envisage towers of quantum-like theories each contained in each other and each accessed by a decoherence-like process. Such a tower has not been developed before and we provide a general framework for constructing an infinite family of them, while ensuring that they exhibit certain properties which make them physically reasonable. The construction is based upon a generalisation of the CPM construction [142, 87] which produces categories of generalised completely positive maps from an underlying symmetric monoidal category equipped with an action by monoidal autofunctors. A particular instance of this construction generates the theory of density hypercubes [90, 96] which we study in some detail to show that it contains quantum theory as an effective subtheory accessed by a hyper-decoherence map. As a result we side-step a no-go result [117] and provide a way in which hyper-decoherence maps could nevertheless exist.

In Part II, we turn our attention to two topics which are challenging to model adequately in standard CQM: spacetime and quantum supermaps. The main categorical tool we investigate is this section is *profunctors* [25, 121], which have seen much use in the functional programming community [134, 47, 137, 5, 102] and in Topological Quantum Field Theory [18, 19, 70, 133, 140], but much more limited use in CQM. Profunctors are generalisations of functors in a similar way to how relations are generalisations of functions. This allows us to consider more general notions compositionality than in more standard approaches in CQM and as a result suggest novel ways of modelling spacetime and supermaps.

In contrast to the compositional view of CQM, to model spacetime we take instead a *de*compositional perspective as was first outlined in [55]. There is an assumption hidden in the foundations of CQM - one of independence of physical systems - which materialises in the types of categories with which CQM concerns itself. Monoidality implies that it is always possible to take the joint system $A \otimes B$ of any pair of systems A and B and that therefore the state of A can be independent of the state of B. When one takes a decompositional view of physics, we mean that we start with some large system in its entirety, break this down into subsystems and then hope to recover some fragments of compositionality. In doing so, one cannot expect that we will recover a monoidal category because there might be some pairs of systems which are incompatible - for instance the tensor $A \otimes A$ of a system A with itself cannot be expected to behave in a bifunctorial fashion. In previous work, *partially* monoidal categories have been suggested as a way of circumventing this problem [55, 85, 91], but such categories are somewhat displeasing to work with, for it is not always clear without additional information about some given systems A and B whether their tensor exists or not. In Chapters 4 and 5, we consider ways of upgrading these structures to richer ones in the form of *pro*monoidal and *pre*monoidal categories. We develop some toy categories of spacetime which exhibit these structures and suggest how one might model fields on spacetime categories with such structure.

In Chapter 6 we make formal the combination of promonoidal and premonoidal structure suggested by our toy spacetime categories. To do so we introduce the notion of a *pro-effectful* category which generalises the effectful categories (also known as Freyd-categories) of Power and Robinson [132, 131, 130]. Along the way we demonstrate an equivalence between pro-effectful categories and prostrong promonads and we fully generalise a result of Power on the closed embeddings of effectful categories [130]. As a result we demonstrate that pro-effectful categories are quite canonical - just as how promonoidal categories are the "shadow" of closed monoidal presheaf categories, pro-effectful categories are the "shadow" of closed effectful central presheaf categories. With this comes the full generalisation of Day convolution and Day's theorem [68, 65] to this setting.

In Chapter 7, the final chapter of this thesis, we develop profunctorial semantics for quantum supermaps - processes that act not on states of systems, but on the processes of quantum theory themselves [43, 44, 45, 126, 21, 22]. Most approaches to supermaps in the quantum literature rely on substantial structure on the category of first-order processes, particularly compact closure [110]. This means that definitions of "supermap" are very particular to quantum theory and not easily generalised to other process theories. We study the possibility of using coend optics as a model for quantum 1-combs (that is, circuits with a single hole) by comparing this definition with a standard definition used in the quantum literature and show that the two coincide for a number of categories relevant to quantum theory. We then consider the categorical semantics of n-combs and discuss how the category of optics has two promonoidal structures which capture the horizontal and vertical composition of holes in monoidal categories. The vertical tensor has been known since the seminal paper on the category of optics [127] but seems to have been neglected in applications. The horizontal tensor has not been discussed before and arises as a generalisation of the symmetric monoidal structure known by the optics community [134]. One intriguing aspect of this tensor is that it does *not* require the underlying category to be symmetric monoidal and this allows us to consider horizontal compositions of optics in a wider setting.

Spurred by our investigations into premonoidal categories we then generalise the category of optics to premonoidal categories. We show that this category also has a promonoidal structure capturing the vertical composition of holes in premonoidal categories. The horizontal composition is more complicated and leads us to our first example of a pro-effectful category.

In the very final section we consider an operationally motivated, process theoretic definition for supermaps given recently in [161] and demonstrate that the natural categorical setting for this definition is the category of Tambara modules (profunctors with strength) [155]. These modules are the presheaf category of the category of optics and as a result we suggest a unifying categorical model for both quantum supermaps and combs which we feel is deserving of much future investigation.

Part I

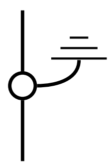
Categorical Approaches to Hyper-decoherence

Chapter 1

Hyper-decoherence in Probabilistic Theories

In quantum theory, the process of *decoherence* leads to the emergence of the classical from the quantum. It is caused by the interaction of the quantum system with an environment system which is inaccessible to the observer. This leads to the irretrievable loss of information to that environment and the effective classicalisation of the quantum system.

From a process theoretic point of view one is interested not in the intricacies of how decoherence occurs in a particular system in the lab, but rather in modelling the overall process as an abstract mathematical object in its own right. One standard approach is to consider the following map [58]:

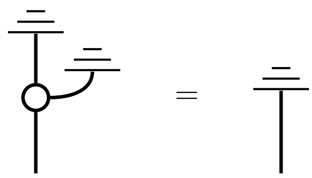


for some special commutative †-Frobenius algebra (†-SCFA) $\circ$ [57]. Such a decoherence map models our physical intuition in two-steps:

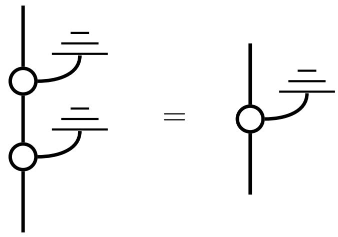
$$\sum_{ij} p_{ij} |i\rangle \langle j| \xmapsto{\text{broadcast}} \sum_{ij} p_{ij} |i\rangle \langle j| \otimes |i\rangle \langle j| \xmapsto{\text{trace}} \sum_{i} p_{ii} |i\rangle \langle i|$$

The †-SCFA acts to broadcast information from the quantum system into an environment coupling the two, and then the trace acts to discard any knowledge of the environment system. The result is that these maps act to zero-out the non-diagonal entries of a density matrix in the basis associated with $\circ$ and thus send a quantum state to a classical probability distribution.

Decoherence maps satisfy a couple of other important physically motivated properties. Firstly they are normalised, so that they correspond to processes that can be made to happen with certainty:



Secondly they are idempotent:



capturing the notion that once a quantum system is decohered to a classical system, further applications of the decoherence map produce no additional effect.

Hyper-decoherence is analogous to decoherence, but one level up: it leads to the emergence of quantum theory from some post-quantum theory, by suppression of the post-quantum part. The existence of hyper-decoherence maps has been considered in the literature as a possible mechanism for our lack of observation of post-quantum effects [164, 63, 64, 118, 117, 90, 96]: perhaps we simply cannot perform experiments accurately enough to see such effects, or perhaps hyper-decoherence happens on time scales shorter than those currently accessible to experimentalists. If quantum theory is to be deemed an effective, as opposed to fundamental theory of nature, hyper-decoherence is one possible mechanism to explain why we have yet to observe post-quantum phenomena.

In the literature, hyper-decoherence maps have been defined analogously to decoherence maps in quantum theory: idempotent and normalised maps taking hyper-quantum states to quantum states [117, 141, 90, 96]. In this Chapter we will develop a notion of decoherence between more arbitrary probabilistic theories, and then provide a way of constructing infinite families of probabilistic theories arranging themselves in towers of decoherence structures. A case of this construction is the theory of density hypercubes which is particularly interesting because it contains quantum theory as an effective sub-theory.

1.1 Probabilistic Theories

We will take our framework of "probabilistic theory" to be that of Categorical Probabilistic Theories (CPTs) [89]. CPTs combine elements of Categorical Quantum Mechanics (CQM) [3] and Operational Probabilistic Theories (OPTs) [42] to allow us to speak about preparations, measurements, probabilities, joint-systems etc. familiar to the OPT approach but in the language of category theory.

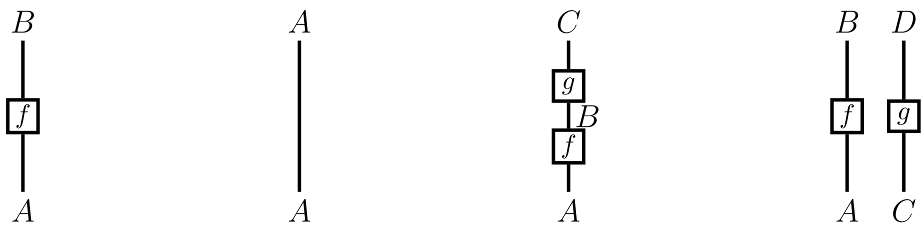
The underlying mathematical object of a CPT is a symmetric monoidal category $\mathcal{C}$. A category has a collection of objects or *systems* A (e.g. qubits, electrons, classical bits) and morphisms or *processes* $f : A \to B$ which evolve a system A into a system B. Processes can be composed so that given $f : A \to B$ and $g : B \to C$ there is a morphism $g \circ f : A \to C$. This composition is associative $(h \circ g) \circ f = h \circ (g \circ f)$ and has two-sided identities: each object is equipped with an identity process $1_A : A \to A$ such that $f \circ 1_A = f$ and $1_B \circ f = f$ for any $f : A \to B$.

Monoidal categories have an additional form of composition allowing us to form joint-systems. For any pair of systems A and B, there is a system $A \otimes B$, and for any two morphisms $f : A \to B$ and $g : C \to D$ there is a morphism $f \otimes g : A \otimes C \to B \otimes D$. More formally, the category $\mathcal{C}$ is equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ sending pairs of objects and morphisms to their tensor products. These tensor products are essentially associative and unital for there exists an object I and natural isomorphisms with components

$$\alpha_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$\lambda_A : I \otimes A \to A$$
$$\rho_A : A \otimes I \to A$$

which much satisfy some coherence conditions known as the triangle and pentagon equations [24, 124, 109] (see Section 4.3 for more on these).

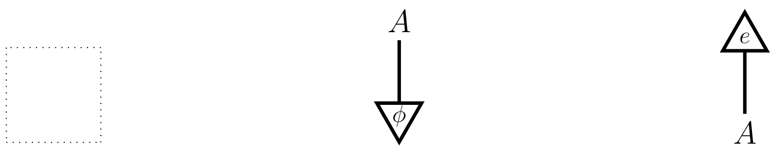
Monoidal categories permit a graphical string diagrammatic calculus which dates back to Penrose's notation for tensor calculus [128] and was formalised in [106, 107]. The main ingredients of this calculus can be summarised as follows:



$f : A \to B$ $\qquad$ $1_A : A \to A$ $\qquad$ $g \circ f : A \to C$ $\qquad$ $f \otimes g : A \otimes C \to B \otimes D$

Morphisms are drawn as boxes to be read bottom-to-top; the identity morphisms are bare wires; composition of morphisms is given by connecting the input and output wires of the boxes; and the tensor product is given by placing boxes horizontally next to each other.

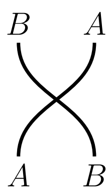
The unit object I is represented by the empty diagram and morphisms with domain or codomain given by I are drawn as triangles showing that they emerge from or disappear into the empty diagram:



Unit object I $\qquad$ $\phi : I \to A$ $\qquad$ $e : A \to I$

Such morphisms play an important role in CQM and are referred to as *states* and *effects* respectively. Those morphisms where the input and output is the unit object $s : I \to I$ are known as *scalars*. We will at times make use of bra-ket notation writing $|\phi\rangle$ for a state and $\langle e|$ for an effect. A most important category for quantum theory is FHilb which has finite-dimensional Hilbert spaces as its objects and linear maps as its morphisms. FHilb is a monoidal category when equipped with the tensor product of Hilbert spaces. The unit object is the complex numbers $I = \mathbb{C}$. In this category the states $\mathbb{C} \to H$ are in bijection with elements of the Hilbert space H and effects $H \to \mathbb{C}$ are in bijection with elements of the dual space H^*. This explains the connection between the notations for states and effects in monoidal categories, our usage of bra-ket notation and the traditional bra-ket notation where $|\phi\rangle \in H$ and $\langle e| \in H^*$.

A monoidal category $\mathcal{C}$ is *symmetric* when it is equipped with a natural isomorphism with components $\sigma_{AB} : A \otimes B \to B \otimes A$ which we depict in the graphical calculus as a crossing:



Like α, λ and ρ, the natural isomorphism σ must satisfy some coherence conditions known as the triangle and hexagon equations [124].

At this point we are in almost in position to define a Categorical Probabilistic Theory. The idea is add additional structure to a symmetric monoidal category so that we have well-defined notions of classical-like system, causality, and probabilistic mixtures. Key to this is to fix a semiring R to act as our generalised probabilities - typically this will be $\mathbb{R}^+$ but could be $\mathbb{Q}^+$, the booleans $\mathbb{B}$ (modelling possibilities), or more wildly, $\mathbb{R}$ (modelling signed probabilities [2]) or $\mathbb{Q}_p$ (p-adic probabilities [86]). The category R-Mat is then a model of classical systems. An object of R-Mat is a natural number n and a morphism $f : m \to n$ is a $n \times m$ matrix with entries from the semiring R. Composition is given by matrix multiplication with identities given by the identity matrices. The symmetric monoidal structure is given on objects by multiplication and on morphisms by the Kronecker product. The unit object is the number 1.

Let us now give the definition of a Categorical Probabilistic Theory in full before unpacking the details.

Definition 1 (Categorical Probabilistic Theory (CPT) [89])**.** Fix a commutative semiring R. An R-probabilistic theory (R-PT) is a symmetric monoidal category $\mathcal{C}$ such that:

- $\mathcal{C}$ is enriched in the category CMon of commutative monoids,
- there is a full symmetric monoidal subcategory $\mathcal{C}_{\mathrm{cl}}$ of $\mathcal{C}$ which is symmetric monoidally and linearly equivalent to R-Mat, that is, the equivalence should preserve the symmetric monoidal structure and the enrichment,
- $\mathcal{C}$ has an *environment structure* which on $\mathcal{C}_{\mathrm{cl}}$ coincides under the equivalence with that of R-Mat.

Enrichment in commutative monoids means that each hom-object $\mathcal{C}(A, B)$ is an object of CMon, a commutative monoid[1]. As a result, it is possible to take the sum $f + g$ of morphisms $f, g : A \to B$ with the same domain and codomain, and this sum is associative, commutative and has a unit. Furthermore, this sum behaves well with composition so that $(h + g)f = hf + gf$ and $h(g + f) = hg + hf$.

Taking a monoidal category and enriching in CMon is not enough though to ensure that sums work well with tensor products. One method for resolving this issue is to ask for more structure in the form of biproducts and duals [100]. Instead, when we say that $\mathcal{C}$ is a "symmetric monoidal category enriched in CMon", we mean that $\mathcal{C}$ is enriched in CMon as a *monoidal* category. So $\mathcal{C}$ is no longer equipped with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, but a tensor product that is also a CMon-functor

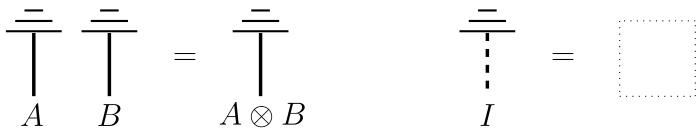
[1] for a more formal treatment of enriched categories see Chapter 4

$\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$, where $\boxtimes$ is the enriched tensor product of categories[2]. For objects A, B, C and D of $\mathcal{C}$, the enriched tensor functor gives a morphism of CMon

$$\mathcal{C}(A, C) \otimes_{\mathsf{CMon}} \mathcal{C}(B, D) \to \mathcal{C}(A \otimes B, C \otimes D)$$

The tensor product $\otimes_{\mathsf{CMon}}$ of CMon is given by taking the free commutative monoid on the product of the sets and quotienting by $(f, g) + (f, h) \sim (f, g + h)$, $(f, h) + (g, h) \sim (f + g, h)$, $(f, 0) \sim 0$ and $(0, f) \sim 0$. As a result, such an enriched category has a tensor product that behaves compatibly with sums: $f \otimes (g + h) = f \otimes g + f \otimes h$, $(f + g) \otimes h = f \otimes h + g \otimes h$, $f \otimes 0 = 0$ and $0 \otimes f = 0$.
By *environment structure* we mean the following:

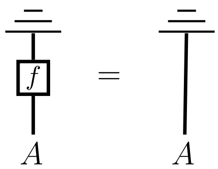
Definition 2 (Environment Structure [48, 58, 50])**.** Let $\mathcal{C}$ be a symmetric monoidal category. An environment structure for $\mathcal{C}$ consists of a choice of an effect $\bar{\top}_A : A \to I$ for each object A of $\mathcal{C}$ such that the following two equalities hold



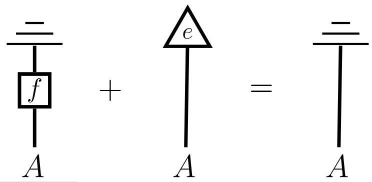
for all A and B.

The environment structure of a CPT equips the theory with a notion of causality which allows us to speak about those processes that happen with certainty and those that can be made to happen probabilistically:

Definition 3 (Normalised/Causal Process)**.** A process $f : A \to B$ is causal (or normalised) when:



A process $f : A \to B$ is *sub-causal* if there exists an effect $e : A \to I$ such that:



[2]such categories have been called $\mathcal{V}$-monoidal categories [68]

The idea of a CPT is to combine together elements of the frameworks of CQM [3] and OPTs [42], allowing us to talk about preparations, measurements, coarse-graining, classical control, etc, all familiar to the operational framework but in the language of category theory. In this way, the constraints on a CPT ensure that it is "physically reasonable" while permitting sufficient scope for theories wildly different than quantum theory. For instance, it is agnostic to the precise probabilistic setting used, allowing for any choice of commutative semiring R of probabilities. It is by demanding that a CPT $\mathcal{C}$ contains a subcategory $\mathcal{C}_{\mathrm{cl}}$ (the *classical* sub-theory) equivalent to R-Mat, and by the CMon-enrichment that we can consider probabilistic mixtures of processes. The category R-Mat of classical systems comes with an environment structure given by the effects $\bar{\top}_n : n \to 1$ which act to sum each column of a matrix $f : m \to n$. Thus, the normalised matrices are those which are stochastic, and the normalised states are the probability distributions.

Many CPTs, including those we consider in this work, arise as the Karoubi envelope of some other category:

Definition 4 (Karoubi Envelope)**.** Let $\mathcal{C}$ be a category. The *Karoubi envelope* $\mathsf{Split}(\mathcal{C})$ is the category with objects of the form (H, e), where H is some object of $\mathcal{C}$ and $e : H \to H$ is an idempotent. The morphisms $f : (H, e) \to (H', e')$ in $\mathsf{Split}(\mathcal{C})$ are exactly the morphisms $f : H \to H'$ in $\mathcal{C}$ which are invariant under the idempotents, such that $f = e' \circ f \circ e$. [143, 59, 89] The *normalised* Karoubi envelope $\mathsf{Split}^{\bar{\top}}(\mathcal{C})$ is the full subcategory of the Karoubi envelope spanned by only the normalised idempotents e.

From an operational perspective, objects (H, e) of the Karoubi envelope capture a situation in which it can be safely assumed that an idempotent process e has taken place between any two operations, e.g. because it happens on time-scales much smaller than those operationally accessible. This is, for example, the way in which classical systems arise from quantum systems by decoherence.

Taking the Karoubi envelope behaves well with probabilistic theories, sending any R-probabilistic theory to another R-probabilistic theory:

Proposition 1. *Let $\mathcal{C}$ be an R-PT. Then $\mathsf{Split}(\mathcal{C})$ is an R-PT.*

Proof. A proof was given in [89] for the normalised Karoubi envelope $\mathsf{Split}^{\bar{\top}}(\mathcal{C})$. To extend this result to the full Karoubi envelope is little work. Like in [89] $\mathsf{Split}(\mathcal{C})$ inherits CMon-enrichment from $\mathcal{C}$ and contains a full subcategory of classical systems equivalent to R-Mat. Indeed, $\mathsf{Split}(\mathcal{C})$ contains $\mathcal{C}$ as a full subcategory given by the objects of the form $(H, 1_H)$ and thus also contains $\mathcal{C}_{\mathrm{cl}}$. It is the environment structure of $\mathsf{Split}(\mathcal{C})$ that requires more care than $\mathsf{Split}^{\overset{=}{\top}}(\mathcal{C})$.

Given an object (H, e) of $\mathsf{Split}(\mathcal{C})$ we can take the discarding map to be given by $\overset{=}{\top}_H \circ e$, that is, the discard for H pre-composed with the idempotent e. Checking that this is a valid environment structure is straightforward and for the normalised idempotents, it is the same environment structure as described in [89] because $\overset{=}{\top}_H \circ e = \overset{=}{\top}_H$. □

The previous proposition is useful because whenever we speak of an R-PT we are able to assume it is of the form $\mathsf{Split}(\mathcal{C})$, for if we have an R-PT not of this form we can utilise Proposition 1 to turn it into one of this form containing the original R-PT as a full subcategory.

1.2 Decoherence and Hyper-decoherence

We can now define decoherence maps in arbitrary R-PTs:

Definition 5 (Decoherence Map)**.** Let $\mathcal{C}$ be an R-PT. A collection of maps $\{\mathrm{dec}\}$ is a *family of probabilistic decoherence maps* when:

1. each map $\mathrm{dec} : H \to H$ is an idempotent,
2. the full subcategory of $\mathsf{Split}(\mathcal{C})$ spanned by objects of the form (H, dec) is symmetric monoidally and linearly equivalent to R-Mat,
3. each map $\mathrm{dec} : H \to H$ can be completed to a normalised process in $\mathcal{C}$. That is, for each dec there exists a process $f : H \to H$ such that $\mathrm{dec} + f$ is causal.

When each map dec is in fact normalised, we call $\{\mathrm{dec}\}$ a *causal family of decoherence maps.*

Condition 2 ensures that the family of decoherence maps really behave like decoherence maps by producing a category equivalent to the category of classical systems. Condition 3 ensures that the decoherence maps can be implemented in the theory: by demanding that there exists a causal completion of each decoherence map we know we can actualise each decoherence map at least probabilistically.

We can extend the notion of decoherence maps to processes that enact transitions between different R-PTs:

Definition 6 (Hyper-decoherence Map)**.** Let $\mathcal{C}$ be an R-PT. A collection of maps $\{\mathrm{hypdec}\}$ is a *family of probabilistic hyper-decoherence maps* when:

1. each map hypdec : $H \to H$ is an idempotent,

2. the full subcategory of $\mathsf{Split}(\mathcal{C})$ spanned by objects of the form (H, hypdec) is an R-PT,

3. each map hypdec : $H \to H$ can be completed to a normalised process in $\mathcal{C}$.

When each map hypdec is in fact normalised, we call $\{\text{hypdec}\}$ a *causal family of hyper-decoherence maps.*

We are now in a position to define what it means for a theory to be *post*-quantum:

Definition 7 (Post-quantum Theory)**.** An $\mathbb{R}^+$-probabilistic theory $\mathcal{C}$ is *post*-quantum if there is a family of probabilistic hyper-decoherence maps $\{\text{hypdec}\}$ (with at least one of the maps non-trivial) such that the full subcategory spanned by hyper-decohered systems of the form (H, hypdec) is equivalent to quantum theory $\mathsf{CPM}(\mathsf{FHilb})$.

From an operational perspective, a post-quantum theory is one such where quantum theory arises as an effective theory by means of hyper-decoherence happening *probabilistically* at time-scales much smaller than those operationally accessible to quantum experiments. Idempotence of hyper-decoherence maps ensures that once a system has collapsed to quantum it remains quantum. Idempotence also ensures that the probabilistic nature of hyper-decoherence manifests exactly once: conditional hyper-quantum collapse having happened at least once, hyper-decoherence is deterministic and does nothing to the quantum system. If observers are for some reason limited to the quantum part of the theory, hyper-decoherence would happen transparently to them: this is not too far removed from what is speculated to happen in string theory and brane cosmology, where the observable world is restricted to a brane within a larger bulk.

One may wonder why we consider *probabilistic* (hyper-)decoherence maps. A known no-go result [117] states that normalised hyper-decoherence maps into quantum theory cannot exist in operational probabilistic theories with purification if some additional assumptions are imposed—namely that pure states in quantum theory be pure in the larger post-quantum theory and that the maximally mixed state of quantum theory be maximally mixed in the larger post-quantum theory.

Theorem 1 ([117])**.** *Let $\mathcal{C}$ be an $\mathbb{R}^+$-probabilistic theory which possesses a causal family of hyper-decoherence maps realising quantum theory. Furthermore suppose that:*

- *tomography is possible, so that any two processes which give the same scalar on all state-effect pairs are in fact equal: $\langle e| f |\phi\rangle = \langle e| g |\phi\rangle, \forall \langle e|, |\phi\rangle \implies f = g$,*

- *all mixed states possess an essentially unique purification,*
- *all states which are pure in quantum theory are pure post-quantumly,*
- *the maximally mixed quantum state is maximally mixed post-quantumly.*

Then the hyper-decoherence maps are identities and thus $\mathcal{C}$ is not truly post-quantum.[3]

As a result, we allow a slightly weaker notion of hyper-decoherence map that can only be implemented probabilistically.

1.3 Post-Quantum Theories from Symmetries

Producing examples of genuinely post-quantum theories is a surprisingly difficult endeavour. Unpacking Definition 7 from an operational perspective, we can see that we would need (roughly):

- a collection of states S_A and effects E_A for each system type A,
- a collection of transformations between state spaces $S_A \to S_B$ for each A and B,
- for each system, a discarding map $\bar{\top}_A$,
- for systems A and B, a composite system $A \otimes B$, with the state, effects and discarding maps being compatible with the composites,
- idempotent hyper-decoherence maps which must be extendable to causal maps.

This is quite a lot of data to specify and as a result there have been only a few attempts namely: *density cubes* [64], *quartic quantum theory* [164] and *density hypercubes* [90, 96].

One point of commonality between these three theories is the usage of higher-dimensional tensors to describe post-quantum states. Quantum theory is a quadratic theory, in the sense that its states are matrices ρ_{ij}. Density cubes on the other hand is cubic with states ρ_{ijk}, while quartic quantum theory and density hypercubes are quartic with states ρ_{ijkl}.

Like quantum theory, where a density matrix obeys $\rho_{ij}^* = \rho_{ji}$ due to Hermicity, it seems reasonable to impose that the higher-dimensional tensors of a post-quantum

[3]the original work states this result in the framework of OPTs, and thus makes two additional assumptions *convexity* and *causality* which are inherent in the definition of probabilistic theory taken in the present work.

theory should satisfy some similar symmetry constraints. The symmetries of a state ρ are given by a permutation of the indices of the tensor and by an automorphism of the underlying field K, that is, an action $G \to \mathrm{Aut}(K)$ by a subgroup $G \leq S_n$ of the symmetric group where n is the number of indices of ρ. The symmetry of a quantum state ρ_{ij} serves a few purposes but foremost it ensures that $\rho_{ii} \in \mathbb{R}$ for all i. If we furthermore demand that $\rho_{ii} \geq 0$ then we can interpret these entries as probabilities of the outcomes i.

Now, it is reasonable to suppose that the post-quantum theory is defined over $\mathbb{C}$, indeed this seems necessary for the post-quantum theory to contain quantum theory.[4] If we assume that our probabilities should be in $\mathbb{R}$, then it is necessary to only consider the symmetries of ρ to be given by automorphisms of $\mathbb{C}$ which fix $\mathbb{R}$, else we would have no guarantee that $\rho_{ii\ldots i} \in \mathbb{R}$. These automorphisms have a special name - they are the Galois automorphisms of the Galois extension $\mathbb{R} \subset \mathbb{C}$ - and there are only two, the identity and complex conjugation $J_{\mathbb{C}}$.

Starting from the symmetries of a theory is a good way of trying to construct post-quantum theories as it ensures a certain amount of consistency. Not only can it be used to ensure that there are sensible probabilities, asking for similar symmetries of the processes in a theory can ensure that the dynamics do not take us outside the state-space. Let us now look at each of density cubes and quartic quantum theory in turn. We leave a thorough investigation of density hypercubes to Chapter 2.

Density Cubes

When we turn to defining density cubes, we need to specify a group homomorphism $G \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ for $G \leq S_3$ to act as the symmetries of the state ρ_{ijk}. S_3 has subgroups isomorphic to S_3, C_3, C_2 and the trivial group $\{*\}$. It is straightforward to show that there are no non-trivial homomorphisms $C_3 \to C_2$. For C_2, there is a single non-trivial homomorphism: the faithful one acting by complex conjugation, and for S_3 the only non-trivial homomorphism is:

$$e, (ijk), (ikj) \mapsto 1 \qquad\qquad (ij), (ik), (jk) \mapsto J_{\mathbb{C}}$$

This is the one chosen by Dakić et. al. - permuting any two indices of ρ_{ijk} conjugates the entry [64] - and we have seen why it is the only reasonable choice.

The issues with the density cubes framework were pointed out in [118], specifically:

[4]One could consider trying to work over, say, a division algebra containing $\mathbb{C}$ for instance the quaternions, or maybe over a field such as the hyper-complex numbers ${}^*\mathbb{C}$, but for now let us entertain the simplest possibility.

- how to form joint systems is not specified and is highly problematic to define,
- transformations between systems are ill-defined. In particular, it is possible to find transformations which produce states with complex probabilities for some outcomes.

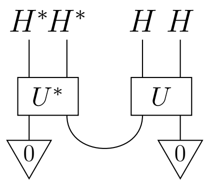
The second point essentially arises because the transformations do not satisfy the same symmetries as the states and as a result there is no guarantee that they should map valid density cubes to density cubes. This is in sharp contrast to quantum theory where the CP-maps between quantum states are also essentially invariant under the action of S_2 by complex conjugation.

As a result, density cubes does not satisfy all the demands of being a post-quantum theory.

Quartic Quantum Theory

In quartic quantum theory the states take the form ρ_{ijkl} and thus their symmetries should be given by an action $G \to \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of a subgroup $G \leq S_4$. S_4 has many more subgroups than S_3 and many more elements with order divisible by 2. As a result there are richer choices of possible symmetries of quartic states than cubic states (assuming an underlying field $\mathbb{C}$ with real probabilities).

The states of quartic quantum theory are linear combinations of states of the form:

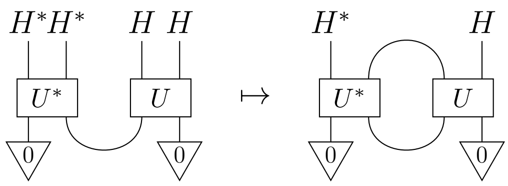


(1.1)

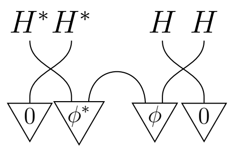
where U is a unitary. Such states exhibit C_2 symmetry – braiding the H's over the H^*'s conjugates the entries – but in general no further symmetries. The transformations are all CP maps which preserve the state-space, i.e. send mixtures of maps of the form (1.1) to mixtures of maps of the form (1.1).

The main issue with quartic quantum theory was discussed in [164, 118]: it is not clear how to define composite systems such that the allowed transformations behave compatibly. It is possible to compose systems, perform a bipartite transformation and then discard one of the systems and find oneself outside the state-space. As a result, quartic quantum theory would fail in its original formulation to form a symmetric monoidal category, and thus cannot be a post-quantum theory. On the other hand, if a suitable restriction on the space of transformations could be given, then it is plausible that quartic quantum theory could form a $\mathbb{R}^+$-probabilistic theory.

One pleasing aspect of quartic quantum theory is that it comes with a hyper-decoherence mechanism, given by partial trace:



which sends a quartic state to a quantum state. Any quantum state $\rho = \mathrm{tr}(|\phi\rangle\langle\phi|)$ can be embedded into a quartic state and recovered by this hyper-decoherence mechanism as:

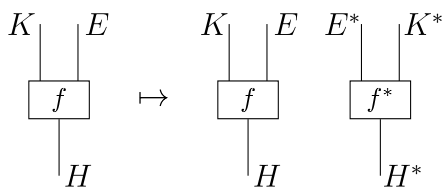


The issue with this hyper-decoherence mechanism is two-fold: is it not clear that it can be implemented as a process (even probabilistically) in the theory and it is not clear whether it can be used to recover all quantum channels from post-quantum ones, in part because the set of allowable transformations of quartic quantum states is not clearly defined. As such, it does not constitute a hyper-decoherence map as specified in Definition 6 and quartic quantum theory cannot be considered a post-quantum theory in the sense of Definition 7.

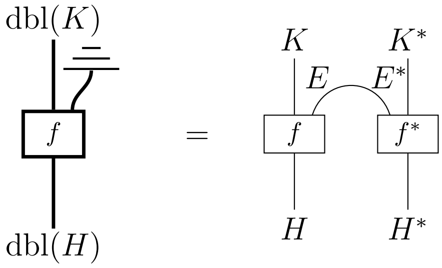
1.4 CPM Categories

Given the discussion in the last section of symmetry as our starting point for constructing post-quantum theories, in this section we will introduce a categorical framework, a generalised CPM construction, for producing theories with rich symmetry structure. This construction builds upon the one presented by Gogioso in [87].

The CPM construction inhabits a prominent position in the study of Categorical Quantum Mechanics [3]. It is the natural categorical generalisation of the transition from pure quantum theory to mixed state quantum theory, taking a †-compact category $\mathcal{C}$ and producing a category $\mathsf{CPM}(\mathcal{C})$ of completely positive maps [142, 143]. The CPM construction can be understood in two steps. Firstly, we "double" the morphisms of the original dagger compact category:



Secondly, we allow for discarding of environment systems:



(1.2)

The discarding is defined in terms of the *cap* arising from the duality of E and E^* and can be thought of as a categorical generalisation of the partial trace. If the underlying category is FHilb (the dagger compact category of finite dimensional Hilbert spaces and complex linear maps), then the maps in the shape of (1.2) are exactly the completely positive maps $\mathrm{dbl}(H) \to \mathrm{dbl}(K)$, where we have defined the *doubling functor* $\mathrm{dbl}(H) := H \otimes H^*$ and $\mathrm{dbl}(f) := f \otimes f^*$. In particular, the states $I \to \mathrm{dbl}(H)$ are Choi-Jamiołkowski isomorphic to positive operators $H \to H$. The dagger compact category with objects in the form $\mathrm{dbl}(H)$ for some finite-dimensional Hilbert space H and morphisms in the shape of (1.2) is called CPM(FHilb).

These "doubled" categories are ubiquitous in the CQM community, for they provide the ideal setting in which to study finite-dimensional quantum theory: physically irrelevant global phases are cancelled out and the category allows for a natural description of the interface between quantum and classical theory. Indeed, by considering the subcategory of the Karoubi envelope of CPM($\mathcal{C}$) spanned by decoherence maps one can produce a category of C*-algebras, known elsewhere as the CP* construction, which unifies quantum theory with classical theory [53, 62, 59].

Example 1. The category Split(CPM(FHilb)) is an $\mathbb{R}^+$-probabilistic theory. The full subcategory spanned by the decohered systems (H, dec) is equivalent to $\mathbb{R}^+$-Mat, i.e. classical theory.

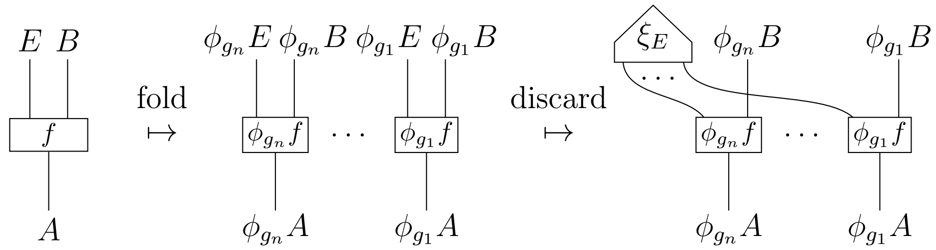
The original CPM construction is a special case of a generalised, "higher-order" CPM construction [87]: from now on, when talking about the CPM construction we shall refer to the latter, generalised version. This generalisation expands upon the inherent C_2 symmetry of the original CPM construction to produce categories with richer symmetries, captured by an essential invariance under the action of a group of monoidal autofunctors. The resulting categories are no longer necessarily the "double" of the original category: varying the generating group can produce for instance "tripled" (C_3) or "quadrupled" (C_4) categories or categories with much more exotic symmetry.

Like the original, the generalised CPM construction proceeds in two-steps. Starting with a symmetric monoidal category $\mathcal{C}$ one:

1. "folds" the objects and morphisms according to a strict left action by monoidal autofunctors $\phi : G \to \mathrm{Aut}(\mathcal{C})$ for a finite group G:

$$A \mapsto \bigotimes_{g \in G} \phi_g A$$

2. discards environment systems E by applying effects ξ_E from a collection Ξ of monoidally closed "discarding maps":



This two-step process generalises the doubling and discarding from the original CPM construction [142, 143], where the discarding maps are chosen to be the caps of the compact closed structure.

In the next two subsections we will formally introduce CPM categories before studying their connection to decoherence structures.

1.4.1 Folding and Equivariant Categories

Consider a symmetric monoidal category $\mathcal{C}$ equipped with a strict left action by monoidal autofunctors $\phi : G \to \mathrm{Aut}(\mathcal{C})$, for some finite group G. We restrict our attention to strict actions, since it is known that any category with a weak G-action (i.e. one with isomorphisms $\phi_g \phi_h \simeq \phi_{gh}$ satisfying certain compatibility conditions) can be strictified and is equivalent to a category with a strict G-action $\phi_g \phi_h = \phi_{gh}$ [145]. From such a symmetric monoidal category with G-action, one can derive a category of G-equivariant morphisms as follows.

Definition 8 (G-equivariant Category [74, 82])**.** Let $\mathcal{C}$ be a category, G be a finite group and $\phi : G \to \mathrm{Aut}(\mathcal{C})$ be a strict left action. The objects of the *G-equivariant category* $\mathcal{C}_G$ are pairs $(A, (\eta^g_A)_{g \in G})$ of an object A of $\mathcal{C}$ and a family of isomorphisms

$\eta_A^g : A \to \phi_g A$ for each $g \in G$, such that the following diagram commutes for all $g, h \in G$:

$$\begin{tikzcd} A \arrow[r, "\eta_A^h"] \arrow[dr, "\eta_A^{hg}"'] & \phi_h A \arrow[d, "\phi_h \eta_A^g"] \\ & \phi_h \phi_g A \end{tikzcd} \tag{1.3}$$

The morphisms $f : (A, (\eta_A^g)_{g\in G}) \to (B, (\eta_B^g)_{g\in G})$ of $\mathcal{C}_G$ are the morphisms of $\mathcal{C}$ which commute with the isomorphisms η_A^g for all $g \in G$:

$$\begin{tikzcd} A \arrow[r, "\eta_A^g"] \arrow[d, "f"'] & \phi_g A \arrow[d, "\phi_g f"] \\ B \arrow[r, "\eta_B^g"'] & \phi_g B \end{tikzcd} \tag{1.4}$$

There is a canonical forgetful functor $\iota : \mathcal{C}_G \to \mathcal{C}$ forgetting the equivariant structure.

Proposition 2. *Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category equipped with a G-action ϕ by strict monoidal autofunctors. Then $\mathcal{C}_G$ is also symmetric monoidal, with tensor product $\boxtimes$ defined as follows:*

$$(A, (\eta_A^g)_{g\in G}) \boxtimes (B, (\eta_B^g)_{g\in G}) := (A \otimes B, (\eta_A^g \otimes \eta_B^g)_{g\in G})$$

Proof. The monoidal structure $\otimes$ of $\mathcal{C}$ induces a monoidal structure $\boxtimes$ on $\mathcal{C}_G$. Strictness of ϕ_g implies that $\eta_A^g \otimes \eta_B^g$ have the correct type. Everything else quickly follows. $\square$

Definition 9 (G-functor [74, 82])**.** Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories equipped with G-actions ϕ and ϕ' respectively. A *G-functor* $(f, \sigma) : (\mathcal{C}, \phi) \to (\mathcal{C}', \phi')$ is a pair of a functor $f : \mathcal{C} \to \mathcal{C}'$ and a family of natural isomorphisms $\sigma_g : f\phi_g \to \phi'_g f$ for each $g \in G$, such that the following diagram commutes:

$$\begin{tikzcd} f\phi_g\phi_h \arrow[r, "\sigma_g\phi_h"] \arrow[dr, "\sigma_{gh}"'] & \phi'_g f \phi_h \arrow[d, "\phi'_g \sigma_h"] \\ & \phi'_g \phi'_h f \end{tikzcd}$$

If f is an equivalence of categories, then we call this a *G-equivalence.*

For any subgroup $H \leq G$, the G-action ϕ descends by restriction to an H-action, so one can form the H-equivariant category $\mathcal{C}_H$. It will often be the case that we have some preferred collection of the isomorphisms $\eta_A^h : A \to \phi_h A$, chosen to be *compatible with the monoidal product $\otimes$ of $\mathcal{C}$*, so that $\eta_A^h \otimes \eta_B^h = \eta_{A\otimes B}^h$ for all objects A and B and group elements $h \in H$. We write $\hat{\mathcal{C}}_{H,\eta}$ for the full subcategory of $\mathcal{C}_H$ spanned by objects involving isomorphisms in this family, those of the form $(A, (\eta_A^h)_{h\in H})$.

Proposition 3. *The category $\hat{\mathcal{C}}_{H,\eta}$ is equivalent to the naturaliser $Nat((\eta^h)_{h\in H})$ of the unnatural isomorphisms η^h in $\mathcal{C}$, i.e. the largest subcategory of $\mathcal{C}$ such that all of the η^h are natural (so that each autofunctor ϕ_h is naturally isomorphic to the identity).*

Proof. $\mathrm{Nat}((\eta^h)_{h\in H})$ is the largest subcategory of $\mathcal{C}$ such that all $\eta^h : 1 \Rightarrow \phi_h$ are natural isomorphisms. It is a wide subcategory containing only the arrows f which make diagram 1.4 commute. The equivalence to $\hat{\mathcal{C}}_{H,\eta}$ follows immediately by forgetting the families $(\eta^h_a)_{h\in H}$ involved in the objects, that is sending $(A, (\eta^h_A)_{h\in H})$ to A in $\mathrm{Nat}((\eta^h)_{h\in H})$. $\square$

Recall that for any group G with subgroup $H \leq G$, a *left transversal* T of H in G is a choice of representative for each left coset of H in G, i.e. a subset $T \subseteq G$ such that every left coset of H intersects T in exactly one point, $|T \cap gH| = 1$ for all $g \in G$. A *right transversal* is defined analogously but with right cosets.

With these definitions in place we can define the first ingredient of the CPM construction:

Definition 10 (Folding Functor)**.** Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category equipped with a G-action ϕ by strict monoidal autofunctors. Let T be a left transversal of some subgroup $H \leq G$. Let $(\eta^h_a)_{h\in H}$ be a collection of isomorphisms compatible with the monoidal product. We refer to the tuple $\tau := (G, H, \eta, T)$ as the *folding data*, and we define the *folding functor* $\mathrm{fld}_\tau : \hat{\mathcal{C}}_{H,\eta} \to \mathcal{C}$ as follows:

$$(A, (\eta^h_A)_{h\in H}) \mapsto \bigotimes_{t\in T} \phi_t \iota A \qquad\qquad f \mapsto \bigotimes_{t\in T} \phi_t \iota f \tag{1.5}$$

Remark. The folding functor from [87] arises as the special case where H is taken to be the trivial group. A transversal T of $H := \{e\}$ in G is then precisely the set $T = G$ of all the elements of G. Since ϕ is strict we have $\phi_e = \mathrm{id}_{\mathcal{C}}$ and hence $\mathcal{C}_{\{e\}} \simeq \mathcal{C}$, so that the folding functor can be considered an endofunctor fld_G on $\mathcal{C}$. We call this a *complete folding functor*, since it uses all the elements of its defining group.

Definition 11 (Folded Category)**.** For given folding data $\tau := (G, H, \eta, T)$, the *folded category* $\mathsf{FLD}_\tau(\mathcal{C})$ is the subcategory of $\mathcal{C}$ formed by the image of the folding functor, with objects and morphisms of the form (1.5). For a complete folding functor fld_G, we write $\mathsf{FLD}_G(\mathcal{C})$.

The folded category has pleasing symmetry properties - it is essentially invariant under each of the autofunctors ϕ_g. This follows because for any $g \in G$ the autofunctor ϕ_g acts on the indices of the tensor product to send a transversal T to another transversal T'. In general, this new transversal differs from the original in two ways - the ordering of the cosets to which each element belongs is permuted and the representative of each coset has been altered. To recover the original transversal we can compose two isomorphisms. Firstly, σ^g which rearranges the cosets back to the original ordering, by a composition of the symmetry isomorphisms arising from the symmetric monoidal structure. Secondly, ρ^g which recovers the representatives of the original transversal and is given by a composition of the isomorphisms η^g. Formally, ρ^g_A is given by

$$\rho^g_A := \bigotimes_{t \in T} \phi_t(\eta^{h_t}_A)^{-1}$$

where each h_t is the element of H which separates the representatives of a given coset of H between the two transversals T and T'.[5] The following proposition establishes this formally:

Proposition 4. *The autofunctors $\phi_g : \mathcal{C} \to \mathcal{C}$ restrict to functors $\phi_g : \mathsf{FLD}_\tau(\mathcal{C}) \to \mathsf{FLD}_\tau(\mathcal{C})$ on the folded category, and are naturally isomorphic to the identity on $\mathsf{FLD}_\tau(\mathcal{C})$.*

Proof. The original proof from [87], valid for complete folding functors, requires slight tweaking for our generalised setting. Any $g \in G$ acts on left transversals of $H \leq G$ by sending a left transversal T to another left transversal T', obtained by permuting the left cosets and altering the choice of element in each coset. There is an isomorphism σ^g_A, given by a suitable composition of the symmetry isomorphisms σ from the symmetric monoidal structure, which arranges the cosets into their original order. There is also an isomorphism ρ^g_A, given by a monoidal product of the isomorphisms η^h_A, which acts to recover the original choices in the transversal. Taken together, σ^g_A and ρ^g_A compose to an isomorphism between the folded object $\mathrm{fld}_\tau A$ and the action of G on the indices in the monoidal product.

$$\phi_g \mathrm{fld}_\tau A = \phi_g \bigotimes_{t \in T} \phi_t \iota A = \bigotimes_{t \in T} \phi_{gt} \iota A = \bigotimes_{t' \in T'} \phi_{t'} \iota A \xrightarrow{\sigma^g_A} \bigotimes_{t \in T} \phi_{th_t} \iota A \xrightarrow{\rho^g_A} \bigotimes_{t \in T} \phi_t \iota A = \mathrm{fld}_\tau A$$

[5]That is, if $t \in T$ is the representative of a coset tH then in the transversal T', the representative of tH is given by th_t for some $h_t \in H$.

where h_t in an element of H for each $t \in T$. We then have:

$$\begin{aligned}
\rho_A^g \circ \sigma_A^g \circ (\phi_g \mathrm{fld}_\tau f) \circ (\sigma_A^g)^{-1} \circ (\rho_A^g)^{-1} &= \rho_A^g \circ \sigma_A^g \circ \bigotimes_{t \in T} \phi_{gt} \iota f \circ (\sigma_A^g)^{-1} \circ (\rho_A^g)^{-1} \\
&= \rho_A^g \circ \sigma_A^g \circ \bigotimes_{t' \in T'} \phi_{t'} \iota f \circ (\sigma_A^g)^{-1} \circ (\rho_A^g)^{-1} \\
&= \rho_A^g \circ \bigotimes_{t \in T} \phi_{th_t} \iota f \circ (\rho_A^g)^{-1} \\
&= \bigotimes_{t \in T} \phi_t \iota f = \mathrm{fld}_\tau f
\end{aligned}$$

$\square$

Remark. For a complete folding functor, the previous proposition shows that there is a canonical embedding of $\mathsf{FLD}_G(\mathcal{C})$ into a G-equivariant category where the structural isomorphisms η^g are given by the permutations σ^g arising from compositions of the symmetry isomorphisms from the symmetric monoidal structure. This remark will be useful for the following propositions.

Proposition 5. *Suppose we have a complete folding functor for the G-action ϕ and suppose there is a subgroup $H \leq G$. Then the complete folding functor factorises into a complete folding functor for the induced H-action followed by the folding functor for any choice of transversal T of H in G.*

Proof. The G-action restricts to an H-action, giving a complete folding functor $\mathrm{fld}_H : \mathcal{C} \to \mathsf{FLD}_H(\mathcal{C})$. The category $\mathsf{FLD}_H(\mathcal{C})$ is H-equivariant with respect to isomorphisms σ^h given by a suitable composition of the symmetry isomorphisms from the symmetric monoidal structure; these isomorphisms merely act by "rearranging" the order of the tensor factors. Therefore, there is an embedding $e : \mathsf{FLD}_H(\mathcal{C}) \to \hat{\mathcal{C}}_{H,\eta}$ of the image of fld_H into $\hat{\mathcal{C}}_{H,\eta}$ given by sending $\mathrm{fld}_H f : \mathrm{fld}_H A \to \mathrm{fld}_H B$ to $\mathrm{fld}_H f : (\mathrm{fld}_H A, \{\sigma_A^h\}) \to (\mathrm{fld}_H B, \{\sigma_B^h\})$. Taking T to be a transversal of H in G, we can form the folding functor $\mathrm{fld}_\tau : \hat{\mathcal{C}}_{H,\eta} \to \mathcal{C}$ and observe that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{fld}_H"] \arrow[d, "\mathrm{fld}_G"'] & \mathsf{FLD}_H(\mathcal{C}) \arrow[d, "e"] \\
\mathcal{C} & \hat{\mathcal{C}}_{H,\eta} \arrow[l, "\mathrm{fld}_\tau"]
\end{tikzcd}$$

$\square$

When $H \trianglelefteq G$ is normal in G, we might expect that a complete folding functor for the G-action factorises through a *complete* folding functor for the quotient G/H-action. The following proposition establishes that this is indeed the case.

Proposition 6. *Suppose we have a complete folding functor for the G-action ϕ. Let $H \trianglelefteq G$ be a normal subgroup of G. Then the complete folding functor for the G-action factorises through complete folding functors for the H-action and the quotient G/H-action.*

Proof. It is known that there exists an induced action of G on $\mathcal{C}_H$ given by taking $\psi_g : \mathcal{C}_H \to \mathcal{C}_H$ to act on morphisms as ϕ_g and on objects as $(A, (\eta^h_A)_{h\in H}) \mapsto (\phi_g A, (\phi_g \eta_A^{g^{-1}hg})_{h\in H})$ [23]. This descends to an action of the quotient by picking a transversal of H in G containing the identity and defining $\hat{\psi}_{gH} := \psi_g$ where g is the representative of the coset gH in the transversal. Any two different choices of transversal T and T' result in different G/H-actions $\hat{\psi}$ and $\hat{\psi}'$, but there is a G/H-equivalence $(\mathrm{id}_{\mathcal{C}_H}, \sigma) : (\mathcal{C}_H, \hat{\psi}) \to (\mathcal{C}_H, \hat{\psi}')$ given by the natural isomorphisms $\sigma_g = \phi_g \eta^h$ where h is such that $g' = gh$ for g and g' the representatives of gH in T and T' respectively. Thus the action is essentially unique. With the G/H-action $\hat{\psi}$ one can form a complete folding functor $\mathrm{fld}_{G/H} : \hat{\mathcal{C}}_{H,\eta} \to \hat{\mathcal{C}}_{H,\eta}$.

Now, consider the complete folding functor $\mathrm{fld}_G : \mathcal{C} \to \mathcal{C}$. As in the previous proposition, this restricts to a folding functor $\mathrm{fld}_H : \mathcal{C} \to \mathsf{FLD}_H(\mathcal{C})$, and we have the embedding $e : \mathsf{FLD}_H(\mathcal{C}) \to \hat{\mathcal{C}}_{H,\eta}$ of the image of fld_H into $\hat{\mathcal{C}}_{H,\eta}$. The following diagram then commutes:

$$
\begin{tikzcd}
 & \mathsf{FLD}_H(\mathcal{C}) \arrow[rr, "e"] & & \hat{\mathcal{C}}_{H,\eta} \arrow[dr, "\mathrm{fld}_{G/H}"] & \\
\mathcal{C} \arrow[ur, "\mathrm{fld}_H"] \arrow[drr, "\mathrm{fld}_G"'] & & & & \hat{\mathcal{C}}_{H,\eta} \arrow[dll, "\iota"] \\
 & & \mathcal{C} & &
\end{tikzcd}
$$

□

1.4.2 Environment Structures

The folded category $\mathsf{FLD}_\tau(\mathcal{C})$ forms the starting point for a family of CPM categories. The second ingredient is the choice of environment structure.

Definition 12 (Environment Structure)**.** An environment structure Ξ is a family of sets Ξ_A of effects $\xi_A : \mathrm{fld}_\tau A \to I$ for each object A of $\mathcal{C}$ satisfying the following conditions:

- $\xi_A \in \Xi_A, \xi_B \in \Xi_B \implies (\xi_A \otimes \xi_B) \circ \pi^{-1}_{A,B} \in \Xi_{A\otimes B}$

- $\Xi_I = \{\mathrm{id}_I\}$
- $\xi_A \in \Xi_A, g \in G \implies \phi_g \xi_A = \xi_A \circ (\sigma_A^g)^{-1} \circ (\rho_A^g)^{-1}$

where σ_A^g and ρ_A^g are the isomorphisms defined in the previous section and $\pi_{A,B}$ is the following permutation (obtained by composition of symmetry isomorphisms):

$$\pi_{A,B} : \bigotimes_{t \in T} \phi_t A \otimes \bigotimes_{t \in T} \phi_t B \longrightarrow \bigotimes_{t \in T} \phi_t (A \otimes B)$$

The constraints of the environment structure ensure that Ξ is essentially closed under the monoidal product and that all the effects are invariant under the action of the autofunctors ϕ_g, up to (well-behaved) natural isomorphism.

Remark. From now on when we refer to an "environment structure" we mean Definition 12 which subsumes Definition 2. Indeed, the latter is a special case of the former where each set Ξ_A contains exactly one element. It is worth noting that it is still necessary to pick a particular effect for each object A (with the choices being closed under the monoidal product) to act as the designated overall discarding maps $\{\bar{\top}_A\}$, bestowing the theory with a notion of causality.

1.4.3 CPM Categories

Definition 13 (CPM Category)**.** The *CPM category* $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$ is the smallest subcategory of $\mathcal{C}$ containing the folded category $\mathsf{FLD}_\tau(\mathcal{C})$, all the effects of Ξ and their monoidal products with the identity morphisms, defined as follows:

$$\xi_A \boxtimes \mathrm{id}_{\mathrm{fld}_\tau B} := (\xi_A \otimes \mathrm{id}_{\mathrm{fld}_\tau B}) \circ \pi_{A,B}^{-1}$$

Remark. The CPM category is essentially invariant under the autofunctors, because so are both the folded category and the effects in the environment structure.

Proposition 7. *The CPM category* $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$ *is a symmetric monoidal category when equipped with the following tensor product, having* $\mathsf{FLD}_\tau(\mathcal{C})$ *as a monoidal subcategory:*

$$F \boxtimes G := \pi_{C,D} \circ (F \otimes G) \circ \pi_{A,B}^{-1}$$

where $F : \mathit{fld}_\tau A \to \mathit{fld}_\tau C$ *and* $G : \mathit{fld}_\tau B \to \mathit{fld}_\tau D$ *are generic morphisms in* $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$.

Proof. This was originally shown in [87] for the complete folding case. $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$ is a symmetric monoidal category with a monoidal product induced by that of $\mathcal{C}$. It is easy to check that the original proof generalises to the construction presented here. $\square$

Proposition 8. *Let $\mathcal{C}$ be a †-compact category equipped with a monoidal G-action ϕ. If there is some $g \in G$ such that $\phi_g = conj_{\mathcal{C}}$ is the conjugating autofunctor, then $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$ is a †-compact category.*

Proof. The dagger of $\mathcal{C}$ extends to the folded category by defining $(f \boxtimes g)^\dagger := f^\dagger \boxtimes g^\dagger$ and $\mathrm{fld}_\tau(f)^\dagger = \mathrm{fld}_\tau(f^\dagger)$. This is precisely the dagger of $\mathsf{FLD}_\tau(\mathcal{C})$ as a subcategory of $\mathcal{C}$.

It is easy to check (cf. [87]) that any morphism b of $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$ has a normal form, namely $b = (\mathrm{id}_{\mathrm{fld}_\tau B} \boxtimes \xi_E) \circ \mathrm{fld}_\tau(a)$ for some morphism $a : A \to B \otimes E$ of $\mathcal{C}$ and some effect $\xi_E \in \Xi_E$. Since $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$ is a subcategory of $\mathcal{C}$, the dagger descends immediately to give $b^\dagger = \mathrm{fld}_\tau(a^\dagger) \circ (\mathrm{id}_{\mathrm{fld}_\tau B} \boxtimes \xi_E^\dagger)$: all we need to show is that the latter is still a morphism in $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$. Using the †-compact closure of $\mathcal{C}$ and writing $\mathrm{cup}_A : I \to A^* \otimes A$ for the unit, it follows that:

$$b^\dagger = [\mathrm{id}_{\mathrm{fld}_\tau A} \boxtimes \xi_E^*] \circ [\mathrm{fld}_\tau a^\dagger \boxtimes \mathrm{id}_{\mathrm{fld}_\tau E^*}] \circ [\mathrm{id}_{\mathrm{fld}_\tau B} \boxtimes \mathrm{fld}_\tau \mathrm{cup}_E^*] \tag{1.6}$$

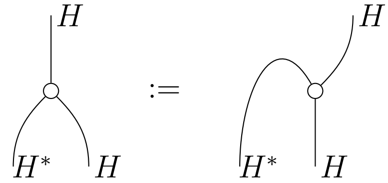
By assumption the conjugation functor is one of the autofunctors ϕ_g, so $\xi_E^* = \xi_E \circ (\sigma_E^g)^{-1} \circ (\rho_A^g)^{-1}$ and so each constituent map of (1.6) is in the normal form. It is fairly straightforward to check the remaining requirements for this to give a valid dagger structure on $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$.

The compact closure of $\mathcal{C}$ then implies that $\mathsf{CPM}_{\tau,\Xi}(\mathcal{C})$ is also compact closed—its unit and counit given by folding those of $\mathcal{C}$—and the discussion above is enough to conclude that the unit and counit are daggers of each other, giving us †-compact closure. □

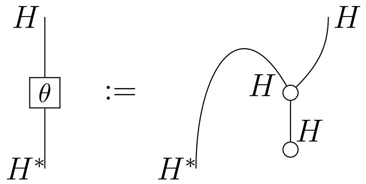
1.5 Environment Structures from Classical Structures

Suppose now that we are working in a symmetric monoidal †-category $\mathcal{C}$ which is rich in special commutative †-Frobenius algebras (†-SCFAs, also known as *classical structures*). In particular, assume that each object A has at least one †-SCFA on it. Classical structures provide an analogy to the category FHilb of finite dimensional Hilbert spaces, where †-SCFAs are in bijection with bases of a given Hilbert space [57]. These †-SCFAs are of vital importance to the categorical study of quantum theory, where they capture, to name but a few, phases [51], Fourier transforms [92] and decoherence.

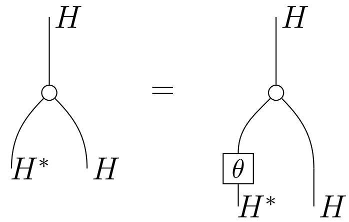
In the original CPM construction, decoherence can be studied by forming the Karoubi envelope and then taking a full subcategory spanned by so-called *decoherence* *maps*. For any †-SCFA, one can use the †-compact closure of FHilb to form the following map:

δ =  (1.7)

The decoherence maps are given by the composition $\mathrm{dec} := \delta^\dagger \circ \delta$. One can show that the full subcategory of the Karoubi envelope $\mathsf{Split}(\mathsf{CPM}(\mathsf{FHilb}))$ spanned by objects of the form $(H^* \otimes H, \mathrm{dec})$ is equivalent to $\mathbb{R}^+$-Mat, the category of positive real valued matrices, while of course the full subcategory spanned by objects of the form $(H^* \otimes H, \mathrm{id})$ is equivalent to $\mathsf{CPM}(\mathsf{FHilb})$. Consequently, the full subcategory of $\mathsf{Split}(\mathsf{CPM}(\mathsf{FHilb}))$ spanned by both decoherence maps and identities captures just enough to have both quantum theory and classical theory live within the same categorical setting. Further to its use in decoherence maps, the morphism δ from (1.7) can be used to construct an isomorphism $\theta : H^* \to H$,



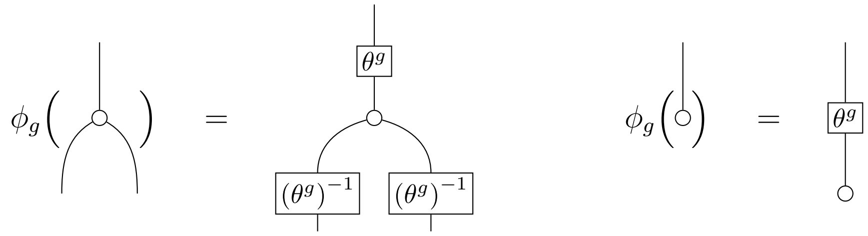
This is to be expected, since the existence of a †-SCFA on an object H implies that H is self-dual. The isomorphism θ is in fact an isomorphism of †-SCFAs, being both a monoid isomorphism for the multiplication (to its conjugate) and a comonoid isomorphism for the comultiplication (to its conjugate). Furthermore, we can rewrite δ in the following form:



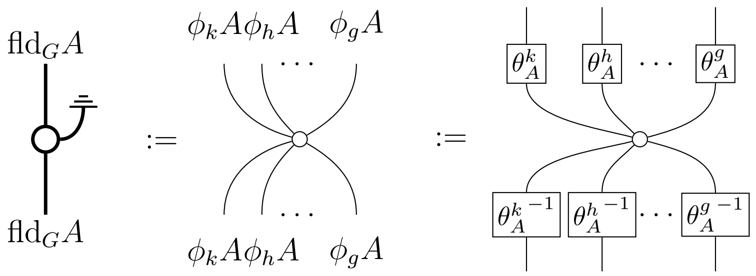
Looking at this from the perspective of higher-order CPM constructions, this suggests that it could be interesting to consider categories $\mathcal{C}$ equipped with a choice of isomorphisms $\theta^g_A : A \to \phi_g A$ for each object A and element $g \in G$, subject to suitable conditions; together with these isomorphisms, the presence of †-SCFAs implies that A is dual to $\phi_g A$ for all $g \in G$.

Definition 14 (Generalised Classical Structures)**.** Let $\mathcal{C}$ be a †-compact category equipped with a G-action ϕ. A *generalised classical structure* on an object A of $\mathcal{C}$

is a pair $(\circ, \theta_A)$ of a †-SCFA $\circ$ on A and a choice of automorphisms $\theta = (\theta^g)_{g \in G}$ on A satisfying the same commutative diagram (1.3) as η^g_A together with the following compatibility conditions:

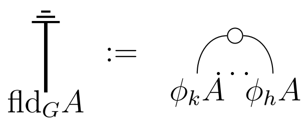


Definition 15 (Complete Decoherence Maps)**.** Let $\mathcal{C}$ be a †-compact category equipped with a G-action ϕ. The *complete decoherence map* for a generalised classical structure $(\circ, \theta)$ is the morphism of $\mathcal{C}$ defined as follows:



(1.8)

This is also a morphism in any CPM category $\mathsf{CPM}_{G,\Xi}(\mathcal{C})$ where the environment structure Ξ contains the following *complete discarding map* $(\circ, \theta)$:



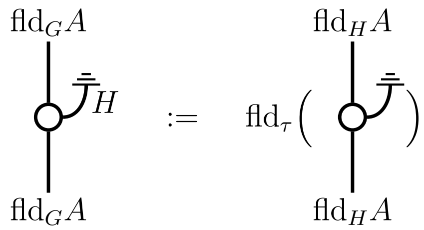
(1.9)

In this case, the spider part of the leftmost diagram in (1.8) is simply the G-folding $\mathrm{fld}_G(\ \ )$ of the comultiplication for the †-SCFA $\circ$.

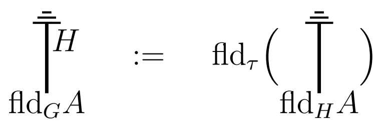
In (1.8) and (1.9), the diagrams with thick lines are in the graphical calculus for the †-compact category $\mathsf{CPM}_{G,\Xi}(\mathcal{C})$, while the diagrams with thin lines are in the graphical calculus for the †-compact category $\mathcal{C}$. This follows the same convention as diagrams for the original CPM construction. Unlike the discarding maps from the original CPM construction, however, the complete discarding maps defined by (1.9) depend on a choice of generalised classical structure $(\circ, \theta)$. This dependence is left implicit in our diagrammatic notation.

Recall, now, the result from Proposition 5: if $H \leq G$, then the complete G-folding factors into the complete H-folding followed by the folding over any transversal T of H in G. Via this mechanism, we can define discarding maps and decoherence maps corresponding to all possible choices of $H \leq G$ and transversal T.

Definition 16 (H-Decoherence Maps)**.** Let $\mathcal{C}$ be a †-compact category equipped with a G-action ϕ. Let $H \leq G$ be a subgroup, T be a transversal for H in G, and $\tau := (G, H, \eta, T)$ be folding data. The *H-decoherence map* for a generalised classical structure $(\circ, \theta)$ and folding data τ is the morphism of $\mathcal{C}$ defined as follows:

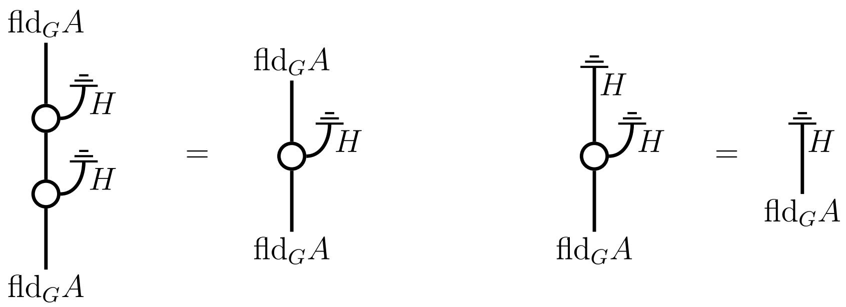


This is also a morphism in any CPM category $\mathsf{CPM}_{G,\Xi}(\mathcal{C})$ where the environment structure Ξ contains the following *H-discarding map* $\bar{\top}_H : \mathrm{fld}_G A \to I$ for $(\circ, \theta)$ and τ:



The dependence of H-decoherence maps and H-discarding maps on both the generalised classical structure $(\circ, \theta)$ and the folding data τ is left implicit in the diagrammatic notation: the subgroup H is the only piece of data that will be of practical interest. Furthermore, by choosing $H := G$ we always recover the complete decoherence maps and discarding maps from Definition 15.

A straightforward observation, following from the Spider Theorem [54, 100, 113], is that the H-decoherence maps are idempotent and causal with respect to the corresponding H-discarding maps:



A similarly straightforward observation is that decoherence maps for subgroups are compositionally well-behaved: if dec_H and dec_K are the H-decoherence and K-decoherence maps for two subgroups $H, K \leq G$, then the following is true, where $H \vee K$ is the group theoretic join of the two subgroups:

$$\mathrm{dec}_H \circ \mathrm{dec}_K = \mathrm{dec}_K \circ \mathrm{dec}_H = \mathrm{dec}_{H \vee K}$$

Now, suppose that we fix some subgroup $H \leq G$, folding data $\tau = (G, H, \eta, T)$ and that we equip each object of $\mathcal{C}$ with a choice of generalised classical structures $\Gamma := ((\bigcirc_A, \theta_A))_{A \in \mathrm{obj}\ \mathcal{C}}$ on all objects, compatibly with its monoidal structure.

Proposition 9. *Under the conditions above, we obtain an environment structure Ξ by associating to each object $A \in \mathrm{obj}\ \mathcal{C}$ the singleton set Ξ_A containing only the H-discarding map.*

Proof. The monoidal product of †-SCFAs is again a †-SCFA so the environment structure is closed under this operation. The autofunctors act essentially trivially on any effect in the canonical environment structure: if $h \in H$ then h just permutes the legs of each spider and if $h \notin H$ then it acts to also permute the spiders. In either case there exists a natural isomorphism given by the symmetric monoidal structure of $\mathcal{C}$ which acts to undo the permutation and this is all we require because we made the assumption that the isomorphisms θ_A^g satisfy the commutative diagram (1.3) and act on the †-SCFA in essentially the same way as the autofunctors ϕ_g. □

The environment structure obtained above can be seen to generalise the standard choice in the original CPM construction: in that case, the group $G = C_2$ only allows a single non-trivial choice, the one corresponding to $H = C_2$ itself which is just the cap.

Chapter 2

A Post-Quantum Theory: Density Hypercubes

In this chapter we will take a close look at a particular instance of the CPM construction known as the theory of density hypercubes [90, 96]. This is the first fully fledged probabilistic theory displaying hyper-decoherence to quantum theory and as such constitutes the first example of a post-quantum theory. The original work [90] demonstrated that density hypercubes form a $\mathbb{R}^+$-probabilistic theory and that there exist idempotent maps recovering quantum theory as a sub-theory. Unfortunately it failed to show that such maps had a well-defined operational interpretation: it was conjectured that they would happen probabilistically as part of some larger process, but it was not known what that process could look like.

In this section we patch the short-comings of [90] and put density hypercubes on solid footing as a probabilistic theory, showing that hyper-decoherence truly has a bona fide operational interpretation as a probabilistic component of a larger deterministic process [96]. This will allow further operational exploration of post-quantum effects in density hypercubes to be carried out with the necessary confidence in its theoretical foundations.

Along the way we will also explore two closely related theories from the literature, those of *double dilation* and *double mixing* [163], developed to describe quantum-like aspects of ambiguity in natural language processing [56, 129]. It was originally believed [90] that density hypercubes and double dilation coincided: we show that not to be the case. We further show that double dilation and double mixing do not feature the hyper-decoherence maps of density hypercubes, nor the same associated phase group.

Here follows a brief summary of known results about double dilation, double mixing and density hypercubes, together with open questions which this section addresses.

- It was originally believed that density hypercubes and double dilation were the same theory [90]. We will show that not to be the case. Conversely, double mixing is a sub-theory of double dilation [163] and a straightforward adaptation of the same argument also shows that double mixing is a sub-theory of density hypercubes.

- It was known that density hypercubes possesses idempotent maps exhibiting classical and quantum theory as full sub-categories of the Karoubi envelope [90]. It was not known whether the hyper-decoherence maps could be completed to causal maps, i.e. whether they were a probabilistic outcome of some larger deterministic process of density hypercubes. We will show that this is indeed the case and thus density hypercubes are a post-quantum theory.

- It was known that the states of a n dimensional density hypercube form a subset of those of a n^2 dimensional quantum state[1]. We will characterise this state-space as precisely the $\mathbb{R}^+$-linear combinations of symmetric pure quantum states.

- The structure of the hyper-phase group—the invertible maps which are quotiented away by hyper-decoherence—for density hypercubes was not known. We will show that there exist non-trivial elements of this group and thus hypercube states are truly an extension of those of quantum theory.

- It was not known whether double dilation and double mixing possess hyper-decoherence maps, or what the associated hyper-phase group would be. We will show that the same choice of maps from density hypercubes would not work and we will provide evidence against the existence of such hyper-decoherence maps by showing that severe restrictions apply to the maps that the associated hyper-phase group can contain.

2.1 Double Dilation, Double Mixing and Density Hypercubes

The states of density hypercubes are given by tensors of the form ρ_{ijkl} and are thus similar to those of quartic quantum theory. In contrast to that theory though, density hypercubes have a higher-order symmetry group: $C_2 \times C_2$ instead of C_2.

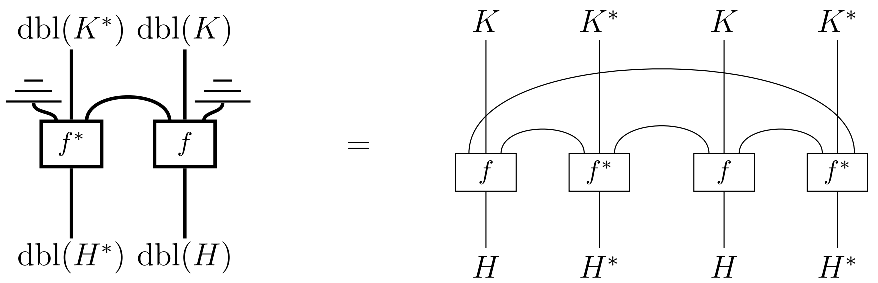
[1] note that this is purely mathematical and not the correct operational interpretation of the theory

Alongside double dilation and double mixing, density hypercubes arises from the CPM construction where the folding is generated by the action

$$\phi_{(a,b)} := \begin{cases} \mathrm{id}_{\mathcal{C}} & \text{if } a \oplus b = 0 \\ \mathrm{conj}_{\mathcal{C}} & \text{if } a \oplus b = 1 \end{cases}$$

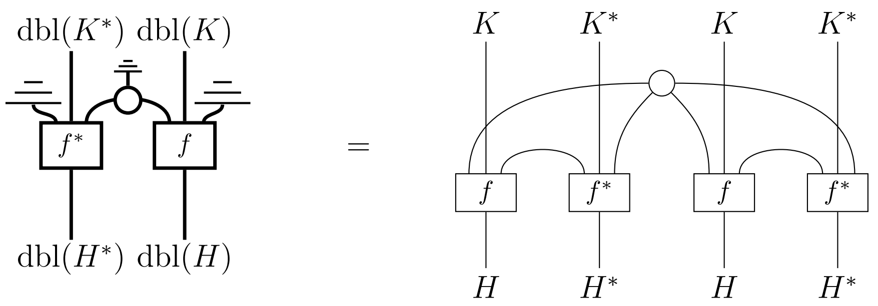
that is, we iterate the original CPM construction twice. Explicitly, the folding functor sends objects H to objects $H \otimes H^* \otimes H \otimes H^*$, and similarly for morphisms. Different choices of environment structures then capture different ways of discarding: the three examples we consider all contain the doubling of the discarding map from $\mathsf{CPM}(\mathsf{FHilb})$ plus one additional effect.

The first choice appearing in the literature is that of *double dilation* [163], also known as *dual density operators* [6]. The environment structure contains the two possible ways in which caps $\epsilon_A : A^* \otimes A \to I$ can be applied to $\mathrm{dbl}(\mathrm{dbl}(E)) = E \otimes E^* \otimes E \otimes E^*$: (i) the doubling $\mathrm{dbl}(\epsilon_E)$ of the discarding maps from $\mathsf{CPM}(\mathsf{FHilb})$ and (ii) the cap $\epsilon_{\mathrm{dbl}(E)}$:



(2.1)

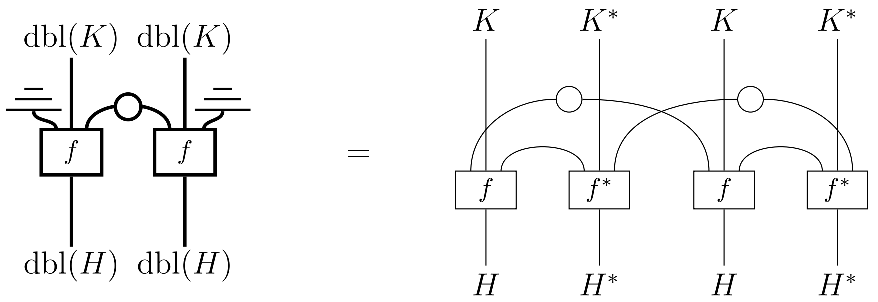
The second choice appearing in the literature is that of *double mixing* [163]. The environment structure contains the doubling $\mathrm{dbl}(\epsilon_E)$ of the discarding maps from $\mathsf{CPM}(\mathsf{FHilb})$ together with a four-legged spider connecting all four environment systems:



(2.2)

It can be shown [163] that double mixing is a sub-theory of double dilation, i.e. that all maps in the form (2.2) can also be put in the form (2.1). Both double dilation and double mixing have found application to the modelling of ambiguity in natural language processing [56, 129].

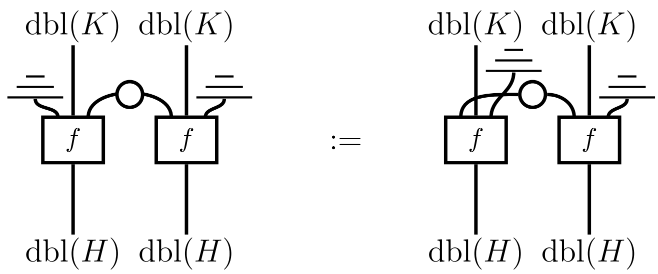
The third choice appearing in the literature is that of *density hypercubes* [90]. The environment structure contains the doubling $\mathrm{dbl}(\epsilon_E)$ of the discarding maps from $\mathsf{CPM}(\mathsf{FHilb})$ together with two-legged spiders resembling the cap $\epsilon_{\mathrm{dbl}(E)}$ from double dilation:



(2.3)

Despite the apparent similarity, density hypercubes are significantly different from double dilation: the two-legged spiders used in the RHS of (2.3) are effects on $E \otimes E$ and $E^* \otimes E^*$, while the two caps forming $\epsilon_{\mathrm{dbl}(E)}$ in (2.1) are both effects on $E \otimes E^*$. Because of this, the objects of density hypercubes are most naturally written in the form $\mathrm{dbl}(H) \otimes \mathrm{dbl}(H)$ (instead of $\mathrm{dbl}(H^*) \otimes \mathrm{dbl}(H)$) and the folded morphisms in density hypercubes are most naturally written in the form $\mathrm{dbl}(f) \otimes \mathrm{dbl}(f)$ (instead of $\mathrm{dbl}(f^*) \otimes \mathrm{dbl}(f)$). This will turn out to make a very significant difference.

On the LHS of (2.3), the wires have been re-arranged to achieve a more pleasant visual effect. In more direct analogy with double dilation and double mixing they would have taken the following form:



We refer to the two legged spider appearing in the middle as a *bridge*. For convenience, we will allow for bridges in any choice of †-SCFA ○: this generalisation yields the same class of morphisms, but with added flexibility when drawing diagrams.

The three choices of environment structure that lead to density hypercubes, double dilation and double mixing can be understood as choices of H-discarding maps for different subgroups $H \leq C_2 \times C_2$. All three models contain the H-discarding maps for the subgroup $H := \{(0,0),(0,1)\} \cong C_2$, while the other effects are K-discarding maps for different choices of subgroup K:

- in density hypercubes, $K_{dh} := \{(0,0),(1,1)\} \cong C_2$;

- in double dilation, $K_{dd} := \{(0,0),(1,0)\} \cong C_2$;
- in double mixing, $K_{dm} := C_2 \times C_2$;

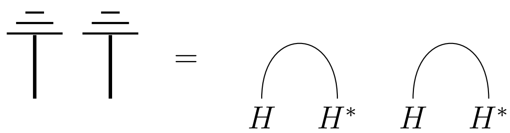
The previously discussed behaviour of H-decoherence maps under composition tells us that:

$$K_{dd} \vee H = G = K_{dm} \quad \Rightarrow \quad \mathrm{dec}_H \circ \mathrm{dec}_{K_{dd}} = \mathrm{dec}_{K_{dd}} \circ \mathrm{dec}_H = \mathrm{dec}_{K_{dm}}$$

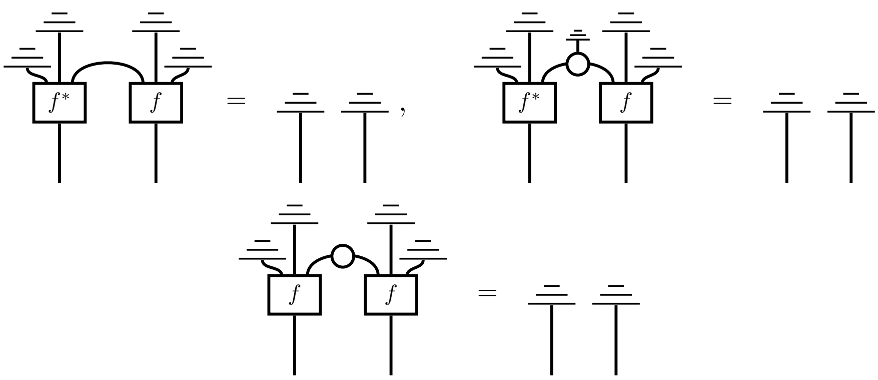
$$K_{dh} \vee H = G = K_{dm} \quad \Rightarrow \quad \mathrm{dec}_H \circ \mathrm{dec}_{K_{dh}} = \mathrm{dec}_{K_{dh}} \circ \mathrm{dec}_H = \mathrm{dec}_{K_{dm}}$$

which recovers the result of [163] showing that double mixing is a sub-theory of double dilation, and the result of [90] showing that double mixing is a sub-theory of density hypercubes. A limitation of this group theoretic approach is that it does not tell us whether double dilation and density hypercubes coincide; we will see shortly, that in fact they do not [96]. This also means that theories generated by the same folding and using discarding maps from isomorphic subgroups of G do not, in general, need to be equivalent: the G-action itself plays an important role in this. An investigation of the exact conditions under which such theories are equivalent is left to future work.

In the generalised CPM construction, a single family of effects has to be chosen within the environment structure to endow the theory with a notion of causality and normalisation [42, 55, 50]. We make the same choice for all three theories:



We refer to the effects above as the *discarding maps* for the theories. The *normalised* morphisms in the three theories are those which respect the choice of discarding maps made above, in the following sense:

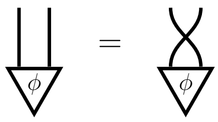


The three theories are also *probabilistic*: they have the non-negative real numbers $\mathbb{R}^+$ as scalars, with morphisms that can be rescaled and added together. In particular, normalised morphisms form a convex set and are interpreted as processes that can be made to happen "with certainty", or "deterministically". More generally, we say that a morphism $f : H \to K$ is *sub-normalised* if there is some $g : H \to K$ such that $f + g$ is normalised: sub-normalised morphisms are interpreted as processes that happen "probabilistically"—with probability dependent on the specific state that they are applied to—as cases of a larger deterministic process. There is a unique normalised scalar, the number 1, and sub-normalised scalars coincide with probabilities $[0, 1]$.

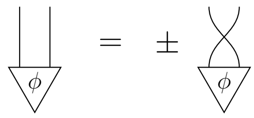
Finally, we can precisely characterise the state-space of density hypercubes.

Proposition 10. *The state-space of density hypercubes is the $\mathbb{R}^+$-linear closure of the symmetric sector of quantum theory.*

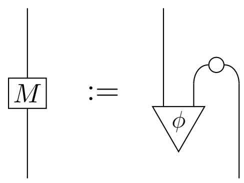
Proof. Consider a pure hypercube state ϕ and note that it is invariant under braiding the outputs.



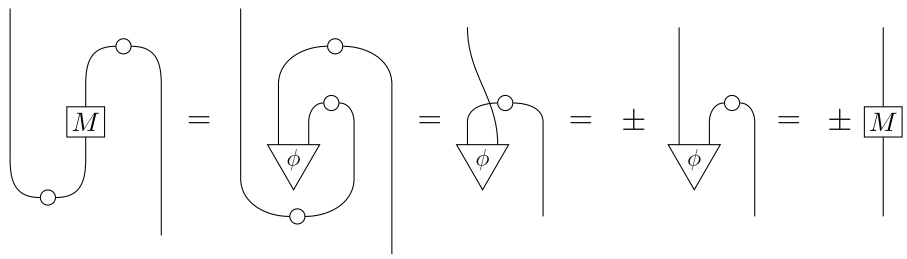
As a consequence the un-doubled version of ϕ must be either symmetric or antisymmetric.



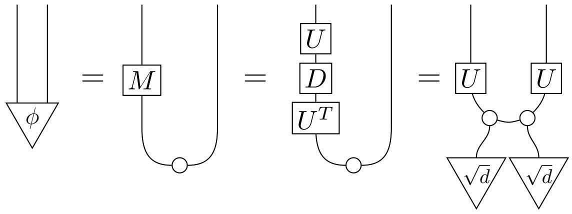
Write M for the matrix corresponding to ϕ by bending down the right leg.



Then it follows that M is either a symmetric or antisymmetric matrix.

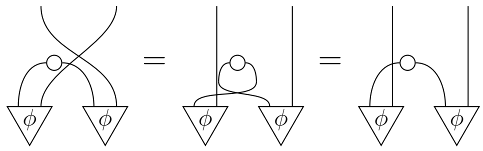


Firstly suppose that M is symmetric. We note an old result of Autonne and Takagi [8, 154] which says that if M is a complex symmetric matrix then there exists a unitary matrix U and a real diagonal matrix D such that $M = UDU^T$. As a result ϕ can be written in the following form which shows it is a valid density hypercube state.



Here, $\left|\sqrt{d}\right\rangle$ is the state $(\sqrt{d_1}, \ldots, \sqrt{d_n})$ with the d_i given by the diagonal of D.

The case where M is antisymmetric is not compatible with the structure of density hypercubes. From the definition, we can see that every pure hypercube state must be symmetric in FHilb.

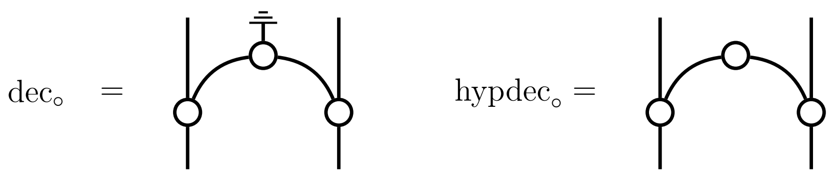


So we have shown that the pure hypercube states are precisely those of symmetric quantum theory. The last thing to note is that the non-pure states arise by discarding some of the outputs of a pure state. The discard maps can be expanded in a basis of one's choosing to see that this is equivalent to taking $\mathbb{R}^+$ linear combinations of the pure states. $\square$

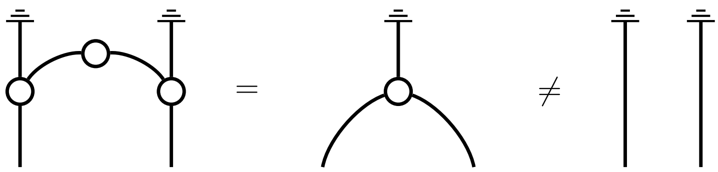
Remark. The previous Proposition characterising the state-space of density hypercubes should be understood as a purely mathematical statement and *not* the intended physical nor operational interpretation of the theory. In the next section we will investigate hyper-decoherence in density hypercubes and we will see how the quantum states arise as a certain sector of the theory accessed by a probabilistic hyper-decoherence process.

2.2 Hyper-decoherence in Density Hypercubes

It has been shown [90] that the theory of density hypercubes has both decoherence maps (collapse to classical theory) and hyper-decoherence maps (collapse to quantum theory), taking the following form:



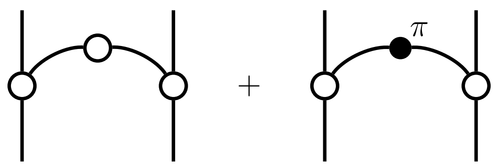
Both maps are idempotent, but unfortunately they are not normalised, meaning that they are not—on their own—bona fide physical processes:



(2.4)

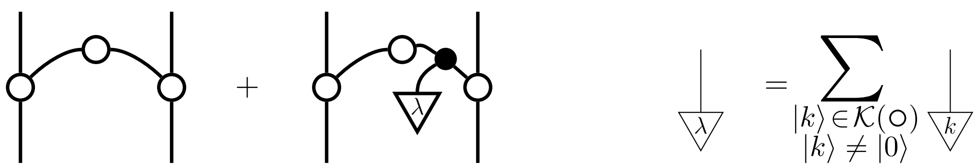
In the seminal [90] it was argued that some completion to a normalised process would indeed be possible in principle, but it was not known whether this be possible *within* the theory of density hypercubes. We now process to show that it is possible indeed, giving hyper-decoherence the operational interpretation of a probabilistic process.

Proposition 11. *The hyper-decoherence map is sub-normalised within the theory of density hypercubes. For qubits, its completion to a normalised process is given as follows:*



(2.5)

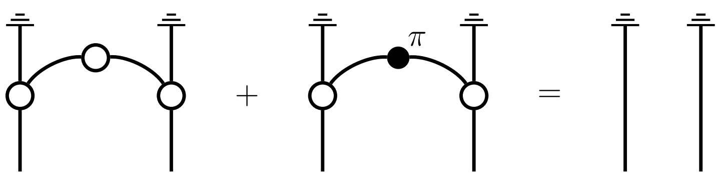
where the white dot is a spider in the Pauli Z basis and the black dot is a phased spider in the Pauli X basis. For higher dimensions, its completion to a normalised process is given as follows:



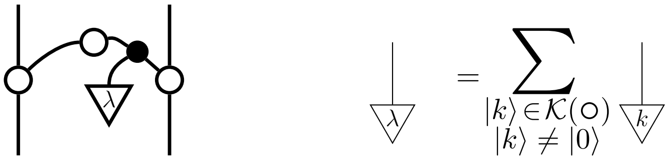
(2.6)

where the white dot is a spider in the computational basis, $\mathcal{K}(\circ)$ *is the set of computational basis states and the black dot is a spider in any Fourier basis.*

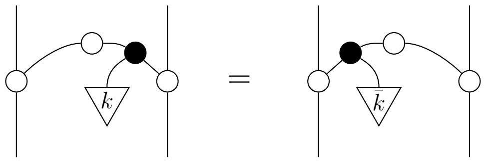
Proof. The second map of (2.5) can be written with two black $\pi/2$ phases on the bridge and is therefore a valid map of density hypercubes. Applying the discarding maps we get:



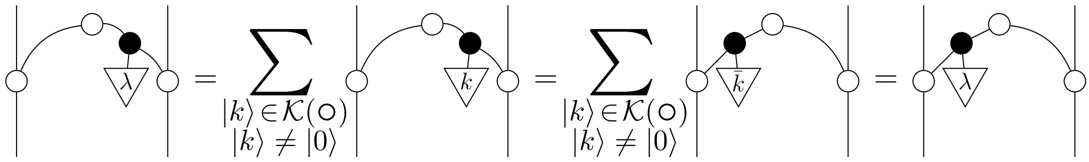
For higher dimensions $d \geq 3$, the completion is slightly more complicated. Let G be a finite abelian group on d elements and $\circ$ correspond to the group element basis in the group algebra $\mathbb{C}[G] \cong \mathbb{C}^d$. Let $\bullet$ correspond to the Fourier basis for G, spanned by the (normalised adjoints of the) characters for G. Let $\mathcal{K}(\circ) = \{ k : k \in G\}$ be the set of classical states for $\circ$. Now consider the following CP map:



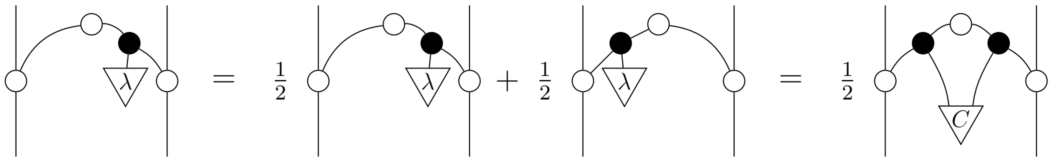
It is easy to check that this gives a completion of the hyper-decoherence map to a normalised CP map, but it is not immediately clear that this is a valid map in the theory of density hypercubes. Indeed, it is not clear that it respects the symmetry required for maps of density hypercubes. However, writing $\bar{k}$ for the inverse of $k \in G$ we note the following equality



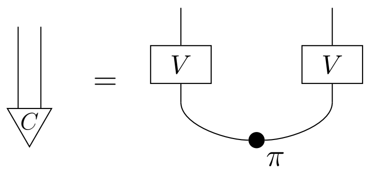
Which in turn implies:



Thus we can write:



The "control state" C on the RHS above is formed as follows:



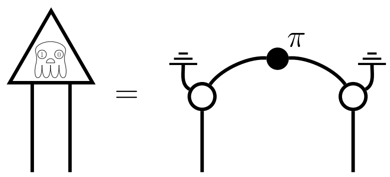
where $V : \mathbb{C}^2 \to \mathbb{C}[G]$ is the isometry in FHilb defined by $|0\rangle \mapsto |0_G\rangle, |1\rangle \mapsto |\lambda\rangle$.

$\square$

Proposition 11 above shows that the theory of density hypercubes splits into two "sectors": a quantum sector accessed by hyper-decoherence and another sector referred to as *the Beyond*. It also shows that hyper-decoherence occurs probabilistically as one outcome of the above normalised process.

Discarding the map which completes the hyper-decoherence map (the second part of (2.6)) gives an important effect which give a special name and symbol.

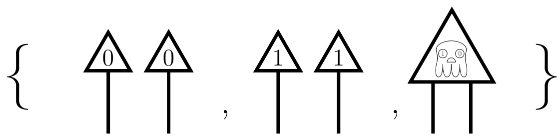
Definition 17. The *Unspeakable Horror from Beyond* (UHfB) is the effect completing the quantum discarding map (LHS of 2.4) to the full discarding map of density hypercubes. In the qubit case we have:



(2.7)

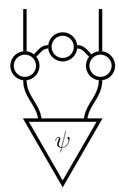
We adopt the same symbol for all dimensions.

The importance of the UHfB is that it can be used to complete quantum measurements/POVMs to genuine measurements/POVMs on density hypercubes. For example, the following completes a computational basis measurement on a qubit to a measurement of density hypercubes:

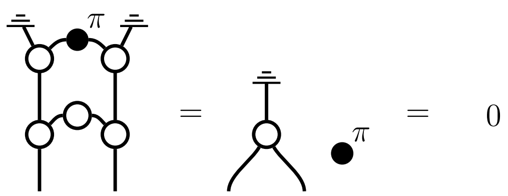


(2.8)

This completion is necessary for a meaningful operational perspective on the larger theory, but is not observable from within quantum theory. Indeed, consider a generic quantum state, taking the following form [90]:



This quantum state has probability zero of yielding the UHfB as a measurement outcome:

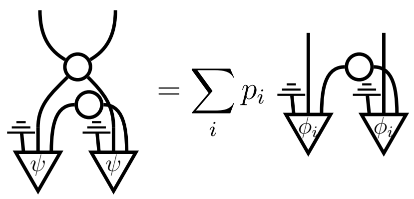


The first equality is by Hopf rule and the second equality is due to the black π dot being the scalar 0.

Although the computational basis states take the product form shown in (2.8), this is not the case for generic quantum states. For example, the state on the left below is the quantum $|+\rangle$ state, while the state on the right is a post-quantum state (sent to the quantum $|+\rangle$ state by hyper-decoherence):

Proposition 12. *There are no post-quantum decompositions of any given quantum state.*

Proof. Suppose there is a decomposition of a hyper-decohered state of the form:



(2.9)

for some $p_i \in \mathbb{R}^+$. Each term ϕ_i of the right hand side is isomorphic (by compact closure) to a positive semi-definite matrix which we may write in the basis corresponding to $\circ$ as $\sum_{ijkl} a_{ij,kl} |ij\rangle \langle kl|$. Such a matrix necessarily has its diagonal coefficients positive real $a_{ij,ij} \in \mathbb{R}^+$. The hyper-decohered state ψ, which can similarly be written as $\sum_{ijkl} b_{ij,kl} |ij\rangle \langle kl|$, must have diagonal coefficients $b_{ij,ij} = 0$ for $i \neq j$ which means that (2.9) cannot hold unless $a_{ij,ij} = 0$ for $i \neq j$.

Now we use Slyvester's criterion: a Hermitian matrix is positive semi-definite if and only if all of its principle minors are non-negative. This allows us to show that many of the entries of ϕ_i are necessarily zero. For instance, take the following principle minor:

$$\begin{vmatrix} a_{00,00} & a_{00,01} \\ a_{01,00} & 0 \end{vmatrix} \geq 0 \implies -a_{00,01}a_{01,00} = -|a_{00,01}|^2 \geq 0 \implies a_{00,01} = 0$$

As a result, it is only the entries $a_{ii,jj}$ which can be non-zero, but these are exactly the entries which are left invariant by hyper-decoherence. Therefore, each ϕ_i must be a quantum state.

□

The previous proposition has a number of important consequences. Firstly, all pure quantum states are pure as post-quantum hypercube states. This was one of the assumptions of the no-go theorem of [117]: if pure quantum states are states of maximal knowledge and hyper-decoherence is the loss of information to an environment, then they should also be pure post-quantumly. Indeed, density hypercubes does not break this assumption.

Secondly, the quantum maximally mixed state is not the maximally mixed state of the post-quantum theory. In this way density hypercubes breaks this assumption of [117]: the maximally mixed post-quantum state is mapped to the maximally mixed quantum state and as a result to a state of more knowledge! At first this seems to be a highly odd situation, but if we slightly shift our perspective then there is a more reasonable explanation. Suppose density hypercubes describes a theory in which the observer is constrained to the quantum world – there is a sector "the Beyond" which is inaccessible to them and they only ever see the quantum "shadows" of the post-quantum states. Such a world is not completely inconceivable, for instance in brane cosmology it is hypothesised that ordinary matter might be constrained to a hypersurface known as a *brane* of a higher-dimensional space known as the *bulk* which exotic matter and gravity might be able to propagate through [116, 37]. The existence of the bulk might provide a way to deal with the extra dimensions predicted by string theory (other than compactification).

Density hypercubes is phenomenologically similar: the hypercube states evolve as a bulk, with the observer constrained to a brane of quantum theory. As a result, although it seems that the maximally mixed post-quantum state is mapped to a state of more knowledge by hyper-decoherence, this is only the effective state seen by the constrained observer - to them the state never changed and their knowledge has remained constant. To a post-quantum observer the hyper-decoherence cannot be implemented deterministically. This explains the knowledge gained by such a post-quantum observer - they would need to implement a measurement and only upon post-selecting for the quantum sector could they be certain they have quantum state.

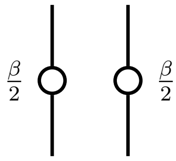
2.3 Phase Group

In quantum theory, the *phase group* for a decoherence map $\mathrm{dec}_{\bigcirc} : H \to H$ is the group formed by all invertible processes $U : H \to H$ which are quotiented away by $\mathrm{dec}_{\bigcirc}$, i.e. such that

$$\mathrm{dec}_{\bigcirc} \circ U = \mathrm{dec}_{\bigcirc}$$

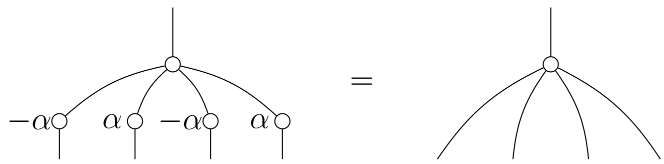
This is formed by the *phase gates*, the unitaries diagonal in the decoherence basis, and is isomorphic to a torus T^{d-1}. For a qubit, it is the circle group T^1.

Proposition 13. *The processes obtained by doubling phase gates from quantum theory are always in the phase group for density hypercubes:*



(2.10)

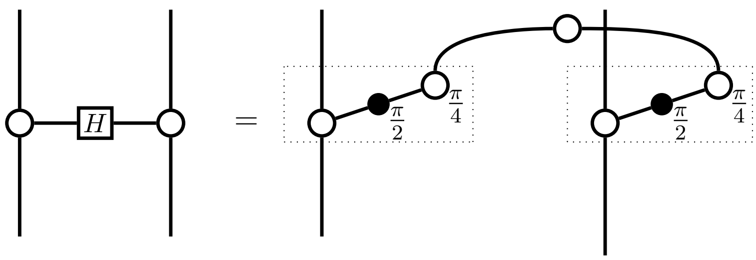
Proof. By spider fusion, it is clear that the doubled phase gates are erased by hyper-decoherence:



□

The maps in (2.10) are natural choices, as they are sent to the usual phase gates of quantum theory by hyper-decoherence. The phase gates of quantum theory themselves are not however in the phase group for density hypercubes, as they are not invertible within the larger theory.

Are there more elements in the phase group? Restricting temporarily to the case where $\dim H = 2$, we can certainly find more:



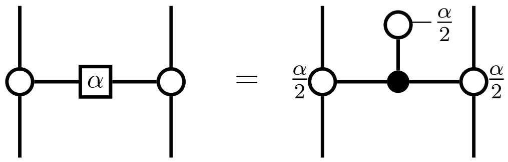
(2.11)

We have used the Euler decomposition of the Hadamard [51] on the right-hand side to demonstrate that this is indeed a valid map for density hypercubes.[2] The map in (2.11) is clearly invertible, and one can check with ease that it is erased by hyper-decoherence.

[2]Note that it is not possible to make the map from (2.11) in the double dilation or double mixing, as the only caps available are those on H and H^*: the conjugation would change the sign of the phases on one side, causing them to vanish.

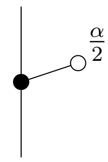
The map also resembles the controlled-Z gate on two qubits, suggesting that there be an entire additional family of maps just like it living in the phase group.

Proposition 14. *The following maps are in the phase group of density hypercubes for* $\dim H = 2$*:*

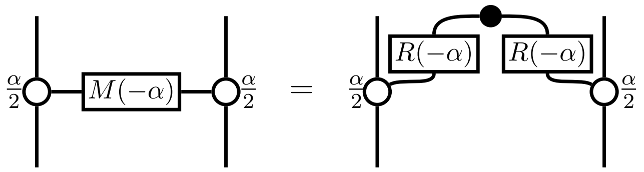


(2.12)

Proof. Showing that the maps (2.12) are erased by decoherence is a simple application of the Hopf rule. The harder part is showing that the maps (2.12) exist in density hypercubes: this boils down to showing that they have a symmetric expansion about the bridge, just as we have previously shown for (2.11). This expansion can be found by taking the square root of the following map in FHilb:

$M(\alpha) =$  $= \sqrt{2}e^{i\alpha/2}\begin{pmatrix}\cos\alpha/2 & 0\\ 0 & -i\sin\alpha/2\end{pmatrix}$

where the matrix on the RHS is written in the Pauli X basis. Since $M(\alpha)$ is diagonal in the Pauli X basis, a square root $R(\alpha)$ is guaranteed to exist, itself diagonal in the Pauli X basis and thus self-transpose in Pauli X basis. Therefore we can safely write the square root on either side of the bridge as follows:

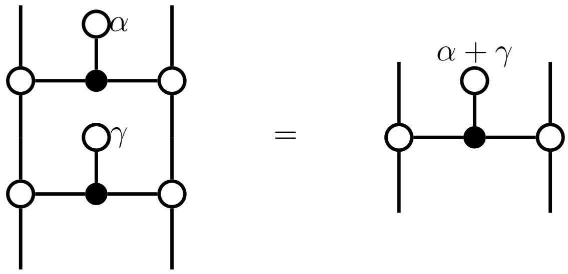


□

It is notable that the maps (2.12) are really a composition of the following two maps:

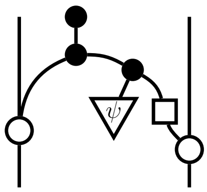
(2.13)

It is then the maps on the LHS of (2.13) that introduce something new to the phase group, since the maps on the RHS are the doubled phase spiders from before. These new maps have been appeared previously in the literature under the name of *phase gadgets* [111]. It is easy to show that composing phase gadgets adds their phases:



Phase gadgets and doubled phase spiders commute, so that the phase group for density hypercubes in dimension $\dim H = 2$ is the torus $T^2 = S^1 \times S^1$.

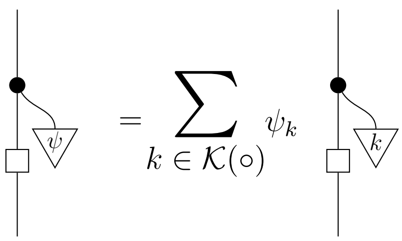
Proposition 15. *The following maps are in the phase group of density hypercubes for arbitrary dimensions, generalising* (2.12)*:*



(2.14)

Above, $|\psi\rangle = \sum_{k \in \mathcal{K}(\circ)} e^{i\theta_k} |k\rangle$ *is a* $\circ$*-phase state with* $\theta_k = \theta_{k^{-1}}$ *and* ⏚ *is the antipode.*

Proof. The doubled phase spiders contribute the usual quantum phase group T^{d-1}. We also have the maps (2.14). One can check that these maps are erased by decoherence and thus are in the phase group. All that is left to check is that they are normalised, invertible and that they exist in the theory of density hypercubes. Existence comes down to showing that they have a symmetric expansion about a bridge, generalising what happened in the proof of Proposition 14. Consider the following map in FHilb:

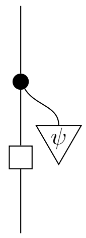


(2.15)

where we have expanded the phase state as its sum over $\circ$-classical states and ψ_k are complex numbers on the unit circle. One can see that (2.15) acts on $\circ$-classical states as $|g\rangle \mapsto \sum_{k \in \mathcal{K}(\circ)} \psi_k |g^{-1}k\rangle$ (up to normalisation). Furthermore, (2.15) acts on $\bullet$-classical states $|\chi\rangle = \sum_{g \in \mathcal{K}(\circ)} \chi(g) |g\rangle$ as follows (up to normalisation):

$$|\chi\rangle \mapsto \left(\sum_{k \in \mathcal{K}(\circ)} \chi(k)\psi_k \right) \overline{|\chi\rangle}$$

where $\overline{|\chi\rangle} = \sum_{g \in \mathcal{K}(\circ)} \chi(g)^* |g\rangle$. (In the above, $\chi \in G^\wedge$ where $G^\wedge$ is the group of multiplicative characters for G.) We are thus able to expand (2.15) as a matrix in the $\bullet$ basis (again up to appropriate normalisation):



$$= \sum_{\chi \in \mathcal{K}(\bullet)} \sum_{k \in \mathcal{K}(\circ)} \chi(k)\psi_k \overline{|\chi\rangle} \langle\chi|$$

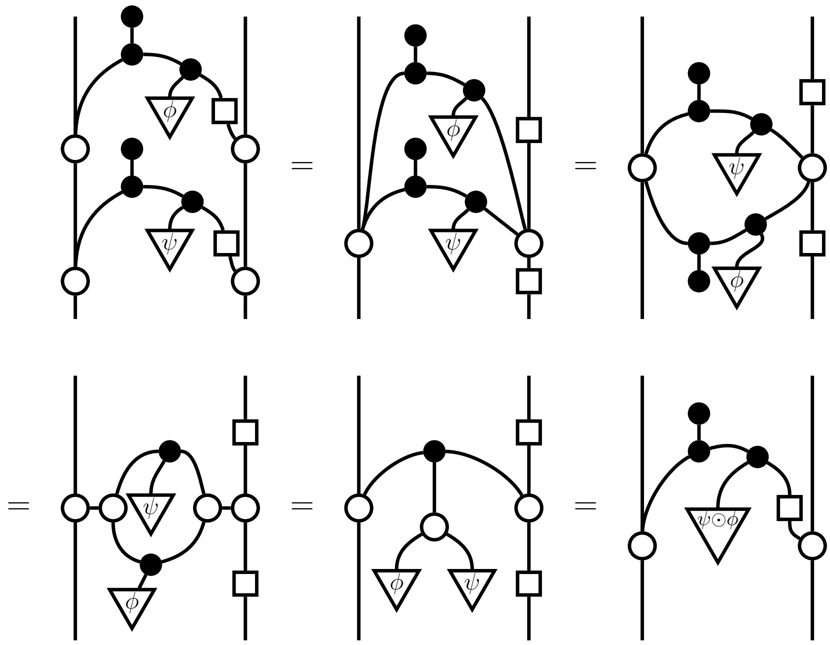
We want this matrix to have a square root and for this square root to be self-transpose with respect to the $\bullet$ basis. Since most entries of the matrix are zero, checking the existence of a square root comes down to looking at the sub-matrices for the terms $|\chi\rangle \overline{\langle\chi|}$ and $\overline{|\chi\rangle} \langle\chi|$, which take the form:

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \tag{2.16}$$

for some a and b, where we have considered the case $\mathrm{ord}(\chi) > 2$ in $G^\wedge$. The case of $\mathrm{ord}(\chi) = 2$ is trivial since it contributes a single non-zero diagonal element to the matrix, which can clearly be square rooted. The matrix (2.16) always has four square roots, since $a, b \neq 0$, but unless $a = b$ none of them are self-transpose in $\bullet$. In order to have $a = b$ we need the following to hold for each χ:

$$\sum_{k \in \mathcal{K}(\circ)} \chi(k)(\psi_k - \psi_{\bar{k}}) = 0$$

Note that the above is the Fourier transform of $f(k) = \psi_k - \psi_{\bar{k}}$ in the finite abelian group G. By inverting the transform, one sees that $\psi_k = \psi_{\bar{k}}$. This is trivially satisfied for those $k \in G$ such that $\mathrm{ord}(k) = 2$. Finally we show that the maps (2.14) are closed under composition:



Above we have have used the Frobenius product $|\psi \odot \phi\rangle = \sum_{k \in \mathcal{K}(\circ)} \psi_k \phi_k \,|k\rangle$. This also shows that the maps (2.14) are invertible. As observed in the proof of Proposition 14 before, the maps are also normalised, by the Hopf law.

$\square$

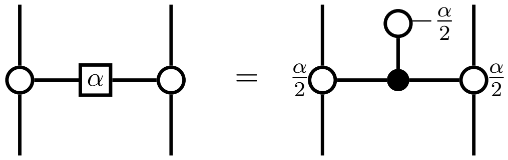
2.4 Hyper-phase Group

Having characterised the phase group, we can now look at its post-quantum generalisation: the *hyper-phase group*, defined below. Because decoherence maps are invariant under pre- or post-composition by hyper-decoherence, the hyper-phase group is always a subgroup of the phase group.

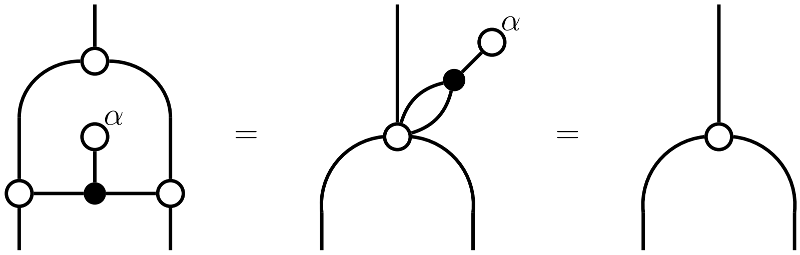
Definition 18. The *hyper-phase group* in density hypercubes for a hyper-decoherence map $\mathrm{hypdec}_{\bigcirc}$ is the group formed by all invertible processes $U : H \to H$ which are quotiented away by $\mathrm{hypdec}_{\bigcirc}$, i.e. such that

$$\mathrm{hypdec}_{\bigcirc} \circ U = \mathrm{hypdec}_{\bigcirc}$$

Proposition 16. *The hyper-phase group of density hypercubes for* $\dim H = 2$ *contains exactly the maps from* (2.12)*:*

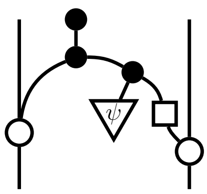


Proof. One can check that the maps (2.13) are erased by hyper-decoherence:



where the final step follows by the Hopf law. On the other hand, the doubled phase spiders are *not* erased by hyper-decoherence. □

Proposition 17. *The hyper-phase group of density hypercubes for arbitrary dimensions contains the maps from* (2.14)*:*



Proof. The proof for Proposition 16 straightforwardly generalises to higher dimensions. □

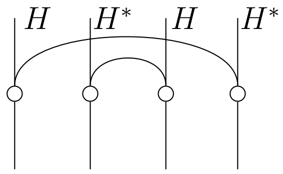
2.5 Double Dilation and Double Mixing

In this section we look at how many of the post-quantum features from density hypercubes are also available in double dilation and double mixing.

To start with, a minor alteration of the proof given in [90] can be used to show that double dilation and double mixing are also probabilistic theories.

Proposition 18. *The decoherence maps of density hypercubes are also decoherence maps for double dilation and for double mixing, i.e. they send double dilated and double mixed systems to classical systems.*

However, the hyper-decoherence maps from density hypercubes are not maps in double dilation or double mixing. The most likely candidate candidate would be the following map (which however does not exist for double mixing):



(2.17)

Proposition 19. *Map* (2.17) *does not give a hyper-decoherence map for double dilation.*

Proof. An arbitrary tripartite pure state on systems A, B and C in double dilation can be written as follows:

$$|\psi_{ABC}\rangle = \sum_{ijk} c_{ijk} \left|e_i^A e_j^B e_k^C\right\rangle$$

for $c_{ijk} \in \mathbb{C}$, where $\{\left|e_i^A\right\rangle\}$ forms an orthonormal basis for A, and similar for B and C. By partial trace, a general state on A in double dilation can be written as follows:

$$\begin{aligned}\rho &= \sum_{pqrs} \left\langle e_r^C e_p^B \middle| \psi_{ABC}\right\rangle \overline{\left\langle e_s^C e_p^B \middle| \psi_{ABC}\right\rangle} \left\langle e_s^C e_q^B \middle| \psi_{ABC}\right\rangle \overline{\left\langle e_r^C e_q^B \middle| \psi_{ABC}\right\rangle} \\ &= \sum_{ijklpqrs} c_{ipr} \left|e_i^A\right\rangle c^*_{ips} \overline{\left|e_j^A\right\rangle} c_{kqs} \left|e_k^A\right\rangle c^*_{lqr} \overline{\left|e_l^A\right\rangle}\end{aligned}$$

Without loss of generality, we can consider the map (2.17) with ○ associated to the basis $\{\left|e_i^A\right\rangle\}$. The result of applying the map to ρ is the following state (written up to Choi-Jamiołkowski isomorphism for convenience):

$$\sum_{ijpqrs} c_{ipr} c^*_{jps} c_{jqs} c^*_{iqr} \left|e_i^A\right\rangle \left\langle e_j^A\right|$$

We see that each of the coefficients is invariant under conjugation:

$$\left(\sum_{pqrs} c_{ipr} c^*_{jps} c_{jqs} c^*_{iqr}\right)^* = \sum_{pqrs} c^*_{ipr} c_{jps} c^*_{jqs} c_{iqr} = \sum_{pqrs} c_{ipr} c^*_{jps} c_{jqs} c^*_{iqr}$$

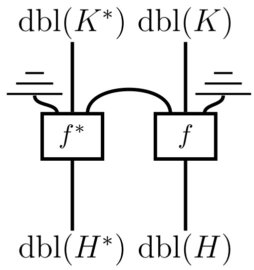
where we relabelled p and q in the final step. As a consequence, the coefficients are all necessarily real and we do not recover all quantum states.

□

A less rigorous but more straightforward way of seeing that map (2.17) cannot possibly be a hyper-decoherence map is to note that it erases the doubled phased spiders. In fact, it is easy to show that the hyper-phase group would be limited to doubled unitaries.

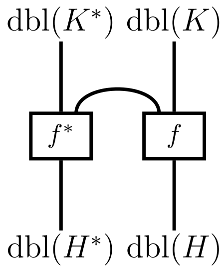
Proposition 20. *In double dilation and double mixing the invertible maps are all doubled unitaries.*

Proof. The maps of double dilation can be written in the following form:



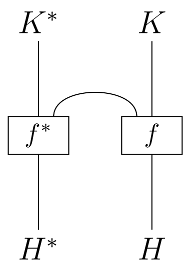
(2.18)

In order for maps in the form above to be invertible, the discarding maps need to be trivial (because of purity in CPM(FHilb)):

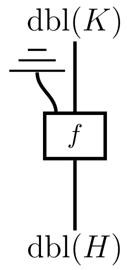


(2.19)

The diagram above is the doubled version of the following diagram in FHilb:

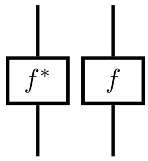


But the diagram above in FHilb corresponds to *another* CP map:



(2.20)

For the CP map (2.20) above to be invertible, the discarding map must be trivial. This in turn implies that the bridge in (2.19) must be trivial and hence that the original map (2.18) must take the following form if it is to be invertible:



The map above is the double of a pure CP map and it is invertible exactly when f is unitary.

$\square$

Proposition 20 above immediately implies that the phase group for double dilation and double mixing is exactly the same phase group of quantum theory. In particular, double dilation is not the same theory as density hypercubes. Furthermore, even if double dilation and/or double mixing *did* possess hyper-decoherence maps, they would not quotient away any non-trivial phases: it may ultimately turn out that one or both are post-quantum theories, but uninteresting ones at best.

2.6 Summary

In this Chapter, we have conclusively shown that density hypercubes possess hyper-decoherence maps with a well-defined operational interpretation. We studied the associated phase group showing that there exist non-trivial phases quotiented away by hyper-decoherence and we have compared our results with analogous statements for double dilation and double mixing.

The probabilistic nature of hyper-decoherence in density hypercubes presents a concrete way around the no-go theorem of Lee and Selby [117]. Simply dropping the constraint that the hyper-decoherence be deterministic allowed the formulation of an operational theory displaying genuinely post-quantum phenomena, together with a mechanism for quantum theory to arise as an effective sub-theory. Density hypercubes also breaks the assumption that the maximally mixed quantum state is not the maximally mixed post-quantum state. This assumption was imposed in [117] because it seemed it would be necessary to ensure that the maximally mixed post-quantum state should not be mapped to a state of greater knowledge by hyper-decoherence. Nevertheless, we offered an interpretation of density hypercubes where the observers are constrained to the quantum sector of the theory and as a result hyper-decoherence happens deterministically and transparently to them. There is no change in knowledge to such an observer. On the other hand, a truly post-quantum observer cannot implement the hyper-decoherence map deterministically, and thus it must be associated with an increase in knowledge.

Chapter 3

CPM Categories Induced by Galois Extensions

In this chapter we will consider a particular case of CPM categories where the symmetry is induced by a finite degree Galois extension. To motivate this, consider the case of decoherence in standard quantum theory. Starting with the category FHilb, we form the category $\mathsf{Split}(\mathsf{CPM}(\mathsf{FHilb}))$ and then we note that classical theory $\mathbb{R}^+$-Mat is recovered as the full subcategory spanned by decoherence maps. A more interesting connection emerges though when we note that there is an equivalence $\mathsf{FHilb} \cong \mathbb{C}$-$\mathsf{Mat}$. We see that there are two fields at play here, $\mathbb{C}$ and $\mathbb{R}$ and that the former is a Galois extension of the latter. Furthermore, the C_2 symmetry of the original CPM construction is isomorphic to the Galois group of the extension $\mathbb{R} \subset \mathbb{C}$, while the positivity of the classical theory arises because the entries of the matrices are in the image of the field norm $N_{\mathbb{C}/\mathbb{R}}(z) = z^*z$.

As a result the quantum and classical theories arrange themselves into a tower mimicking those arising from the Galois correspondence:

$$\begin{array}{ccc} \mathbb{C} & \{*\} & \mathsf{CPM}(\mathsf{FHilb}) \\ | & | & | \\ \mathbb{R} & C_2 & \mathbb{R}^+\text{-}\mathsf{Mat} \end{array}$$

If we now replace the extension $\mathbb{R} \subset \mathbb{C}$ with another finite degree Galois extension $k \subset K$, we could imagine that applying the CPM construction to K-Mat equipped with the canonical action by its Galois group $\mathrm{Gal}(K/k)$ might produce a theory with decoherence maps exhibiting k^+-Mat as the "classical" sub-theory of the "quantum" $\mathsf{CPM}(K\text{-}\mathsf{Mat})$. Moreover, if there exists an intermediate field $k \subseteq F \subseteq K$ then perhaps it might be possible to produce a tower of three probabilistic theories over each field with decoherence maps exhibiting transitions between them? This will be the focus of this chapter.

3.1 A Gentle Introduction to the Classical Galois Theory of Fields

Let us start with a summary of the required tools, techniques and results from the classical Galois theory of fields; for a more detailed discussion, we refer the reader to standard texts [139, 35], although any good undergraduate course on the subject would suffice for the core concepts. This section can be safely skipped by those readers who are already familiar with this most beautiful area of mathematics.

Galois theory concerns itself with understanding field extensions by studying a certain group of field automorphisms known as the *Galois group*. A field extension $k \subset K$ is simply another way of saying that k is a subfield of K and we will often refer to k as the *base field* and K as the *extension field*. K always forms a vector space over k and we refer to the dimension of this vector space as the *degree* of the extension, denoting it by $[K : k]$. An extension is *finite* if the degree is finite; extensions of degree 2 are in particular called *quadratic*. Given an extension $k \subset K$ and a collection of elements $z_i \in K$, we write $k(z_i)$ for the smallest sub-field of K containing k and all of the z_i.

Example 2. Here are some examples of field extensions:

- $\mathbb{R} \subset \mathbb{C}$ is a quadratic field extension: all $z \in \mathbb{C}$ can be written in the form $z = a + ib$ for $a, b \in \mathbb{R}$, so that $1, i$ forms a basis of $\mathbb{C}$ over $\mathbb{R}$.

- $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive fifth root of unity, is a fourth degree extension: since roots of unity sum to 0, every element $z \in \mathbb{Q}(\zeta_5)$ can be written in the form $z = a + b\zeta_5 + c\zeta_5^2 + d\zeta_5^3$ for some $a, b, c, d \in \mathbb{Q}$.

- $\mathbb{Q} \subset \mathbb{Q}(\pi)$ is an infinite degree extension.

An element $a \in K$ is *algebraic* over k if there exists a polynomial with coefficients from k that has a as a root. In other words, there is a polynomial $p \in k[x]$, the ring of polynomials over k, such that $p(a) = 0$. A field extension $k \subset K$ is algebraic if every element of K is algebraic over the base field k. In particular, all finite degree extensions are algebraic because for any $a \in K$ there must be some linear dependence between $1, a, a^2, \ldots, a^m$ as m grows sufficiently large—or else K cannot be a finite dimensional vector space over k—giving a polynomial with a as its root. An element $a \in K$ which is not algebraic over k is known as *transcendental*; extensions which are not algebraic are also called transcendental.

Example 3. $\mathbb{R} \subset \mathbb{C}$ and $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$ are algebraic extensions while $\mathbb{Q} \subset \mathbb{Q}(\pi)$ is transcendental. On the other hand, $\mathbb{R} \subset \mathbb{R}(i\sqrt{\pi})$ is an algebraic extension because $i\sqrt{\pi}$ is a root of $x^2 + \pi$.

A polynomial $p \in k[x]$ is *irreducible* if p is not a constant polynomial (that is, $\deg p > 0$) and whenever $p = qr$ for $q, r \in k[x]$ then one of q or r must be a constant polynomial. An irreducible polynomial is said to *split* over K if it factorises into linear factors over K. In general, it is possible that a polynomial could have a root in K but not all of its roots in K: the polynomial would factor, but not split. An algebraic field extension $k \subset K$ is *normal* if every irreducible polynomial over k which has at least one root in K splits, i.e. has all of its roots in K.

Example 4. The polynomial $x^3 - 2$ has one of its roots in the field $\mathbb{Q}(\sqrt[3]{2})$ but not its other two. Thus this polynomial does not split and this field is not normal. Conversely, the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}, \omega)$ where ω is a primitive third root of unity is normal and the polynomial $x^3 - 2$ now splits.

So far we have implicitly viewed extensions such as $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$ as already embedded into some algebraic closure, in this case $\mathbb{C}$. When constructing the field $\mathbb{Q}(\zeta_5)$, we adjoin ζ_5 to $\mathbb{Q}$ and by knowing which polynomial ζ_5 is a root of we are able to give a general expression for an element of the field. There is another way of looking at this: we could start with a polynomial and try to construct a field from it directly. In particular, if R is a commutative ring and $I \subset R$ is an ideal, then the quotient R/I is a field if and only if I is maximal. One can show that the ideal (f) generated by a polynomial $f \in k[x]$ is maximal if and only if f is irreducible, in which case we can take the quotient $k[x]/(f)$ and form a proper field. If a is a root of f, then there is an isomorphism $k[x]/(f) \simeq k(a)$: a is algebraic over k and f is the *minimal polynomial* of a over k.

At this point we are almost in a position to define Galois extensions, but there is a technical pathology which we must rule out: if an irreducible polynomial over k splits in K, is it the always case that all of its roots are distinct? We will not delve too far into the specifics, other than to say that this is always the case for algebraic extensions of finite fields and fields of characteristic zero. Many extensions—in particular, all those considered in this work—have the property of being *separable*, which rules out the aforementioned pathology: an irreducible polynomial $f \in k[x]$ is called separable if all of its roots are distinct in some extension field, an element $a \in K$ is separable if its minimal polynomial is separable, and an extension is separable if all its elements are separable.

A finite degree field extension $k \subset K$ is *Galois* if it is normal and separable. It can be shown that this is equivalent to K being the *splitting field* of a separable polynomial $p \in k[x]$, that is K is the minimal degree extension of k such that p splits.

Example 5. The extension $\mathbb{R} \subset \mathbb{C}$ is Galois extension, because it is the splitting field of $x^2 + 1$. Similarly, the extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$ is Galois, because it is the splitting field of $1 + x + x^2 + x^3 + x^4$. The extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})$, on the other hand, cannot be Galois, because it is not normal. But we have already seen that we get a normal extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}, \omega)$ if we add ω: this is indeed Galois, because it is the splitting field of $x^3 - 2$.

Galois extensions have attracted a lot of attention in the study of number fields because of a key property, known as the Fundamental Theorem of Galois Theory. To understand it, we first need to introduce a few more notions. If $k \subset L$ and $k \subset K$ are both extensions of k, a *k-homomorphism* $\tau : L \to K$ is a field homomorphism $L \to K$ which fixes k, i.e. one such that $\tau(a) = a$ for all $a \in k$. The k-automorphisms $K \to K$ for a field extension $k \subset K$, i.e. the field automorphisms which fix the base field, always form a group $\mathrm{Aut}(K/k)$: for a Galois extension, this is known as the *Galois group* and is written $\mathrm{Gal}(K/k)$. Importantly, the order of $\mathrm{Gal}(K/k)$ coincides with the degree of the extension, i.e. $|\mathrm{Gal}(K/k)| = [K : k]$.

Theorem 2 (Fundamental Theorem of Galois Theory)**.** *If $k \subset K$ is a Galois extension, then there is a bijection between subgroups $H \leq Gal(K/k)$ and intermediate fields $k \subseteq F \subseteq K$. Each subgroup $H \leq Gal(K/k)$ is sent to the field fixed by all elements of H:*

$$H \mapsto Fix(H) := \{a \in K : \tau(a) = a, \ \forall \tau \in H\}$$

Conversely, each intermediate field $k \subseteq F \subseteq K$ is sent to the group of F-automorphisms of K:

$$F \mapsto Gal(K/F) := \{\tau \in Gal(K/k) : \tau(a) = a, \ \forall a \in F\}$$

Example 6. The extension $\mathbb{R} \subset \mathbb{C}$ has Galois group $\{\mathrm{id}, \mathrm{conj}\} \simeq C_2$ generated by complex conjugation. The extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$ has Galois group $\langle \sigma \mid \sigma^4 = \mathrm{id} \rangle \simeq C_4$ where $\sigma :: \zeta_5 \mapsto \zeta_5^2$. There is only one non-trivial subgroup, $\langle \sigma^2 \rangle \simeq C_2$: one can check that $\sigma^2 :: \zeta_5 \leftrightarrow \zeta_5^4, \zeta_5^2 \leftrightarrow \zeta_5^3$ fixes $\zeta_5 + \zeta_5^4 = (1 + \sqrt{5})/2$ and $\zeta_5^2 + \zeta_5^3 = (1 - \sqrt{5})/2$. Therefore, $\mathrm{Fix}(\langle \sigma^2 \rangle) = \mathbb{Q}(\sqrt{5})$.

For a Galois extension $k \subset K$ with intermediate field $k \subset F \subset K$, it is always the case that $F \subset K$ is a Galois extension. A corollary of the Fundamental Theorem of Galois Theory also tells us that $k \subset F$ is Galois if and only if $\Lambda := \mathrm{Gal}(K/F)$ is normal in $\Gamma := \mathrm{Gal}(K/k)$. As a consequence, $\mathrm{Gal}(F/k)$ isomorphic to the quotient Γ/Λ.

Example 7. The extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$ has an abelian Galois group, so that the subgroup $\Lambda := \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}(\sqrt{5})) = \langle\sigma^2\rangle \simeq C_2$ is automatically normal in $\Gamma := \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \simeq C_4$. As a consequence, $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) \simeq C_4/C_2 \simeq C_2$ and can be explicitly characterised by restricting the automorphisms of Γ to the intermediate field $\mathbb{Q}(\sqrt{5})$:

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) = \{\mathrm{id}, \sigma\mid_{\mathbb{Q}(\sqrt{5})} =: \tau\}$$

where $\tau :: \sqrt{5} \mapsto -\sqrt{5}$ as expected.

The Fundamental Theorem of Galois theory is one of the two notions from Galois theory that will play a major role in this work; the other is the field norm. All finite extensions $k \subset K$ admit a multiplicative map $N_{K/k} : K \to k$, known as the *field norm*, which sends elements of the extension back to the base field. To understand how the field norm is defined, note that every element $a \in K$ induces a map $m_a :: x \mapsto ax$ by left multiplication. Because K is a finite-dimensional vector space over k, this map has a matrix representation: the field norm of a is defined to be the determinant of this matrix, that is $N_{K/k}(a) := \det(m_a)$. If $k \subset E$ is Galois, the field norm can be written explicitly as follows:

$$N_{K/k}(a) = \prod_{\sigma \in \mathrm{Gal}(K/k)} \sigma(a) \tag{3.1}$$

More generally, if $k \subset E$ is a finite separable extension, then the field norm can be written explicitly as follows:

$$N_{E/k}(a) = \prod_{\sigma \in T} \sigma(a) \tag{3.2}$$

where T is a left transversal of $\mathrm{Gal}(\hat{E}/E)$ in $\mathrm{Gal}(\hat{E}/k)$ [139] and $\hat{E}$ is the normal closure of E—the separable field extension of E of smallest degree that is normal (and hence also Galois).

The field norm is a group homomorphism for the multiplicative groups of K and k, $N_{K/k} : K^\times \to k^\times$, meaning expressions like $N_{K/k}(ab) = N_{K/k}(a)N_{K/k}(b)$ hold. Additionally, field norms behave well with towers of extensions, factorising via the intermediate fields. That is, if we have a tower of field extensions of finite degree $k \subset K \subset L$, then $N_{L/k} = N_{K/k} \circ N_{L/K}$.

3.2 Galois CPM Categories

In this section, we develop the connection between Galois theory and CPM constructions more fully. Let S-Mat denote the category of matrices over a commutative semiring S, with the positive natural numbers $n \in \mathbb{N}^+$ as objects and the $n \times m$ matrices with entries from S as morphisms $m \to n$. It is well known that S-Mat is a symmetric monoidal category enriched in commutative monoids, with a wealth of additional structure (e.g. biproducts). If $k \subset K$ is a Galois extension, then its Galois group $\Gamma := \mathrm{Gal}(K/k)$ induces the following action ϕ on K-Mat by linear monoidal autofunctors:

- on objects, $\phi_\gamma(A) := A$ for all $\gamma \in \Gamma$;
- on morphisms, entrywise application of the automorphism, $\phi_\gamma\left((M_{i,j})_{j=1,\dots,m}^{i=1,\dots,n}\right) := (\gamma(M_{i,j}))_{j=1,\dots,m}^{i=1,\dots,n}$.

It is worth noting that the complete folding functor fld_Γ for this action acts as the field norm on the scalars of the category.

$$\mathrm{fld}_\Gamma(x) = N_{K/k}(x) \tag{3.3}$$

Definition 19 (Galois CPM Category)**.** Let $k \subset K$ be a Galois extension with Galois group Γ. The *Galois CPM category* associated to this extension is the CPM category $\mathsf{CPM}_{K/k}(K\text{-}\mathsf{Mat}) := \mathsf{CPM}_{\Gamma,\Xi}(K\text{-}\mathsf{Mat})$ obtained by considering the Γ-action induced by the Galois group on K-Mat and the complete folding functor fld_Γ. The environment structure Ξ is obtained by taking the H-discarding maps for all subgroups $H \leq \Gamma$:

$$\Xi_n := \left\{ \bar{\top}_H : \mathrm{fld}_\Gamma n \to I \,\middle|\, H \leq \Gamma \right\}$$

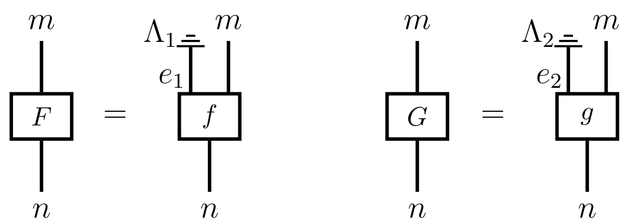
with respect to the classical structure $\circ$ defined by the standard orthonormal basis $(|i\rangle)_{i=1,\dots,n}$ and closing the sets Ξ_n under the tensor product.

For any sub-semiring $S \subset K$ we also introduce the notation $\mathsf{CPM}_{K/k}(S\text{-}\mathsf{Mat}) := \mathsf{CPM}_{\Gamma,\Xi}(S\text{-}\mathsf{Mat})$ for the restriction of the Galois CPM category $\mathsf{CPM}_{K/k}(K\text{-}\mathsf{Mat})$ to the semiring S. The category has the same folding and environment structure as the Galois CPM category but on S-Mat embedded into K-Mat. Note that because the folding is complete, and thus contains all possible embeddings of K into its algebraic closure, the choice of embedding of S into K is not important.

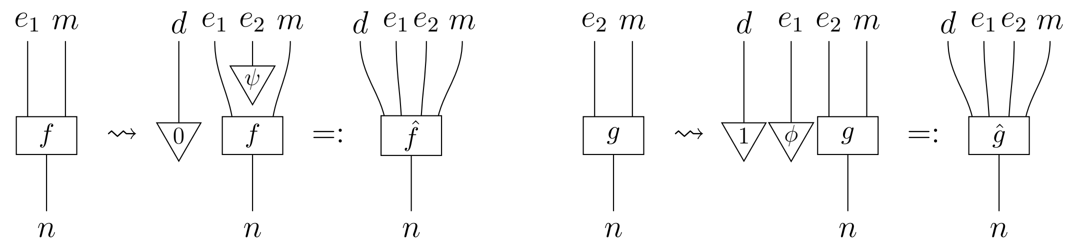
Proposition 21. $\mathsf{CPM}_{K/k}(S\text{-}\mathsf{Mat})$ *is* CMon*-enriched.*

Proof. This follows by the fact that S-Mat is CMon-enriched and the fact that the complete discarding map $\bar{\top}$ is included in the environment structure. Given a collection $\{f_i : \text{fld}_\Gamma n \to \text{fld}_\Gamma m\}_{i=1}^{l}$ of morphisms in $\mathsf{CPM}_{K/k}(S\text{-}\mathsf{Mat})$ we can sum them using the CMon-enrichment of S-Mat but we need to show that the resulting morphism is still in $\mathsf{CPM}_{K/k}(S\text{-}\mathsf{Mat})$. For simplicity we will prove the case of summing two morphisms, but the general case is a straightforward generalisation.

Consider maps F and G of $\mathsf{CPM}_{K/k}(S\text{-}\mathsf{Mat})$ involving possibly different environments e_1 and e_2 and possibly different discarding maps:



We can append additional environments to f and g in S-Mat so that their environments are the same:



where $|0\rangle$ and $|1\rangle$ are orthonormal basis elements associated to the †-SCFA defining the discarding maps, and $|\phi\rangle$ and $|\psi\rangle$ are chosen suitably so that $\bar{\top}^{\Lambda_1}_{e_1} \circ \text{fld}_\Gamma |\phi\rangle = 1$ and $\bar{\top}^{\Lambda_2}_{e_2} \circ \text{fld}_\Gamma |\psi\rangle = 1$. It is always possible to find suitable states, for instance just take a basis element of the basis associated to the †-SCFA defining the discarding maps $\bar{\top}$.

Now we are able to sum these maps in S-Mat and we have that $F + G = \left(\bar{\top}^{\Gamma}_{d} \boxtimes \bar{\top}^{\Lambda_1}_{e_1} \boxtimes \bar{\top}^{\Lambda_2}_{e_2}\right) \circ \text{fld}_\Gamma(\hat{f} + \hat{g})$ showing that this sum is still a map of $\mathsf{CPM}_{K/k}(S\text{-}\mathsf{Mat})$. $\square$

Now, for any Galois CPM category, because the folding functor acts as the field norm (3.3), the pure scalars $\text{fld}_\Gamma(x)$ are always elements of the base field k. Moreover, because k is closed under addition, the mixed scalars which are obtained by applying complete discarding maps, i.e. those in the form:

$$\bar{\top} \circ \text{fld}_\Gamma (|v\rangle) = \sum_{i=1}^{n} \text{fld}_\Gamma(v_i)$$

are also elements of k.

Similarly, for generic scalars obtained by using discarding maps for intermediate subgroups $\{1\} < \Lambda < \Gamma$, we can write the following:

$$\begin{aligned}\bar{\top}_\Lambda \circ \mathrm{fld}_\Gamma\left(|v\rangle\right) = \prod_{t\in T}\left(\sum_{i=1}^{n}\prod_{\mu\in t\Lambda}\phi_\mu(v_i)\right) &= \prod_{t\in T}\phi_t\left(\sum_i\prod_{\lambda\in\Lambda}\phi_\lambda(v_i)\right) \\ &= N_{F/k}\left(\sum_i N_{K/F}(v_i)\right)\end{aligned} \tag{3.4}$$

Since for any $v_i \in K$, $N_{K/F}(v_i) \in F$ and F is closed under addition, we have $\alpha := \sum_i N_{K/F}(v_i) \in F$. Consequently $N_{F/k}(\alpha) \in k$. Indeed, any generic scalar of the Galois CPM category must be an element of the base field because it is fixed by all elements of the Γ-action, just as all morphisms are. As noted, the automorphisms of the action descend on the scalars to the Galois automorphisms of K by k.

It is not necessarily the case that the set $\mathrm{End}(I)$ of all scalars of the Galois CPM category is isomorphic to k itself but rather it must form a sub-semiring. We have the additive and multiplicative units, 0 and 1, since they are pure scalars and closure under multiplication follows immediately from the fact we are working in a monoidal category. The complete discarding maps $\bar{\top}$ act to allow us to take arbitrary sums of any chosen scalars (by proposition 21) and consequently $\mathrm{End}(I)$ is also closed under addition.

Precisely which sub-semiring of k the scalars form can be difficult to decide in general. For instance we see from equation (3.4) that we will have to consider not only the closure of the norm from the top field to the base field, but also the iterative closure of the norm for each intermediate field where we take the norm to an intermediate F, close it under addition, and then take the norm of any element of this to the base field.

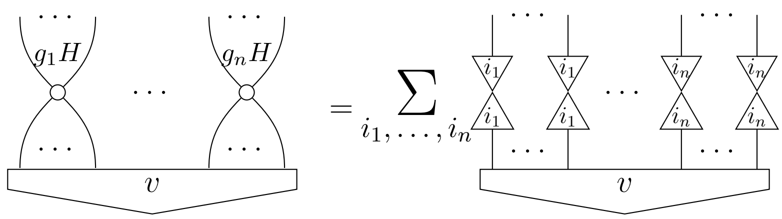
Nevertheless, for many large classes of extensions we will see that we can say fair amount about the semiring of scalars and in many cases fully characterise it.

3.3 Decoherence Structures

By construction, Galois CPM categories come with a family of decoherence maps which mimic the decoherence of quantum theory to classical theory. Each subgroup Λ of the Galois group Γ induces a decoherence map via its corresponding effect in the environment structure Ξ. The similarity with quantum theory is two-fold:

1. The decoherence maps reduce the Γ-folding to one given by a transversal of the subgroup in the overall group. In the case of a normal subgroup, this gives a full folding by the quotient Γ/Λ. This serves to kill-off "interference terms" as one moves down the tower of subgroups, progressively reducing the degrees of freedom that can be used to describe the state of a system. This is akin to how traditional decoherence kills-off non-diagonal terms in the transition from density matrices to classical probability distributions.
2. The decoherence maps also reduce the degree of the field extensions, producing theories over a series of sub-fields given by the Galois correspondence. This is akin to the transition from $\mathbb{C}$ to $\mathbb{R}$ in quantum theory.

The second point follows immediately by considering the symmetries of any decohered map. For simplicity, consider a state v of a Galois CPM category. We can expand the H-decohered v in the orthonormal basis $(|i_n\rangle)_{i_n}$ as



One sees that a generic term of this state is invariant under all autofunctors ϕ_h for $h \in H$ while the remaining autofunctors do not, in general, fix it. Thus the entries of the matrix for the decohered v are fixed by all $h \in H$, i.e. a subgroup of the Galois group and thus must belong to the corresponding intermediate field of the extension.

The hard part, in exactly the same way as for the scalars, is characterising how much of the intermediate field a given decoherence hits, since this depends on properties of the fields in question. Nevertheless, it is clear that it is always a sub-semiring of the intermediate field. We will study many examples in the next sections where, for large classes of extensions, we will be able to fully characterise the sub-semirings forming the images of the decoherences.

3.4 Examples of Galois CPM Categories

3.4.1 Number Fields

An interesting class of Galois extensions to consider are those which are also *number fields*: finite algebraic extensions of $\mathbb{Q}$. Our main aim will be to characterise the closures of field norms and in doing so constrain the semirings at each level of decoherence. We start with a few new definitions.

A field K is *ordered* if there exists a subset $P \subset K$ which is closed under addition and multiplication, with K equal to the disjoint union $P \sqcup \{0\} \sqcup -P$ where $-P := \{-p : p \in P\}$. In such an ordered field one writes $a > b$ if and only if $a - b \in P$. A field K is *formally real* if -1 is not a sum of squares in K which is equivalent to K being ordered [103]. There is a bijection between orderings of K and embeddings (field homomorphisms) of K into its real closure (one can just think of $\mathbb{R}$ for the fields in this work) and for a Galois number field the embeddings are equivalent to considering the $\mathbb{Q}$-automorphisms contained in the Galois group.

An element $a \in K$ is *totally positive* if $a > 0$ for all orderings of K, or equivalently if $\sigma(a) > 0$ for all real embeddings σ of K. We write K^+ for the set of totally positive elements of K, which forms a semiring if we additionally include 0. The semiring of totally positive elements actually has the structure of a *semifield* where every non-zero element has a multiplicative inverse. If K has no orderings then it is vacuously true that any element is positive for all orderings and we say that all elements of K are totally positive. It is the case that total positivity is preserved under field norms.

A number field is called *totally real* if all embeddings into the complex numbers lie within the real numbers. If, on the other hand, no embeddings lie within the real numbers, then the extension is known as *totally imaginary* (or sometimes as *totally complex*). All Galois number fields are either totally real or totally imaginary and for them, being totally real is equivalent to being formally real and being totally imaginary is equivalent to not being formally real.

The well-known Waring's problem asks whether for each $d \in \mathbb{N}$, every natural number is the sum of some finite number $n \in \mathbb{N}$ of naturals raised to the d^{th} power, and was proven by Hilbert in 1909. The result implies that the same is true for the rationals $\mathbb{Q}$.

One can ask a similar question of a general field. If K is a field then we say that *Waring's problem of exponent d* holds if every totally positive element $a \in K$ can be written as a finite sum of d^{th} powers of totally positive elements of K. That is:

$$a = \sum_{i=1}^{n} a_i^d \qquad a_i \in K$$

where the a_i are all totally positive and n is bounded above by some finite $g(K, d)$ dependent only on K and d. Ellison [75, 76] reduces this problem to being able to write all totally positive elements of K as a finite sum of squares alongside a constraint about the density of d^{th} powers in K. By a classical result of Siegel [146] the former of these is possible for number fields: every totally positive element of K is a sum of at most four squares in K.

For us, the outcome of this discussion is the following useful result:

Theorem 3 (Waring's Problem for Fields [75, 76])**.** *If either of the following hold:*

1. *K is a non-real field of characteristic 0*
2. *K is formally real, every totally positive element of K can be written as a sum of at most s squares for some s, and d^{th} powers are suitably dense in K*

then Waring's problem holds for all exponents. In particular, if K is a number field then Waring's problem holds for all exponents.

As a consequence we are able to show that for any extension $k \subset K$ where Waring's problem holds, all totally positive elements of k are contained in the closure of the image of the norm from K to k.

Proposition 22. *Let $k \subset K$ be a finite extension with Waring's problem holding in k. Then $k^+ \subset \overline{N}_{K/k}$.*

Proof. Say $[K : k] = d$, then for any $a \in k \hookrightarrow K$ we have $N_{K/k}(a) = a^d$. Thus all finite sums $\sum_i a_i^d \in \overline{N}_{K/k}$ for $a_i \in k$. By Waring's problem this is all totally positive elements. $\square$

In particular $k^+ \subset \overline{N}_{K/k}$ for all number fields. It is worth pointing out that the upper bound $g(K, d)$ on the number of terms needed in Waring's problem places an upper bound on the dimension of ancillary systems we require to form all the totally positive elements of k in any Galois CPM category.

A simple implication of Proposition 22 is that when k has characteristic 0 and is not formally real, the closure of the norm coincides with k.

Corollary 22.1. *Let $k \subset K$ be a finite extension where k has characteristic 0 and is not formally real. Then $\overline{N}_{K/k} = k$.*

Proof. Since k is not formally real, every element of k is totally positive. $\square$

Proposition 23. *Let $\mathbb{Q} \subset k \subset K$ be a tower of extensions where K and k are both Galois over $\mathbb{Q}$. If K is totally imaginary and k is totally real then $\overline{N}_{K/k} = k^+$.*

Proof. The embeddings of a totally imaginary field always come in pairs, for if $e : K \hookrightarrow \mathbb{C}$ is an embedding then complex conjugation $J_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ composed with e gives another embedding of K. Indeed, $J_{\mathbb{C}}$ induces an automorphism J of K which acts like complex conjugation on K. This automorphism is not necessarily independent of the choice of embedding into $\mathbb{C}$ (and in particular will not commute with the other elements of the Galois group, unless for instance, the field is CM), but nevertheless it forms a subgroup of $\Gamma := \mathrm{Gal}(K/\mathbb{Q})$ isomorphic to C_2.

Now, k is totally real and so must be fixed by J implying that $J \in \Lambda := \mathrm{Gal}(K/k)$. Thus there is at least one embedding of k into $\mathbb{R}$ where $N_{K/k}(a) > 0$ for all $0 \neq a \in K$.

Fix this embedding and consider any $\sigma \in \mathrm{Gal}(k/\mathbb{Q})$. We have:

$$\sigma N_{K/k}(a) = \prod_{\lambda \in \Lambda} \sigma\lambda(a) = \prod_{\mu \in \sigma\Lambda} \mu(a) = \prod_{\mu \in \Lambda\sigma} \mu(a) = N_{K/k}(\sigma(a)) > 0$$

where we used that fact that Λ must be normal in Γ and thus left cosets and right cosets coincide. So $N_{K/k}(a)$ is positive for all embeddings of k. Therefore $\overline{N}_{K/k} \subset k^+$. Proposition 22 gives the other containment $k^+ \subset \overline{N}_{K/k}$. □

Corollary 23.1. *Let $\mathbb{Q} \subset K$ be a totally imaginary Galois extension. Then $\overline{N}_{K/\mathbb{Q}} = \mathbb{Q}^+$*

Remark. Corollary 23.1 shows that the scalars of the folded category $\mathsf{FLD}_{\Gamma}(K\text{-}\mathsf{Mat})$ for totally imaginary K are always elements of $\mathbb{Q}^+$.

We can also go some way to dealing with decoherences from totally real Galois number fields.

Proposition 24. *Let $\mathbb{Q} \subset K$ be a totally real Galois extension. Then $\overline{N}_{K/\mathbb{Q}} = \mathbb{Q}$.*

Proof. This is immediate if $[K : \mathbb{Q}]$ is odd: just note $N(-1) = -1$ and combine with proposition 22. If $[K : \mathbb{Q}]$ is even then a different argument is needed (which still holds for the odd case).

By the normal basis theorem we know that there exists some $\alpha \in K$ such that $\{\sigma(\alpha) : \sigma \in \mathrm{Gal}(K/\mathbb{Q})\}$ forms a $\mathbb{Q}$-basis of K. This means that α is distinct under all Galois automorphisms σ_i. Since K is totally real, there is an ordering on K given by the ordering of $\mathbb{R}$.

If there are an odd number of σ_i such that $\sigma_i(\alpha) < 0$ then $N(\alpha) < 0$ and the result follows.

If there are an even number $2m$ of σ_i such that $\sigma_i(\alpha) < 0$ then, ignoring the σ_i where $\sigma_i(\alpha) > 0$, we have an ordering, say $\sigma_1(\alpha) < \cdots < \sigma_{2m}(\alpha) < 0$. There exists $q \in \mathbb{Q}$ such that $-\sigma_{2m}(\alpha) < q < -\sigma_{2m-1}(\alpha)$ which implies that $\sigma_1(\alpha + q) < \cdots < \sigma_{2m-1}(\alpha + q) < 0 < \sigma_{2m}(\alpha + q)$. Thus $N(\alpha + q) < 0$ and the result follows. □

Remark. Proposition 24 shows that for CPM categories generated by complete foldings for totally real Galois number fields, the mixed scalars arising from the complete discarding map are enough to capture all of $\mathbb{Q}$, and thus $\mathrm{End}(I) \simeq \mathbb{Q}$.

The case of general decoherences from totally real Galois number fields currently seems to be more tricky and is left to future work.

3.4.2 Examples

Cyclotomic Extensions

Cyclotomic extensions are obtained by adjoining a primitive n^{th} root of unity ζ_n to the rationals. These extensions are always Galois, since they are the splitting fields of the cyclotomic polynomials $\Phi_n(x)$. They are of interest to us because they give easily constructible examples of Galois extensions with abelian Galois groups, isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$. Moreover, the Kronecker-Weber theorem guarantees that every finite abelian extension of the rational numbers is a subfield of some cyclotomic field.

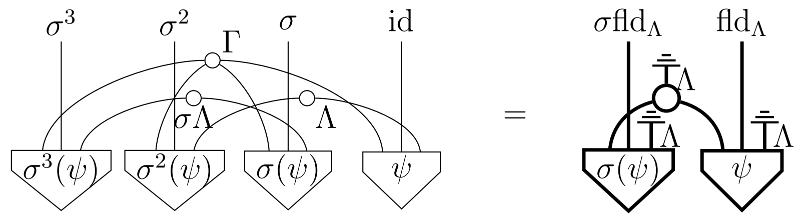
All cyclotomic extensions are totally imaginary: they are given by an imaginary quadratic extension of the totally real field $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ (notably, they are CM-fields). This means that they inherit the results of section 3.4.1: in particular we can constrain decoherences with Corollary 22.1 and Proposition 23.

Let us now present our first explicit example of a Galois CPM theory. Consider the cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$ where ζ_5 is a primitive fifth root of unity. As discussed in Examples 6 and 7, the Galois group of this extension is $\Gamma := \langle \sigma \mid \sigma^4 = \mathrm{id} \rangle \simeq C_4$, where $\sigma :: \zeta_5 \mapsto \zeta_5^2$. The Galois group has a single non-trivial subgroup $\Lambda := \langle \sigma^2 \rangle \simeq C_2$. This subgroup is in correspondence with the field $\mathrm{Fix}(\Lambda) = \mathbb{Q}(\sqrt{5})$, which is a Galois extension of $\mathbb{Q}$ since Λ is normal in Γ.

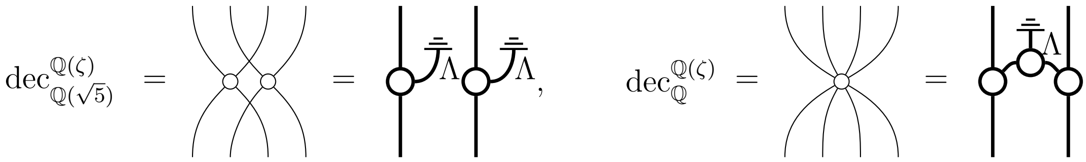
We have the following equalities of closures of norms consistent with the results of Section 3.4.1:

$$\overline{N}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}} = \mathbb{Q}^+ \qquad\qquad \overline{N}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}(\sqrt{5})} = \mathbb{Q}(\sqrt{5})^+$$

The states of $\mathsf{CPM}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(\mathbb{Q}(\zeta_5)\text{-}\mathsf{Mat})$ take the following form:



where the discarding maps on the left have been labelled by the cosets that induce them. On the left we have used the string diagrams of $\mathbb{Q}(\zeta_5)$-Mat and on the right those of $\mathsf{CPM}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\mathbb{Q}(\sqrt{5})^+\text{-}\mathsf{Mat})$. There are two decoherence maps, corresponding to the subgroups Λ and Γ respectively:



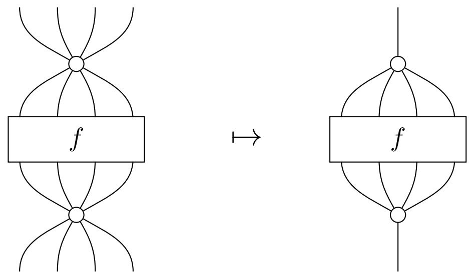
Proposition 25. *Write* $\mathsf{Quant}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}} := \mathsf{Split}(\mathsf{CPM}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(\mathbb{Q}(\zeta_5)\text{-}\mathsf{Mat}))$ *for the Karoubi envelope of the Galois CPM category from the previous example. The full sub-categories of the Karoubi envelope* $\mathsf{Quant}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}$ *spanned by the decoherence maps are characterised as follows:*

- *dec*$^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}}$ *decoherence maps* $\Rightarrow$ *equivalent to* $\mathbb{Q}^+$-Mat*,*
- *dec*$^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}(\sqrt{5})}$ *decoherence maps* $\Rightarrow$ *equivalent to* $\mathsf{CPM}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\mathbb{Q}(\sqrt{5})^+\text{-}\mathsf{Mat})$*.*

Proof. We will directly construct the functors and show that they are full, faithful and essentially surjective on objects and thus witness the equivalences.

Start with the decoherences $\mathrm{dec}^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}}$. On objects send $(\mathrm{fld}_\Gamma n, \mathrm{dec}^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}})$ to n in $\mathbb{Q}^+$-Mat. This is clearly essentially surjective on objects.

On morphisms $f : (\mathrm{fld}_\Gamma n, \mathrm{dec}^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}}) \to (\mathrm{fld}_\Gamma m, \mathrm{dec}^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}})$ we do the following:



which is clearly faithful. We are left to show that the functor is full, which follows by the results on the closures of norms. The symmetries of the maps are sufficient to show that the elements of the matrices are in $\mathbb{Q}$ - they are fixed by the Galois group. The fact that $\overline{N}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}} = \mathbb{Q}^+$ shows that it is enough to sum pure maps of the form $\mathrm{fld}_\Gamma g$ for $g : n \to m$ using the complete discarding maps in order to get every matrix of $\mathbb{Q}^+$-Mat. That we can get nothing more than this is a consequence of the preservation of totally positive elements under norms. A general fully decohered map of $\mathsf{Quant}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}$ may contain Λ-discarding maps but the matix elements of such a map can always be written in the form $\sum_i N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\alpha_i)$ for $\alpha_i \in \overline{N}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}(\sqrt{5})} = \mathbb{Q}(\sqrt{5})^+$.

At which point it is enough to note that totally positive elements of $\mathbb{Q}(\sqrt{5})$ are sent to totally positive elements of $\mathbb{Q}$ by the norm.

A similar argument holds for the $\mathrm{dec}^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}(\sqrt{5})}$ decoherences. On objects we send $(\mathrm{fld}_\Gamma n, \mathrm{dec}^{\mathbb{Q}(\zeta_5)}_{\mathbb{Q}})$ to $\mathrm{fld}_{\Gamma/\Lambda} n$ and on morphisms do similar to the previous case of combining input or output legs on each spider. The symmetries of the maps show that the elements of the matrices live in $\mathbb{Q}(\sqrt{5})$ and that we can only reach the totally positive ones is a consequence of the closure of the norm. $\square$

As a result there is the following correspondence between Galois CPM categories, fields and groups.

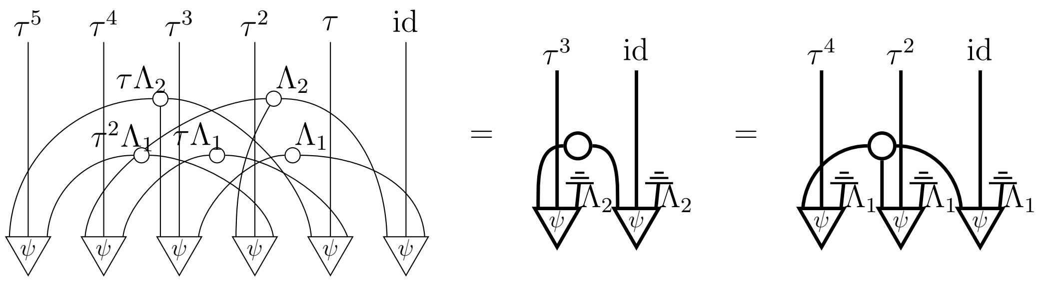
$$
\begin{array}{ccc}
\mathsf{CPM}_{\mathbb{Q}(\zeta_5)/\mathbb{Q}}(\mathbb{Q}(\zeta_5)\text{-}\mathsf{Mat}) & \mathbb{Q}(\zeta_5) & \{*\} \\
| & | & | \\
\mathsf{CPM}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\mathbb{Q}(\sqrt{5})^+\text{-}\mathsf{Mat}) & \mathbb{Q}(\sqrt{5}) & \Lambda \\
| & | & | \\
\mathbb{Q}^+\text{-}\mathsf{Mat} & \mathbb{Q} & \Gamma
\end{array}
$$

Let us now present a second explicit example of a Galois CPM theory. The cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_7)$, where ζ_7 is a primitive seventh root of unity, is our first example of an extension where the decoherence structures do not just form a linear tower. This is because the Galois group has two non-trivial subgroups with trivial intersection. The Galois group of $\mathbb{Q} \subset \mathbb{Q}(\zeta_7)$ is $\Gamma := \langle \tau \mid \tau^6 = \mathrm{id}\rangle \simeq C_6$, where $\tau :: \zeta_7 \mapsto \zeta_7^3$. There are two non-trivial subgroups $\Lambda_1 := \langle \tau^3 \rangle \simeq C_2$ and $\Lambda_2 := \langle \tau^2 \rangle \simeq C_3$. The subgroups give rise to two intermediate fields $\mathrm{Fix}(\Lambda_1) = \mathbb{Q}(\zeta_7 + \zeta_7^6)$ and $\mathrm{Fix}(\Lambda_2) = \mathbb{Q}(\sqrt{-7})$, both of which are Galois extensions of $\mathbb{Q}$ (note that Γ is abelian).

We have the following equalities of closures of field norms:

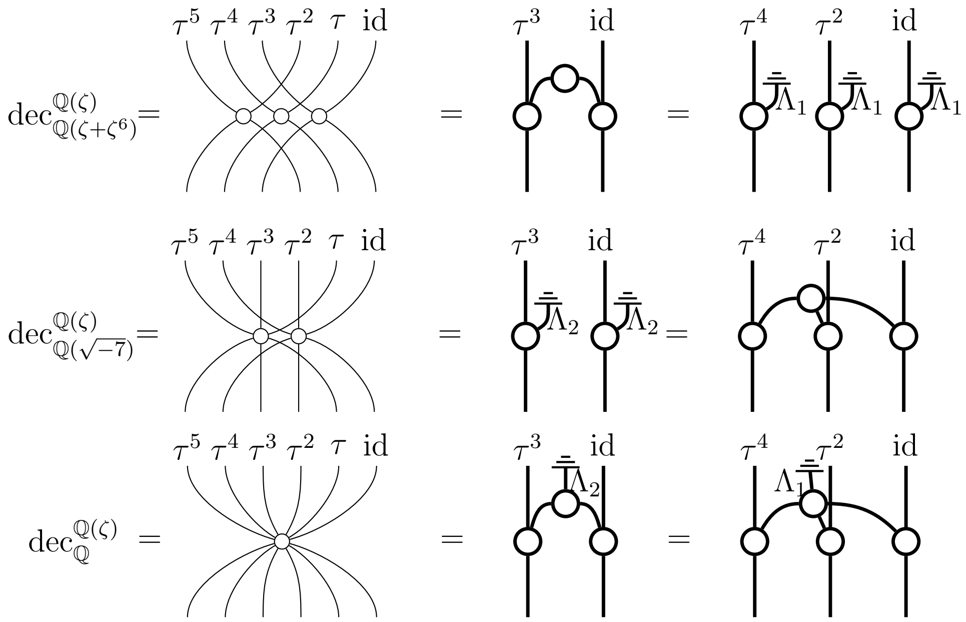
$$
\begin{aligned}
\overline{N}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}} &= \mathbb{Q}^+ & \qquad \overline{N}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}(\sqrt{-7})} &= \mathbb{Q}(\sqrt{-7}) \\
\overline{N}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}(\zeta_7+\zeta_7^6)} &= \mathbb{Q}(\zeta_7 + \zeta_7^6)^+ & \qquad \overline{N}_{\mathbb{Q}(\sqrt{-7})/\mathbb{Q}} &= \mathbb{Q}^+
\end{aligned}
$$

The Galois CPM category induced by this extension has states of the following form:



where the cosets that give rise to each discarding map have been indicated and the automorphisms on ψ have been suppressed for readability.

There are three decoherence maps, the partial decoherences to the intermediate theories and the full decoherence down to the base field:

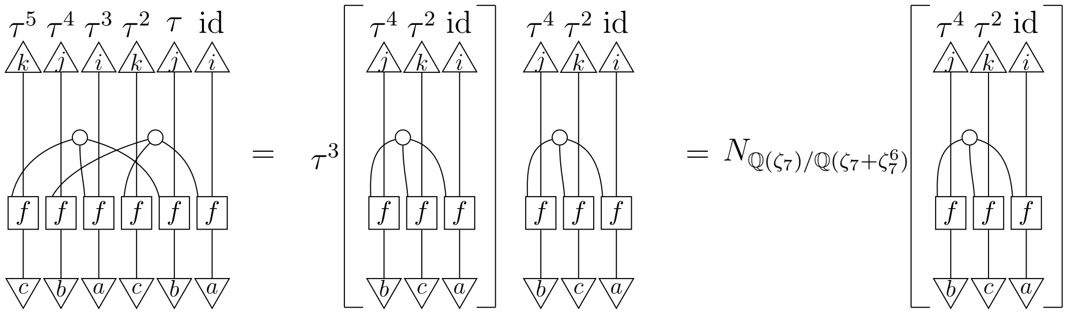


Proposition 26. *Write* $\mathsf{Quant}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}} := \mathsf{Split}(\mathsf{CPM}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}}(\mathbb{Q}(\zeta_7)\text{-}\mathsf{Mat}))$ *for the Karoubi envelope of the Galois CPM category from the previous example. The full sub-categories of the Karoubi envelope* $\mathsf{Quant}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}}$ *spanned by the decoherence maps are characterised as follows:*

- *$dec_{\mathbb{Q}}^{\mathbb{Q}(\zeta_7)}$ decoherence maps $\Rightarrow$ equivalent to* $\mathbb{Q}^+\text{-}\mathsf{Mat}$*,*
- *$dec_{\mathbb{Q}(\sqrt{-7})}^{\mathbb{Q}(\zeta_7)}$ decoherence maps $\Rightarrow$ equivalent to* $\mathsf{CPM}_{\mathbb{Q}(\sqrt{-7})/\mathbb{Q}}(\mathbb{Q}(\sqrt{-7})\text{-}\mathsf{Mat})$*,*
- *$dec_{\mathbb{Q}(\zeta_7+\zeta_7^6)}^{\mathbb{Q}(\zeta_7)}$ decoherence maps $\Rightarrow$ equivalent to* $\mathsf{CPM}_{\mathbb{Q}(\zeta_7+\zeta_7^6)/\mathbb{Q}}(\mathbb{Q}(\zeta_7+\zeta_7^6)^+\text{-}\mathsf{Mat})$*.*

Proof. The proof is very similar to Proposition 25 - the construction of the functors is analogous and the arguments about norms hold here too, so we leave the reader to fill in those details. There is one major additional point which needs to be considered: what happens when we have a morphism with both Λ_1 and Λ_2-discarding maps? We must demonstrate that these do not give rise to any additional maps (i.e. a matrix with some non-positive entries).

For instance consider an entry of a Λ_1-decohered morphism with a Λ_2-discarding map and note that the following holds:



so that such a term is in the image of the expected norm. A similar result holds reversing Λ_1 and Λ_2 and both are precisely because of the second isomorphism theorem, also known as the diamond theorem. This tells us that $\Gamma/\Lambda_1 \simeq \Lambda_2/\{*\} \simeq \Lambda_2$ and so the folding due to Λ_2 is precisely the same as the quotient folding Γ/Λ_1. In other words, the folding "left over" after decohering by Λ_1 is that of Λ_2.

We have dealt with the decoherences to the intermediate fields, but there is one more issue which may raise some concern - the full decoherence to $\mathbb{Q}$ but on a morphism with both Λ_1 and Λ_2-discarding maps. In fact, by a similar method to the one outlined above one can show that an entry in such a matrix must be an element of both the closure of the norms to both intermediate fields, while of course also being an element of $\mathbb{Q}$ by the symmetries. Thus it must be an element of $\mathbb{Q} \cap \mathbb{Q}(\sqrt{-7}) \cap \mathbb{Q}(\zeta_7 + \zeta_7^6)^+ = \mathbb{Q}^+$. $\square$

The result above gives the following correspondence between fields, groups and Galois CPM constructions, respectively:

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\zeta_7) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\zeta_7+\zeta_7^6) \arrow[dr, dash] & & \mathbb{Q}(\sqrt{-7}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \{*\} \arrow[dl, dash] \arrow[dr, dash] & \\
\Lambda_1 \arrow[dr, dash] & & \Lambda_2 \arrow[dl, dash] \\
 & \Gamma &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
 & \mathsf{CPM}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}}(\mathbb{Q}(\zeta_7)\text{-}\mathsf{Mat}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathsf{CPM}_{\mathbb{Q}(\zeta_7+\zeta_7^6)/\mathbb{Q}}(\mathbb{Q}(\zeta_7+\zeta_7^6)^+\text{-}\mathsf{Mat}) \arrow[dr, dash] & & \mathsf{CPM}_{\mathbb{Q}(\sqrt{-7})/\mathbb{Q}}(\mathbb{Q}(\sqrt{-7})\text{-}\mathsf{Mat}) \arrow[dl, dash] \\
 & \mathbb{Q}^+\text{-}\mathsf{Mat} &
\end{tikzcd}
$$

In the third diagram, the categories are "connected" by the decoherence maps.

Remark. Similar proof methods to those of Propositions 25 and 26 will work for a much larger class of field extensions. In particular, for Dedekind groups where all subgroups are normal, all intermediate fields are Galois and many of the results constraining closures of norms will be useful. In such a case, the second isomorphism theorem will also come into play allowing one to make similar arguments about morphisms with discarding maps arising from different subgroups.

Quadratic Fields

A quadratic extension $\mathbb{Q} \subset K$ is a degree two extension of the rationals, i.e. $[K : \mathbb{Q}] = 2$. Any quadratic field is isomorphic to one of the form $\mathbb{Q}(\sqrt{d})$ for d square-free. If $d > 0$ then we have a real quadratic field, while $d < 0$ gives an imaginary quadratic field. Imaginary quadratic fields are clearly CM, and constitute the motivating examples for the theory of CM-fields.

Quadratic fields are Galois extensions of $\mathbb{Q}$ (they are the splitting fields of $x^2 - d$) with Galois group isomorphic to C_2, generated by the map $\sigma :: \sqrt{d} \mapsto -\sqrt{d}$. Since standard quantum theory has underlying C_2 folding symmetry, the CPM categories induced by quadratic fields look a lot like standard quantum theory.

In the case of imaginary quadratic fields, the similarities are substantial. Writing $\sqrt{d} = i\sqrt{c}$, the field norm $N(x + \sqrt{d}y) = x^2 + cy^2 \geq 0$ is elliptic and non-negative (consistently with our previous results on folding of scalars in CM-fields). The scalars form the semiring $\mathbb{Q}^+$ and the phases form a subgroup of the standard quantum phases. By Hilbert's Theorem 90, these phases take the form $\sigma(b)/b$ for some $b \in \mathbb{Q}(\sqrt{d})$.

In the case of real quadratic fields, there are instead substantial differences from quantum theory. The scalars become the entire field $\mathbb{Q}$ and the norm $N(x + \sqrt{d}y) = x^2 - dy^2$ is hyperbolic. Hilbert 90 allows for the same description of the phases and the theory has similarities to hyperbolic quantum theory [86]: the only multiplicative characters are the real ones, so that hidden subgroup problems [156, 88] can only be efficiently solved for the groups $\mathbb{Z}_2^n$.

Finite Fields

Galois CPM categories induced by finite fields provide simple and nice examples of the structures we have seen so far. For any power of a prime $q = p^n$ there exists a unique finite field $\mathbb{F}_q$ of order q. The non-zero elements $\mathbb{F}_q^\times$ form a cyclic group of order $q - 1$, generated by some element a. Extensions of the form $\mathbb{F}_q \subset \mathbb{F}_{q^m}$ are always Galois with cyclic Galois group generated by the Frobenius automorphism $\phi_p :: t \mapsto t^p$ for $t \in \mathbb{F}_{q^m}$.

Of particular interest to us is the fact that the field norm is surjective. Indeed, by taking a to generate $\mathbb{F}_{q^m}^{\times}$, one always has that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a) = a^{1+q+\cdots+q^{m-1}} = a^{(q^m-1)/(q-1)}$, immediately implying that $N(a)$ has multiplicative order $q-1$; therefore $N(a)$ generates $\mathbb{F}_q$. Because the field norm is surjective, the endomorphisms of the unit object $\mathrm{End}(I)$ of $\mathsf{CPM}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\mathbb{F}_{q^m}\text{-}\mathsf{Mat})$ forms a field isomorphic to $\mathbb{F}_q$, and all intermediate theories spanned by decoherence maps in $\mathsf{Quant}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ are Galois CPM categories for finite fields, not just categories over sub-semirings.

Proposition 27. *Let* $\mathsf{Quant}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ *be the subcategory of the Karoubi envelope of* $\mathsf{CPM}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\mathbb{F}_{q^m}\text{-}\mathsf{Mat})$ *spanned by the decoherence maps induced by the subgroups of the Galois group. Then for any subgroup* $\Lambda \leq \Gamma$, *the subcategory of* $\mathsf{Quant}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$ *spanned by the decoherence maps* dec_Λ *is equivalent to the category* $\mathsf{CPM}_{\mathbb{F}_{q^l}/\mathbb{F}_q}(\mathbb{F}_{q^l}\text{-}\mathsf{Mat})$ *where* $\mathbb{F}_{q^l} = Fix(\Lambda)$.

Proof. The proof is very similar to Propositions 25 and 26. By observing that the field norms are surjective for finite fields, so that their images are isomorphic to the entire codomain field, one does not need to be nearly as careful as for the aforementioned propositions. The simple symmetry argument will suffice. □

3.4.3 CPM Categories for Separable Extensions

Let us now consider a generalisation of the construction presented in the previous section, where we have an extension $k \subset E$ which is separable but not necessarily normal. As noted in section 3.1, such an extension still has a concise expression for the field norm $N_{E/k}$, but we are forced to work with the normal closure $\hat{E}$ of E. Both $k \subset \hat{E}$ and $E \subset \hat{E}$ are then Galois extensions, and the norm $N_{E/k}$ is given by a product over a transversal of $\mathrm{Gal}(\hat{E}/E)$ in $\mathrm{Gal}(\hat{E}/k)$ (see equation (3.1)).

The generalisation of the CPM construction to group transversals outlined in section 1.4 gives the necessary machinery to treat the case of separable extensions. One way of producing an interesting CPM category (with suitably constrained scalars) from E is to "upgrade" E-Mat to $\hat{E}$-Mat, which has a canonical group action ϕ by its Galois group $\Gamma := \mathrm{Gal}(\hat{E}/k)$. Upon picking a left transversal T of $\Lambda := \mathrm{Gal}(\hat{E}/E)$ in Γ, one can then take the folding functor $\mathrm{fld}_\tau : E\text{-}\mathsf{Mat} \to \hat{E}\text{-}\mathsf{Mat}$, where E-Mat is equivalent to the subcategory of the equivariant category $\hat{E}\text{-}\mathsf{Mat}_\Lambda$ spanned by the identity isomorphisms $\eta_A^g := \mathrm{id}_A$ (because ϕ_g is the identity on objects for all $g \in \Lambda$).

One consequence of this generalisation is that we are now able to consider Galois CPM categories induced by extensions with Galois groups which are not *Dedekind*: that is, Galois groups with subgroups which are not normal. For such an extension there exist intermediate fields which are not Galois over the base field (they are in bijection with the non-normal subgroups) and complete foldings are not enough to deal with them in a rigorous way.

As an example, consider the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha, \omega)$, where ω is a primitive third root of unity and $\alpha^3 = 2$. This is the splitting field of $x^3 - 2$ over $\mathbb{Q}$, and is therefore Galois. It has Galois group $\Gamma \simeq S_3$ generated by the automorphisms $\sigma :: \omega \mapsto \omega^2$ and $\tau :: \alpha \mapsto \alpha\omega$, with the following lattice of subgroups:

$$\begin{tikzcd}
& \boldsymbol{\{*\}} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\boldsymbol{\langle \tau \rangle} \simeq C_3 \arrow[dr, dash] & \langle \sigma \rangle \simeq C_2 \arrow[d, dash] & \langle \tau\sigma \rangle \simeq C_2 \arrow[dl, dash] & \langle \sigma\tau \rangle \simeq C_2 \arrow[dll, dash] \\
& \boldsymbol{\langle \sigma, \tau \rangle} \simeq S_3 & &
\end{tikzcd}$$

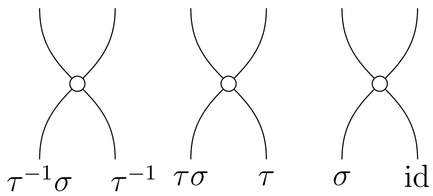
The normal subgroups of Γ are shown here in bold font. This lattice is in bijection with the following lattice of intermediate fields:

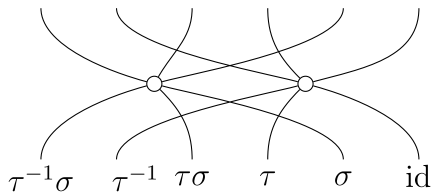
$$\begin{tikzcd}
& \mathbb{Q}\boldsymbol{(\alpha, \omega)} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\mathbb{Q}\boldsymbol{(\omega)} \arrow[dr, dash] & \mathbb{Q}(\alpha) \arrow[d, dash] & \mathbb{Q}(\alpha\omega^2) \arrow[dl, dash] & \mathbb{Q}(\alpha\omega) \arrow[dll, dash] \\
& \mathbb{Q} & &
\end{tikzcd}$$

The Galois extensions of $\mathbb{Q}$ are shown here in bold font. With the generalisation of the CPM construction given in this section, we are now able to consider decoherences to the non-normal intermediate fields, such as $\mathbb{Q}(\alpha)$. The folding of $\mathbb{Q}(\alpha, \omega)$ over $\mathbb{Q}(\alpha)$ is straightforward, since this extension is Galois. The folding of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ is more tricky, and requires picking a left transversal of $\langle \sigma \rangle$ in Γ; for instance, one can pick $T := \{\mathrm{id}, \tau, \tau^{-1}\}$. These foldings mimic the form of the field norms and act to suitably constrain the scalars of the theories.

$$N_{\mathbb{Q}(\alpha,\omega)/\mathbb{Q}(\alpha)}(a) = a\sigma(a) \qquad N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(a) = a\tau(a)\tau^{-1}(a)$$

Although it was possible to pick T to be a subgroup of Γ this need not be the case in general. A transversal can always be given the structure of a quasigroup, and if the transversal contains the identity then the algebraic stucture is stronger and forms a loop. These structures are not associative, making them problematic to study internally to a category which is why we take the route of describing the foldings directly at the level of the transversal. Nevertheless, there are many occasions when a particular choice of transversal does form a group - yet one must take care since this group is not, even in the case where Λ is normal in Γ, necessarily a subgroup of Γ (for instance consider the quaternion group Q_8 which has centre $Z(Q_8) \simeq C_2$ and quotient $Q_8/Z(Q_8) \simeq C_2 \times C_2$ which is not isomorphic to any subgroup of Q_8). Finally, we can write down the decoherence maps in all their glory:

$\mathrm{dec}_{\mathbb{Q}(\alpha)}^{\mathbb{Q}(\alpha,\omega)} =$ 

$\mathrm{dec}_{\mathbb{Q}}^{\mathbb{Q}(\alpha)} =$ 

3.5 Summary

In this chapter we have presented an infinite family of probabilistic theories induced by Galois extensions. We have seen that these categories contain rich decoherence structures, in correspondence with subgroups of the Galois group, producing intricate towers of CPM categories with progressively reduced degrees of freedom. The fundamental theorem of Galois theory acts to provide the bridge between these subgroups and intermediate fields, from which we were able to constrain the semirings that each intermediate theory is over. Totally imaginary number fields are particularly physically relevant because the scalars of the theory are constrained to be strictly positive and thus have an immediate and standard operational interpretation. In this case, it is also fairly straightforward to fully characterise the semirings: if the intermediate field k is also totally imaginary then decoherence hits all elements of k and if the intermediate field is totally real then decoherence is restricted to hit only the totally positive elements which form a semifield. The total positivity of these elements is preserved under the field norm so subsequent decoherences to lower fields is also restricted to only the totally positive elements. The case of cyclotomic extensions is rich enough to produce non-trivial decoherence towers while being relatively simple fields to work with - they are abelian and constructible by adjoining roots of unity.

More generally, it is the case that the folding of a CPM construction is compositional. In essence, this is because each folding produces a category with morphisms essentially invariant under the action of the group used to generate it. Thus, one can factorise a folding into a series of lower-order foldings by subgroups and transversals of the overall group. Of course in the case of normal subgroups (which is particularly relevant for Galois extensions) this is equivalent to foldings by subgroups and quotient groups.

Afterword, Conclusion and Future Work

In Part I we studied hyper-decoherence between CPTs. We furthered the study of the theory of density hypercubes showing that it is a bona-fide probabilistic theory exhibiting hyper-decoherence to quantum theory. Then we introduced Galois CPM categories which generalise the transition from quantum to classical theory, producing examples of towers of probabilistic theories connected by generalised decoherence maps. There are several directions for future work that follow directly from this.

- One could study any of the myriad of theories presented here from a GPT/CPT [42, 41, 89] perspective. Are all processes purifiable in a theory and how do they interplay with the decoherence tower? Are pure maps in a subtheory pure in a larger theory? Do purifications exist within the same subtheory or do we sometimes require enlarging to a larger theory?

- There is also the question of computational advantage. Can we find computational advantage in some extension theory over its base theory and do we see this advantage decrease as one decoheres through the tower of intermediates?

- What phases are quotiented away through the decoherence tower of a Galois CPM category? Can these phases be estimated and are they of computational use?

- The issue of causality needs to be taken seriously if one wants to discuss the operational semantics of the theories in this work. One can always make all the decoherence maps causal by taking the overall discarding map to be the spider induced by the entire underlying folding group. However, there is a balancing act between making the decoherences causal and allowing for bona fide and interesting measurements, which arise as convex decompositions of the discarding map. This issue was investigated in the particular case of density hypercubes [96] but the general case has not been considered.

- A proof is known against the existence of a causal and idempotent hyper-decoherence map (post-quantum to quantum decoherence map) in a theory with purifications, where pure quantum states are pure post-quantumly and where the maximally mixed state is maximally mixed post-quantumly [117]. It seems necessary then to consider dropping at least one of these assumptions in the search for plausible hyper-decoherence structures. The theories presented here could be interesting grounds to explicitly investigate the interplay between decoherence, purifications and causality and could lead to refinements of the no-go result. Does a version hold in a wider class of theories? Can we weaken any of the assumptions or indeed find novel ways to defeat the no-go?

- There is initial evidence suggesting that density hypercubes would exhibit higher-order interference [90], that is, multi-slit interference which cannot be decomposed into lower slit interference patterns [149, 148, 14, 118]. Final confirmation will require a fully fledged simulation of the triple and quadruple slit interference experiments within density hypercubes. Confirming higher-order interference would then immediately lead to investigation of computational advantage in density hypercubes. Is higher-order interference also present in any of the Galois CPM categories?

- Gaining a better understanding of the structure of the Beyond would also be an interesting route forward. In general, it seems likely that the Beyond is a non-trivial sector of density hypercubes, which could define its own operational theory, perhaps (though we think it is unlikely) equivalent to quantum theory again. Investigations of this nature might also help shed some light on whether there are more elements of the hyper-phase group than we discovered in this work.

- From a foundational perspective, we are also interested in exploring variations on the current formulation of density hypercubes, e.g. by describing it directly from the perspective of quantum observers. Perhaps this could yield a post-quantum theory with deterministic hyper-decoherence maps, but where pure quantum states would become fundamentally mixed as states of density hypercubes.

- On the technical side of things, it could be interesting to try to extend the proofs of Propositions 25 and 26 to a wider class of Galois CPM categories. The difficulties arise when one has a morphism involving many discarding maps for different subgroups - by symmetry the decoherences will always produce morphisms over the corresponding intermediate field but it is not clear precisely what sub-semiring is produced by the interacting discarding maps in general. There is also the challenge of dealing with intermediate fields which are not Galois. We outlined the general idea in section 3.4.3 hinging upon our generalisation of the CPM construction to group transversals and demonstrated that such a construction would constrain scalars suitably by capturing the field norm of these separable extensions. Nevertheless, the general case of what sub-semiring forms the image of a decoherence is not known. Dealing with the case of totally real Galois number fields could be a good starting point.

Part II

Profunctorial Methods for Spacetime and Quantum Supermaps

The second half of this work will concern itself with the application of *profunctorial* methods to models of physics, focusing on spacetime and quantum supermaps. Profunctors are a strict generalisation of functors allowing us to study more complex compositional structure between categories. Whereas a functor $F : \mathcal{C} \to \mathcal{D}$ assigns an object FC of $\mathcal{D}$ to each object C of $\mathcal{C}$, a profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ takes each pair (D, C) of objects C from $\mathcal{C}$ and D from $\mathcal{D}$, and assigns them a set $P(D, C)$. Formally, P is itself a functor $P : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$.

One of the complexities of profunctors lies in their composition, which is a fairly abstract procedure that can seem quite opaque to those at the start of their profunctorial journey. Their composition can be understood to be a generalisation of the tensor product of bimodules over a ring and of the composition of relations between sets. This is, at least in part, why profunctors are also known as distributors, bimodules and relators. To see the bimodular nature of a profunctor note that the sets $P(D, C)$ come with commuting left and right actions by the morphisms of $\mathcal{D}$ and $\mathcal{C}$ respectively. To see the relational nature of a profunctor note that a relation between sets A and B is a function $B \times A \to \{0, 1\}$.

Nevertheless, the complexity of the composition of profunctors also contributes to their strength and utility, since it is this that allows for richer compositional structure than otherwise. In particular, working with profunctors comes with a significant conceptual shift: instead of working inside the categories we now work externally studying the action of the categories on the category of sets. This is akin to the shift from group theory to group representation theory, where instead of studying a group G itself one studies how the group acts on vector spaces. A representation is nothing more than a functor from G as a one object category into the category of vector spaces that is, a presheaf $G^{\mathrm{op}} \to \mathsf{Vect}$. Similarly, profunctors are generalisations of presheaves: $P : \mathcal{C} \nrightarrow \mathcal{D}$ is a presheaf of the product category $\mathcal{D} \times \mathcal{C}^{\mathrm{op}}$.

Over the next few chapters, we will see how profunctors can be used to deal with a number of modelling problems in physics. Firstly, there arise categories which fail to be straightforwardly monoidal. For instance, decompositional approaches to physics where one starts with a global system (e.g. a fixed spacetime $\mathcal{M}$) and breaks it up into subsystems (e.g. spacelike subsets of the spacetime) will not in general be monoidal. There is no guarantee that given two spacelike subsets X and Y, their union $X \cup Y$ will still be spacelike. This means our category lacks the necessary objects to define a monoidal structure. From a physical point of view there are obstructions to this monoidal structure due to the dependency of the systems: the state of a field defined at Y is certainly not independent of the state at some X in the past light cone of Y. In Chapter 5 we will demonstrate how these issues can be overcome by weakening monoidal structure to promonoidal structure. Roughly speaking, we replace the functorial structures of monoidal categories with profunctorial structures allowing us to assign "virtual" tensor products to otherwise problematic objects.

In Chapter 7 we will look at some similar problems that arise in the study of quantum combs and more general quantum supermaps. Given a category of first-order processes, it is surprisingly difficult to construct a category of higher-order processes. Most known methods rely on substantial structure of the category of first-order processes (such as compact closure, or an explicit assumption that we are working with FHilb). We compare two methods for constructing a category of combs, the first inspired by quantum foundations and the second from the study of bidirectional data accessors in computer science. These categories do not have natural monoidal structures because pairs of holes in a circuit may not be equivalent to a single hole, and similar to the spacetime models, our category lacks sufficient objects to describe these. We will see that promonoidal structure can once again be used to equip these categories with two "virtual" tensors (roughly horizontal and vertical composition of combs) and we study how these tensors interact.

Chapter 4

Profunctors

For the remainder of this thesis (apart from Chapter 5) we will work with categories enriched over an arbitrary cosmos. We take our cosmos $\mathcal{V}$ to be a complete, cocomplete, closed symmetric monoidal category, writing $\boxtimes$ for its tensor product and $I_\mathcal{V}$ for its unit object.

Definition 20 ($\mathcal{V}$-Category)**.** A $\mathcal{V}$-enriched category $\mathcal{C}$ consists of the following data:

- a collection $\mathrm{Ob}(\mathcal{C})$ of objects,
- (hom-objects) for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, an object $\mathcal{C}(A, B)$ of $\mathcal{V}$,
- (composition) for each triple of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, an arrow of $\mathcal{V}$

$$\mathcal{C}(B, C) \boxtimes \mathcal{C}(A, B) \xrightarrow{\circ_{ABC}} \mathcal{C}(A, C)$$

- (identities) for each object $A \in \mathrm{Ob}(\mathcal{C})$, an arrow of $\mathcal{V}$

$$I_\mathcal{V} \xrightarrow{j_A} \mathcal{C}(A, A)$$

such that a family of diagrams commute ensuring associativity and unitality of composition (see e.g. [108]).

We can conceptualise a $\mathcal{V}$-category as generalising a usual category by allowing the homs $\mathcal{C}(A, B)$ to have additional structure. This additional structure is captured by saying that the homs are in fact objects of some other category $\mathcal{V}$. For instance, $\mathcal{V}$ could be taken to be CMon (as was the case in Part I), which means that each hom-object has a commutative monoid operation allowing us to sum morphisms. In the case that $\mathcal{V} = \mathsf{Set}$, a Set-enriched category is precisely a (locally small) category in the usual sense.

Definition 21 ($\mathcal{V}$-Functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- a function $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ sending objects of $\mathcal{C}$ to objects of $\mathcal{D}$,
- for each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, an arrow of $\mathcal{V}$,

$$\mathcal{C}(A, B) \xrightarrow{F_{AB}} \mathcal{D}(FA, FB)$$

such that a family of diagrams commute ensuring functoriality of F, so that F respects both composition and identities (see e.g. [108]).

Definition 22 ($\mathcal{V}$-Natural Transformation)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be $\mathcal{V}$-functors. A $\mathcal{V}$-natural transformation $\eta : F \Rightarrow G$ consists of an arrow $I_{\mathcal{V}} \to \mathcal{D}(FA, GA)$ of $\mathcal{V}$ for each object $A \in \mathrm{Ob}(\mathcal{C})$ such that a family of diagrams commute ensuring naturality of η (see e.g. [108]).

Enriched categories, functors and natural transformations assemble into a 2-category $\mathcal{V}$-Cat generalising the 2-category Cat. This 2-category is monoidal when equipped with the following tensor product of enriched categories.

Definition 23 (Enriched Tensor Product)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. Their tensor product $\mathcal{C} \boxtimes \mathcal{D}$ has as objects, pairs (C, D) of an object C of $\mathcal{C}$ and D of $\mathcal{D}$. The hom-objects are given by taking the tensor product in $\mathcal{V}$ of the hom-objects of $\mathcal{C}$ and $\mathcal{D}$,

$$(\mathcal{C} \boxtimes \mathcal{D})\big((C, D), (C', D')\big) := \mathcal{C}(C, C') \boxtimes \mathcal{D}(D, D')$$

Composition and identities in $\mathcal{C} \boxtimes \mathcal{D}$ are induced by those of $\mathcal{C}$ and $\mathcal{D}$, together with the symmetry $\sigma_{\mathcal{V}}$ of the cosmos $\mathcal{V}$.

For ease of reading, unless otherwise indicated, "category," "functor," "natural transformation" etc. should from now on be taken to mean $\mathcal{V}$-category etc.

4.1 Profunctors

Let us start with some key definitions concerning profunctors and their composition. A more comprehensive study can be found in e.g. [121].

Definition 24 (Profunctor)**.** A profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $P : \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{C} \to \mathcal{V}$.

There are a variety of ways to conceptualise profunctors. Firstly, they generalise bimodules over rings. P is an assignment of an object $P(D, C)$ of $\mathcal{V}$ to each pair of objects C in $\mathcal{C}$ and D in $\mathcal{D}$, which for our intended applications to physics, it may be helpful to think of as an object of "generalised processes" $p : D \rightsquigarrow C$. These generalised processes come equipped with a left action $g \obslash p := P(g, C)(p)$ by arrows $g : D' \to D$ of $\mathcal{D}$ and a right action $p \oslash f := P(D, f)(p)$ by arrows $f : C \to C'$ of $\mathcal{C}$. These actions commute to make the generalised compositions $\oslash$ and $\obslash$ associative: $(g \obslash p) \oslash f = g \obslash (p \oslash f)$.

Secondly, profunctors generalise functors in a similar way to how relations generalise functions between sets - profunctors are like "relations between categories," (note that a relation $A \sim B$ is equivalently a function out of the cartesian product of the sets $B \times A \to \{0, 1\}$). Just as every function is a special type of relation, every functor is a special type of profunctor. The Yoneda lemma guarantees the existence of two embeddings of a category into its presheaves:

Definition 25 (Presheaf)**.** A presheaf on $\mathcal{C}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$. There is a category $\widehat{\mathcal{C}} := [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ whose objects are presheaves on $\mathcal{C}$ and whose hom-objects are the natural transformations. There exist two embeddings

$$\begin{aligned} \text{よ}_{-} &: \mathcal{C} \to \widehat{\mathcal{C}} :: C \mapsto \text{よ}_C := \mathcal{C}(-, C) \\ \text{よ}^{-} &: \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{C}^{\mathrm{op}}} :: C \mapsto \text{よ}^C := \mathcal{C}(C, -) \end{aligned}$$

The presheaves of the form よ_C and よ^C for any C are known as the *representable* and *co*representable presheaves respectively. Profunctors share a close relationship with presheaves, for any $P : \mathcal{D}^{\mathrm{op}} \boxtimes \mathcal{C} \to \mathcal{V}$ can be curried to see that it is equivalently a functor $P : \mathcal{C} \to \widehat{\mathcal{D}}$. This allows us to compose any functor with the Yoneda embeddings to produce examples of profunctors:

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\text{よ}} \widehat{\mathcal{D}} \tag{4.1}$$

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{D}^{\mathrm{op}} \xrightarrow{\text{よ}} \widehat{\mathcal{D}^{\mathrm{op}}} \tag{4.2}$$

Up to currying, the profunctors (4.1) and (4.2) are $\mathcal{D}(-, F{=}) : \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathcal{D}(F-, =) : \mathcal{D} \nrightarrow \mathcal{C}$ respectively. Profunctors of this form are also known as (co)representable.

The composition of profunctors is somewhat more complicated than for functors. Before we can discuss it we need the following technical construction.

Definition 26 (Extranatural Transformation, Coend)**.** Given a profunctor $P : \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C} \to \mathcal{V}$, a family of arrows $w_C : P(C, C) \to W$ in $\mathcal{V}$ is extranatural if the following diagram commutes for all C and C'.

$$\begin{tikzcd}
\mathcal{C}(C,C') \boxtimes P(C',C) \arrow[r, "r"] \arrow[d, "l"'] & P(C',C') \arrow[d, "w_{C'}"] \\
P(C,C) \arrow[r, "w_C"'] & W
\end{tikzcd}$$

Here the arrows l and r are the left and right actions of the hom on P, given by transporting the arrows defining P as an enriched functor along the adjunction due to the closed monoidal structure of $\mathcal{V}$. In the case where $\mathcal{V} = \mathsf{Set}$, the extranaturality condition reduces to the usual "cowedge" diagram.

$$\begin{tikzcd}
P(C',C) \arrow[r, "P(C',f)"] \arrow[d, "P(f,C)"'] & P(C',C') \arrow[d, "w_{C'}"] \\
P(C,C) \arrow[r, "w_C"'] & W
\end{tikzcd}$$

The *coend* of P is a universal extranatural transformation $\mathrm{copr}_C : P(C,C) \to \int^C P(C,C)$: this is the extranatural transformation such that all other extranatural transformations factorise uniquely through it.

Remark. It is worth noting that it not necessarily the case that the coend should exist, though we make the assumption for the remainder of this work that it does. Nevertheless, in the case that $\mathcal{C}$ is small, then $\int^C P(C,C)$ does exist and is given by a certain coequaliser [108]. See also [121] for discussion of existence of coends and iterated coends.

Coends have a series of nice properties which help to justify the use of an integral symbol to notate them. Firstly, they satisfy a Fubini-style law allowing us to commute coends:

$$\int^{C\in\mathcal{C}} \int^{D\in\mathcal{D}} P(C,C,D,D) \cong \int^{(C,D)\in\mathcal{C}\boxtimes\mathcal{D}} P(C,C,D,D) \cong \int^{D\in\mathcal{D}} \int^{C\in\mathcal{C}} P(C,C,D,D)$$

Secondly, the Yoneda lemma implies the following identities, sometimes known as the ninja Yoneda lemma:

$$\int^{C} \mathcal{C}(-,C) \boxtimes F(C) \cong F(-) \qquad\qquad \int^{C} G(C) \boxtimes \mathcal{C}(C,-) \cong G(-) \tag{4.3}$$

for any functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$ and $G : \mathcal{C} \to \mathcal{V}$. So the hom-profunctor behaves "like a Dirac-delta function".

Definition 27 (Composition of Profunctors)**.** Given profunctors $P : \mathcal{C} \nrightarrow \mathcal{D}$ and $Q : \mathcal{D} \nrightarrow \mathcal{E}$, their composite is given by taking the coend

$$(QP)(-,=) = \int^{D} Q(-,D) \boxtimes P(D,=) : \mathcal{C} \nrightarrow \mathcal{E}$$

When $\mathcal{V} = \mathsf{Set}$ this coend can be characterised as the coequaliser:

$$\bigsqcup_{f:D \to D'} Q(-,D) \times P(D',=) \rightrightarrows \bigsqcup_{D} Q(-,D) \times P(D,=) \longrightarrow \int^{D} Q(-,D) \times P(D,=)$$

where the coequalised pair are given by the actions $Q(-,f)$ and $P(f,-)$ on the left and right under the profunctors. Explicitly, the coend is given by a quotient of the coproduct:

$$\int^{D} Q(-,D) \times P(D,=) \cong \bigsqcup_{D} \big(Q(-,D) \times P(D,=)\big)/\sim$$

Given our interpretation of profunctors as containing generalised processes, we want $(QP)(E,C)$ to contain compositions of processes in $Q(E,D)$ and $P(D,C)$, e.g. the pair (q,p) means the composition

$$E \overset{q}{\rightsquigarrow} D \overset{p}{\rightsquigarrow} C$$

Suppose now that we have $E \overset{q}{\rightsquigarrow} D$ and $D' \overset{p}{\rightsquigarrow} C$ together with an arrow $D \xrightarrow{f} D'$ of $\mathcal{D}$. Then we can form the following composites

$$(q \otimes f, p) \simeq (E \overset{q}{\rightsquigarrow} D \xrightarrow{f} D') \overset{p}{\rightsquigarrow} C$$
$$(q, f \otimes p) \simeq E \overset{q}{\rightsquigarrow} (D \xrightarrow{f} D' \overset{p}{\rightsquigarrow} C)$$

We would like these two composites to be equivalent and this is precisely what the quotient of the coend achieves: it is the smallest equivalence relation generated by equivalences of the form $(q \otimes f, p) \sim (q, f \otimes p)$. We will refer to these as the "sliding" relations since it is as though we can slide f from one side to the other (up to changing Q and P).

We will, at times, use a shorthand Einstein-style notation for profunctors writing $P(D,C) = P^{D}_{C}$, with subscripts for covariant variables and superscripts for contravariant ones. The composition of profunctors will be written as $(QP)(E,C) = Q^{E}_{D} P^{D}_{C}$ in the Einstein notation, where instead of the summation convention we have a "coend convention" - repeated indices, once covariant and once contravariant, are to be coended out. In this way, one also sees the similarity between profunctor composition and matrix multiplication.

Categories, profunctors and natural transformations form a monoidal bicategory $\mathcal{V}$-Prof where the monoidal product acts as $\mathcal{C} \boxtimes \mathcal{D}$ on 0-cells and as $(P \boxtimes Q)(C, D, E, F) = P(C, D) \boxtimes Q(E, F)$ on 1-cells. The hom-profunctors play a special role in $\mathcal{V}$-Prof: they are the identity 1-cells by the ninja Yoneda lemma (4.3).

By dualising the definition of a coend one can define ends which we also write with an integral symbol, but with the limit at the bottom $\int_C P(C, C)$. Ends will play a small role in some proofs in this work for they are closely connected to the space of natural transformations between any two functors $F, G : \mathcal{C} \to \mathcal{D}$ by the following identity.

$$\mathrm{Nat}(F, G) \cong \int_C \mathcal{D}(FC, GC) \tag{4.4}$$

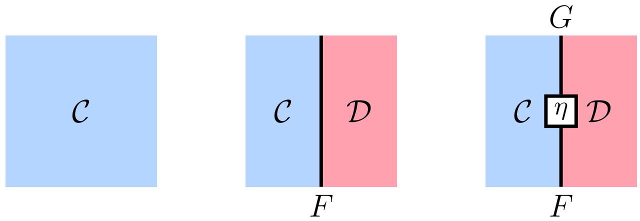
This means ends can be used to talk about the 2-hom, the space of 2-cells, in $\mathcal{V}$-Prof. Since the hom-functor preserves limits, and ends and coends can be shown to be (co)limits, we have the following useful identities.

$$\begin{aligned} \mathcal{D}\left(\int^C P(C, C), D\right) &\cong \int_C \mathcal{D}\left(P(C, C), D\right) \\ \mathcal{D}\left(D, \int_C P(C, C)\right) &\cong \int_C \mathcal{D}\left(D, P(C, C)\right) \end{aligned} \tag{4.5}$$

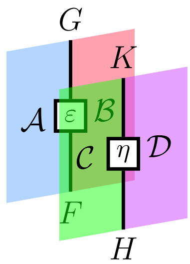
The identities (4.3), (4.4) and (4.5) form what is sometimes known as the *coend calculus*.

4.2 String Diagrams

Monoidal bicategories permit a couple of graphical calculi. The first, and perhaps most familiar, is very similar to the string diagrams for monoidal categories. 1-cells are represented by wires and 2-cells as boxes between the wires, leaving us with the regions which we shade to represent the 0-cells.

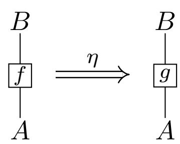


The monoidal structure is drawn by layering the sheets on top of each other:

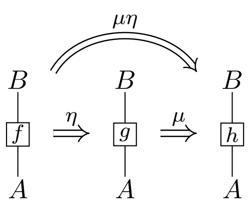


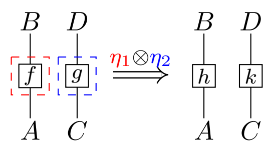
There is good reason for the similarity between this calculus and that for monoidal categories: simply, every monoidal category is a bicategory with a single 0-cell.

There is also another graphical calculus for monoidal bicategories known as *wire diagrams* [17] which is the one we will employ in this work. In this calculus we have string diagrams akin to those of a monoidal category where 0-cells are wires and 1-cells are boxes between wires. The 2-cells are now represented by arrows between these diagrams:

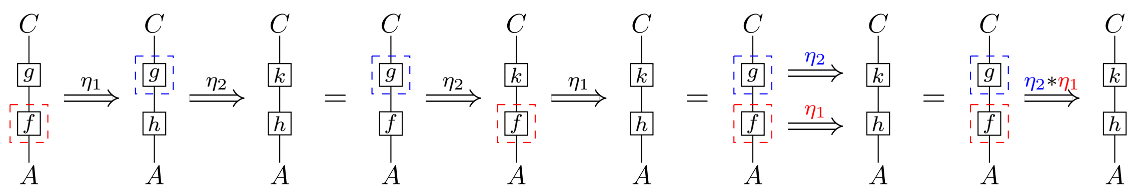


Vertical composition of 2-cells is represented by composition of the arrows between diagrams while the tensor product is formed by tensoring the string diagrams in the domain and codomain:





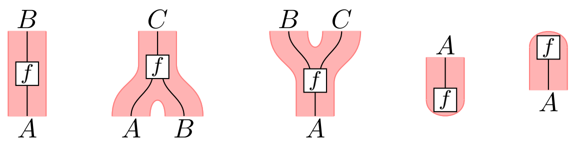
The 2-cells act locally on the string diagrams so that 2-cells on independent parts of diagram interchange giving us the horizontal composition of 2-cells.



As a monoidal bicategory, $\mathcal{V}$-Prof permits the usage of wire diagrams and we will make some use of these in the subsequent sections.

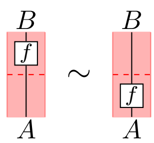
It will often be useful for us to be able to "look inside" the profunctors to work with the generalised processes that they contain. There is yet another diagrammatic calculus known as *internal string diagrams* that will let us do this. Internal string diagrams were first introduced in the Vect-enriched case in [19] and further explored in [101]. The same sort of graphical calculus was described in [136] where the author shows that they form a bicategory of pointed profunctors Prof^*. A pointed category is a pair $(\mathcal{C}, C)$ of a category $\mathcal{C}$ and a specified object C of $\mathcal{C}$. A pointed profunctor $(P, p) : (\mathcal{C}, C) \twoheadrightarrow (\mathcal{D}, D)$ is a profunctor $P : \mathcal{C} \twoheadrightarrow \mathcal{D}$ and a specified element $p \in P(D, C)$. Finally, a pointed natural transformation $\eta : (P, p) \Rightarrow (Q, q)$ is one that preserves the specified elements, $\eta_{DC}(p) = q$. Composition, units and tensor products are defined in the obvious way.

Internal string diagrams can be seen to be wire diagrams for the bicategory Prof^*. We will focus on the particular case of internal string diagrams for representable profunctors (and compositions thereof). These consist of the usual string diagrams for monoidal categories bounded inside cobordisms. Given a monoidal category $\mathcal{C}$, the identity, the Yoneda embeddings of the tensor product and the Yoneda embeddings of the tensor unit are drawn as follows:

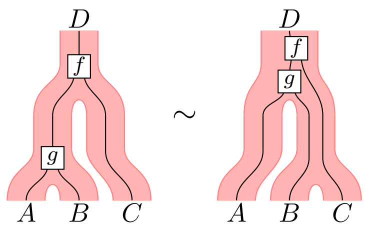


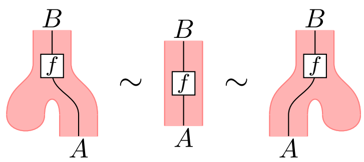
(4.6)

The internal diagrams can be manipulated and composed as usual, but they are constrained by the topology of the cobordisms. Moreover, when we compose these diagrams together, we are allowed to slide morphisms between them as follows:

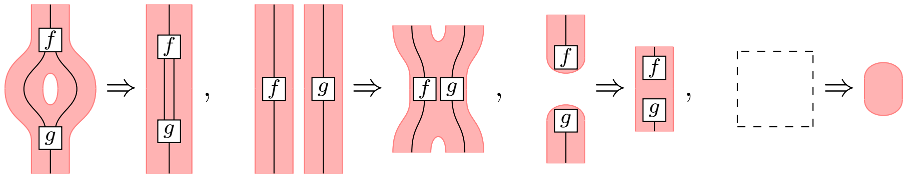


The shapes in (4.6) are associative monoids and comonoids





Since the two Yoneda embeddings of any functor are adjoint in Prof, there exist the following 2-cells which allow us to "pop bubbles":



(4.7)

There is much more to say about pointed profunctors, but we will omit the technical discussion and refer the interested reader to [19] and [136] for a more in-depth discussion.

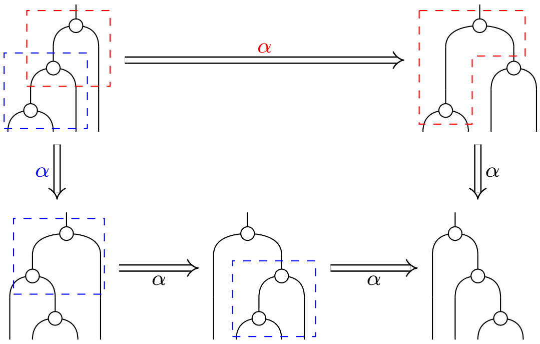
4.3 Presentations

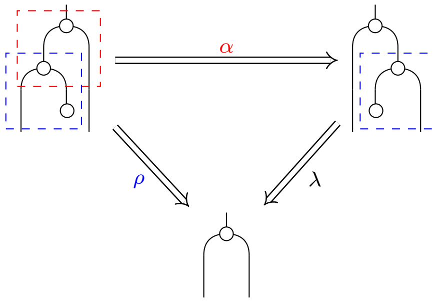
There will be many monoidal-like structures introduced in the following sections. It will therefore be beneficial to have a high-level abstraction that allows us to compare these structures and see in what ways they are or are not alike. Presentations are a useful way of considering some abstract algebraic data in a myriad of different categories - by considering representations of certain freely generated symmetric monoidal bicategories which encapsulate the algebraic data [140, 19]. By doing so we are able to interpret the generating data in alternative bicategories and thus more easily see the connections between the various notions introduced.

A *presentation* $\mathfrak{G}$ of a symmetric monoidal bicategory consists of giving a finite collection of generating 0-cells, 1-cells and 2-cells (where the domains and codomains of the generating 1-cells and 2-cells are built from those of lower dimension), together with a collection of coherence equations between composites of generating 2-cells. There is a symmetric monoidal bicategory $F(\mathfrak{G})$ freely generated by a presentation $\mathfrak{G}$ [140] and a *representation* of $\mathfrak{G}$ in a symmetric monoidal bicategory $\mathcal{B}$ is a strict symmetric monoidal functor $F(\mathfrak{G}) \to \mathcal{B}$.

Important to us will be the monoid presentation $\mathfrak{P}$ as described in [19]. It consists of a single generating 0-cell, generating 1-cells (,), together with invertible generating 2-cells witnessing the unitality and associativity of the monoid.

These must satisfy the triangle and pentagon coherence equations.



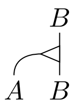


Representations of $\mathfrak{P}$ are known as *pseudomonoids* [105, 69, 152, 71]. Pseudomonoids in Cat are nothing more than the monoidal categories we have been working with all along[1]. In subsequent sections, we will see how by altering Cat for other symmetric monoidal bicategories, we can formulate other "monoidal-like" structures on categories.

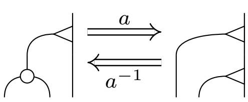
In addition to the monoid presentation we will be interested in the *module presentation*.

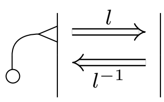
Definition 28 (Module Presentation)**.** The left module presentation $\mathfrak{M}^L$ is a 0-extension of the monoid presentation $\mathfrak{P}$, in addition to the data $(A, \ , \ , \alpha, \lambda, \rho)$ of $\mathfrak{P}$ we have:

- another generating 0-cell B,
- a generating 1-cell:



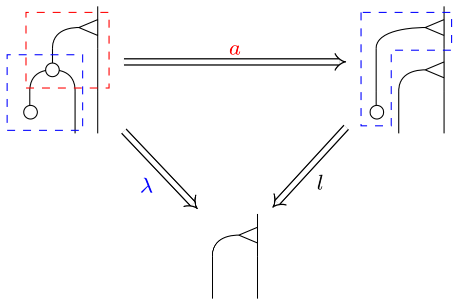
- invertible generating 2-cells:

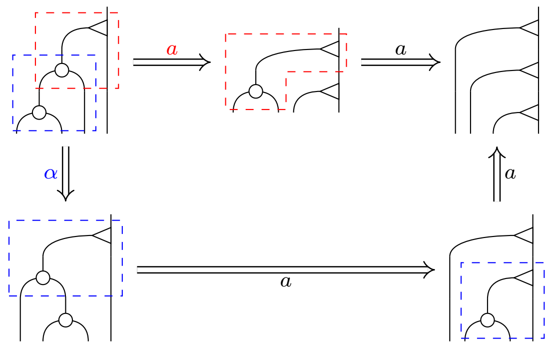




Such that the following coherences hold.

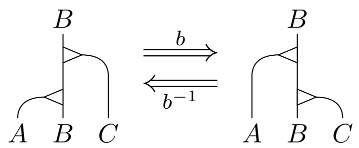
[1]note that in comparison to [19, 101] these are arbitrary monoidal categories as we make no assumption of Cauchy completeness



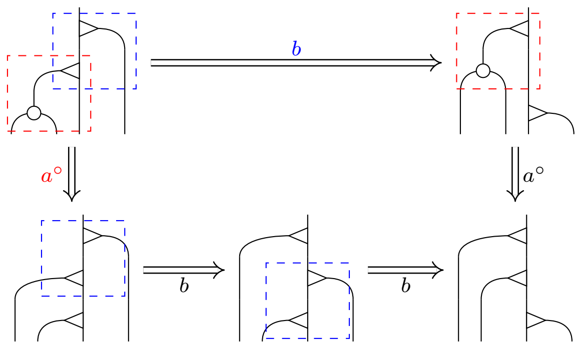


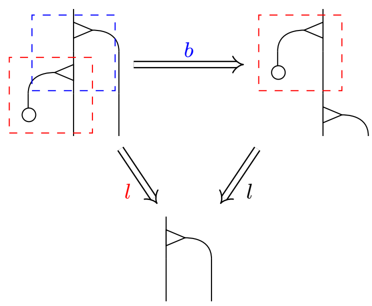
The right module presentation $\mathfrak{M}^R$ is defined analogously.

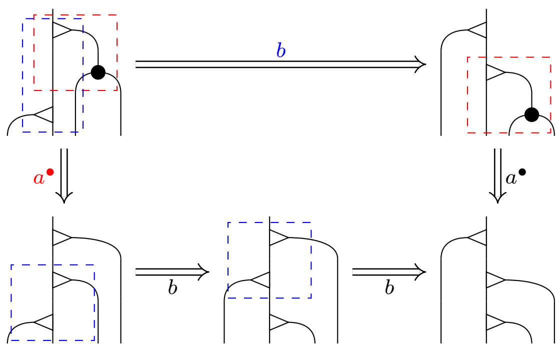
The bimodule presentation $\mathfrak{M}$ consists of two module presentations $\mathfrak{P}$, with different generating 0-cells A and C, together with the data of $\mathfrak{M}^L$ and $\mathfrak{M}^R$ on the same generating 0-cell B. Additionally we ask for the following invertible generating 2-cell.

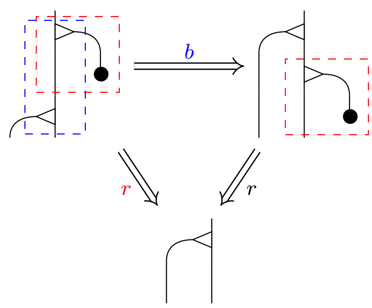


Together with the following coherence diagrams.









The module presentation allows us to abstract some other well-known categorical structures.

Definition 29 (Actegory)**.** A left actegory is a representation of $\mathfrak{M}^L$ in $\mathcal{V}$-Cat. Explicitly this means we have a monoidal category $(\mathcal{C}_0, \otimes, i)$ and a category $\mathcal{C}_1$ equipped with a left action by $\mathcal{C}_0$: a functor $\ltimes : \mathcal{C}_0 \boxtimes \mathcal{C}_1 \to \mathcal{C}_1$ and natural isomorphisms $c \ltimes (c' \ltimes d) \cong (c \otimes c') \ltimes d$ and $d \cong i \ltimes d$ satisfying the coherence diagrams outlined by $\mathfrak{M}$. Right actegories are representations of $\mathfrak{M}^R$ in $\mathcal{V}$-Cat.

Definition 30 (Biactegory)**.** A biactegory is a representation of $\mathfrak{M}$ in $\mathcal{V}$-Cat. This means we have monoidal categories $\mathcal{C}$ and $\mathcal{D}$ and a category $\mathcal{A}$ equipped with a left $\mathcal{C}$-action $\ltimes$ and a right $\mathcal{D}$-action $\rtimes$ together with a natural isomorphism $c \ltimes (a \rtimes d) \cong (c \ltimes a) \rtimes d$ such that the coherence diagrams commute.

Example 8. A monoidal category $\mathcal{C}$ is canonically a left (and right) $\mathcal{C}$-actegory where the action is given by the tensor.

Given a presentation $\mathfrak{G}$ there is an extension $\mathfrak{G}^{\dashv}$, the *adjoint* extension, given by freely adding right adjoints for each generating 1-cell, and additional 2-cells satisfying the snake or "zig-zag" equations witnessing the adjunction [18, 19]. For instance, for a 1-cell $f : A \to B$ we add a new 1-cell $f^* : B \to A$ and 2-cells $\eta : 1_A \to f^*f$ and $\varepsilon : ff^* \to 1_B$ satisfying the snake equations. In an adjoint presention $\mathfrak{G}^{\dashv}$, it is possible to transport the generating 2-cells of $\mathfrak{G}$ along the adjunction to yield 2-cells between the newly added right adjoint 1-cells [18]. For instance, in $\mathcal{P}^{\dashv}$ we can transport α along the adjunction to get a 2-cell $\breve{\alpha}$ witnessing the co-associativity of the comonoid adjoint to .

For us, whenever we have a representation in Cat of a presentation $\mathfrak{G}$, this yields a representation in Prof of the adjoint presentation $\mathfrak{G}^{\dashv}$ precisely because the two Yoneda embeddings are adjoints. For instance a monoidal category is a pseudomonoid in Cat and thus an adjoint pseudomonoid in Prof. In fact, a monoidal category is a representable pseudomonoid in Prof but one should be careful to note that this is a strictly stronger statement than the previous. Without the assumption of Cauchy completeness, adjoint pseudomonoids are *not* necessarily representable [34], though all representable pseudomonoids are adjoint. This discussion can also of course be generalised to the other presentations including the module presentation. See [19] for further discussion of the issue of adjoints in the context of presentations.

4.4 Partially Monoidal Categories

In this section we formally introduce the partially monoidal categories of [55, 115, 85].

Definition 31 (Partial Functor [85])**.** A partial functor $\mathcal{C} \rightharpoonup \mathcal{D}$ is a span of functors $\mathcal{C} \overset{i}{\hookleftarrow} \mathcal{S} \xrightarrow{F} \mathcal{D}$ where i is an opisofibration, embedding S as a subcategory of $\mathcal{C}$ (so i is full, faithful and $\mathcal{S}$ is a replete subcategory of $\mathcal{C}$). Composition of partial functors is by pullback. A morphism of partial functors $(\phi, \eta) : (i, F) \to (j, G)$ is a pair of a functor $\phi : \mathcal{S} \to \mathcal{S}'$ between the apexes of the spans and a natural transformation $\eta : F \implies G\phi$,

$$\begin{tikzcd}
& \mathcal{S} \arrow[ddl, hook', bend right, "i"'] \arrow[d, "\phi"] \arrow[ddr, bend left, "F"] & \\
& \mathcal{S}' \arrow[dl, hook', "j"] \arrow[dr, "G"'] \arrow[r, phantom, "\overset{\eta}{\Longleftarrow}"] & \\
\mathcal{C} & & \mathcal{D}
\end{tikzcd} \tag{4.8}$$

Categories, partial functors and morphisms of partial functors form a monoidal bicategory PCat where the tensor is given pointwise by taking the product of categories and the product of the underlying functors in the spans. Note that full and faithful opisofibrations are closed under composition and stable under pullback.

Definition 32 (Partially Monoidal Category [85])**.** A partially monoidal category $\mathcal{C}$ is a representation of $\mathfrak{P}$ in PCat. In particular, a category $\mathcal{C}$ is partially monoidal if it is equipped with:

- A tensor product partial functor $\boxtimes : \mathcal{C} \times \mathcal{C} \rightharpoonup \mathcal{C}$
- A unit object I

together with associativity and unit natural isomorphisms such that the triangle and pentagon equations hold. A partially monoidal category is *strict* when the coherence isomorphisms are equalities.

The partially monoidal categories introduced here are inspired chiefly by applications in physics - we ask that the left legs of our partial functors are opisofibrations so as to ensure that partially monoidal categories have a tensor that is defined on a subcategory (faithfulness) and that whenever the tensor of objects is defined then the tensor of all morphisms between those objects is defined (fullness). Repleteness simply ensures that if $A \otimes B$ exists and $B \cong B'$ then $A \otimes B'$ also exists. There might be interest in weakening some of these demands on the left legs of partial functors[2], for instance by dropping the fullness requirement, but the physical motivation of such weakened partially monoidal categories is not so clear.

Partially monoidal categories do not seem to have been explored much in the literature. This is perhaps in part because they are somewhat unnatural from a mathematical perspective – most categories arising in mathematics have a canonical tensor product or else they are not monoidal at all. Of course monoidal categories give straightforward examples of partially monoidal categories:

Example 9. Every monoidal category is also trivially a partially monoidal category.

Yet it is perhaps not immediate whether there exist interesting partially monoidal categories which are not monoidal? The lack of examples may also come down to the compositionality versus decompositionality philosophical underpinnings. Many monoidal categories sit firmly in the former - one considers the category of *all* vector spaces, or *all* rings, for instance, we can then build up the larger objects from the smaller atomistic blocks.

The few examples of partially monoidal categories in the literature fit more neatly into the decompositional paradigm.

Example 10. [10, 12, 11] Fix a set N and let $\mathsf{Set}(N)$ be the category where objects are finite subsets of N and morphisms are functions. The $\mathsf{Set}(N)$ has a strict partial monoidal structure given by the union, defined only when the sets are disjoint.

Here, we start with some global system (a set N), which we decompose to produce the objects of the category. As a result there is a restriction placed on the types of objects we can consider - a restriction imposed by the universe in which we live (the set N in this case). Partially monoidal categories seem to be a more natural construction when thinking in this decompositional way, for instance we can generate other partially monoidal categories along similar lines:

Example 11. Let $[0, 1]$ be the poset given by the ordering on the elements of the interval. $[0, 1]$ has a strict partial monoidal structure given by addition $x + y$ when $x + y \leq 1$.

Example 12. Fix a finite dimensional vector space V. Let $\mathsf{Sub}(V)$ be the category whose objects are subspaces of V and morphisms are linear maps. The sum of subspaces $U + W$ gives another subspace of V but it is not a monoidal product on $\mathsf{Sub}(V)$ because

[2]while ensuring that they are still closed under composition and stable under pullback

it is not well-defined on morphisms. For instance suppose $U \cap W \neq \{0\}$. Then if $f : U \to U'$ and $g : W \to W'$ are such that there is a $u \in U \cap W$ with $f(u) \neq g(u)$ then it is not clear how to define the tensor product of these morphisms. $\mathsf{Sub}(V)$ does however have a partially monoidal structure given by the sum of subspaces only defined when the subspaces are disjoint.

4.5 Promonoidal Categories

Before we introduce the formal definition of a promonoidal category let us comment on the intuition we hope to capture.

In a monoidal category $\mathcal{C}$, the tensor product of two objects of $\mathcal{C}$ returns another object in $\mathcal{C}$, that is, it is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Returning to the example of a category of spacetime slices, it is problematic to assign an object of $\mathcal{C}$ to the tensor product whenever the regions of spacetime are timelike separated. The best we could hope for would be a *partial* monoidal structure which is only defined when regions are spacelike separated. Perhaps it might be possible though to assign the tensor of timelike separated regions to be a different sort of object, one that lives outside the category $\mathcal{C}$? What is a sensible choice of such "external" objects and how can we ensure that they work together compatibly such that we might describe the overall structure as something like a tensor product?

We will investigate the usage of promonoidal categories to deal with the aforementioned issues. Rather than assign an object of $\mathcal{C}$ to the tensor product, we assign it a presheaf: a functor $\mathcal{C}^{\text{op}} \to \mathcal{V}$. Recall from Definition 25 that presheaves are nicely-behaved mathematical objects: they form a category $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ where the morphisms are natural transformations between the presheaves, and the Yoneda lemma provides a way of embedding $\mathcal{C}$ fully and faithfully into its presheaves $よ : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathcal{V}]$. The image of this functor consists of the representable presheaves which are of the form $よ_A \cong \mathcal{C}(-, A)$ for some object A of $\mathcal{C}$.

By working with promonoidal categories we are able to assign the tensor a presheaf $(A \otimes B)(-) : \mathcal{C}^{\text{op}} \to \mathcal{V}$, and in doing so, work with otherwise undefinable tensor products. Since $\mathcal{C}$ embeds into its presheaves, we do not lose any ability to still assign some tensor products to essentially be objects of $\mathcal{C}$. Indeed, when the tensor product of objects of $\mathcal{C}$ is a representable presheaf, $(A \otimes B)(-) \cong \mathcal{C}(-, C)$ we can identify $A \otimes B$ with C under the Yoneda embedding. In this way, promonoidal categories are like partially monoidal ones - when the presheaf is representable we essentially have an object of $\mathcal{C}$ again - but rather than the tensor being undefined elsewhere we can still assign otherwise "untensorable" objects a non-representable presheaf.

We are now in a position to define promonoidal categories.

Definition 33 (Promonoidal Category [68, 65]). A promonoidal category is a representation of $\mathfrak{P}$ in $\mathcal{V}$-Prof. In particular, a category $\mathcal{C}$ is promonoidal if it is equipped with

- a tensor product profunctor $\otimes : \mathcal{C} \boxtimes \mathcal{C} \nrightarrow \mathcal{C}$
- a unit profunctor $I : 1 \nrightarrow \mathcal{C}$, i.e. a presheaf $I : \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$

together with natural isomorphisms $\otimes(\otimes \boxtimes 1) \overset{\alpha}{\cong} \otimes(1 \boxtimes \otimes)$ and $\otimes(\otimes \boxtimes I) \overset{\rho}{\cong} 1 \overset{\lambda}{\cong} \otimes(I \boxtimes \otimes)$ subject to the triangle and pentagon coherence conditions similar to a monoidal category. A promonoidal category is *strict* when the coherence isomorphisms are identities. A promonoidal category is *symmetric* when there is a natural isomorphism $\sigma_{ABC} : \otimes^A_{BC} \to \otimes^A_{CB}$ satisfying the hexagon and triangle equations.

We will mostly think of the tensor product profunctor $\otimes : \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{V}$ in its curried form as a functor into presheaves, $\otimes : \mathcal{C} \boxtimes \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ and in an abuse of notation we freely switch between using $\otimes$ for the tensor product in its three different forms (as a profunctor, a functor into $\mathcal{V}$ and a functor into presheaves) so long as it is clear which we mean.

There are many similarities between the definitions of promonoidal and monoidal categories. One can think of promonoidal categories as what we get when we "upgrade" the functors of a monoidal category to profunctors. This really is an upgrade since every functor induces two profunctors by taking its covariant or contravariant Yoneda embeddings. Furthermore, by the following result we can consider promonoidal categories as strictly more general than monoidal ones.

Theorem 4 ([68, 65]). *All monoidal categories $(\mathcal{C}, \otimes, I)$ are promonoidal categories where we define the tensor profunctor as $(A \tilde{\otimes} B)(-) := \mathcal{C}(-, A \otimes B)$ and the unit profunctor as $\tilde{I}(-) := \mathcal{C}(-, I)$. Conversely, a promonoidal category whose tensor and unit are everywhere representable is a monoidal category.*

The connection between promonoidal and monoidal categories goes even deeper due to the following theorem:

Theorem 5 ([68, 65]). *There is an equivalence between promonoidal structures on $\mathcal{C}$ and closed monoidal structures on the presheaf category $[\mathcal{C}^{op}, \mathcal{V}]$.*

The induced closed monoidal structure on $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is known as Day convolution and is given by

$$(F * G)(-) := \int^{CD} \otimes(-, C, D) \boxtimes F(C) \boxtimes G(D)$$

with unit object given by the presheaf $I(-)$.

Promonoidal categories can also be seen to be special types of co-multicategories. A co-multicategory $\mathfrak{C}$ consists of a collection of objects and for any objects A and $B_0, \ldots, B_n$, a collection of morphisms $\mathfrak{C}(A; B_0, \ldots, B_n)$. Given $f : A \to B_0, \ldots, B_n$ and $g : B_i \to C_0, \ldots, C_m$ there is a composite

$$g \circ_{B_i} f : A \to B_0, \ldots, B_{i-1}, C_0, \ldots, C_m, B_{i+1}, \ldots, B_n$$

Each object is also equipped with an identity morphism $1_A : A \to A$. The compositions should be associative, unital and satisfy certain interchange laws, see [119] for more detailed discussion of these issues.

Monoidal categories are examples of co-multicategories where the morphisms of type $f : A \to B_0, \ldots, B_n$ are given by those $f : A \to B_0 \otimes \cdots \otimes B_n$. Promonoidal categories are also examples of co-multicategories where the collection of morphisms $\mathfrak{C}(A, B)$ is given by $\mathcal{C}(A, B)$ and $\mathfrak{C}(A; B_0, B_1)$ is given by $\otimes(A; B_0, B_1)$. The higher-arity collections $\mathfrak{C}(A; B_0, \ldots, B_n)$ are defined inductively, for instance $\mathfrak{C}(A; B_0, B_1, B_2)$ is given by $\int^X \otimes(A, B_0, X) \boxtimes \otimes(X, B_1, B_2)$. Note that associativity of the promonoidal structure means it does not matter which of the two isomorphic objects we pick here.

So can think of a promonoidal category in two different ways, either as a (closed) monoidal structure on presheaf categories or as a special co-multicategory. In the former mindset we can think of the tensors $(A \otimes B)(-)$ as assigning external "virtual" objects to tensor products of objects of our category. In the latter mindset we can think of the promonoidal structure as an assignment of the multi-arity morphisms over our original category, so that for instance $\otimes(A; B_0, B_1)$ defines the collection of morphisms $A \to B_0, B_1$. We will revisit these ideas again in Chapter 5 and 7.

4.5.1 Partially Monoidal Categories as Promonoidal Categories

In this Section we will take a slight detour to study some connections between partially monoidal and promonoidal categories. While this Section is not necessary to understand the rest of this work, we hope that it will further elucidate some of the reasons for considering promonoidal categories as a replacement for partially monoidal ones, and thus further explain some connections between the approach we take in Chapter 5 based on [97], and the works [55, 85, 91].

In general, partially monoidal (as defined here and in [85]) and promonoidal categories are *not* the same thing, though there is a special type of the former that can be turned into the latter. There exists a class of partial functors where the left leg is not only an opisofibration but a proper discrete opfibration. This makes the left leg a *cosieve* which coincides with the definition of partial functor given by [25]. Demanding that the left leg is a cosieve ensures that the subcategory on which the tensor is defined is closed under post-composition with morphisms of $\mathcal{C} \times \mathcal{C}$. This captures the following physical intuition: if $X \otimes Y$ exists and there is a morphism $X \to X'$ then $X' \otimes Y$ exists too. Thus we maintain the intuition that if one applies a local map to X then the tensor product should still exist afterwards. From a mathematical perspective, when the left leg of the tensor product partial functor is a cosieve, the partially monoidal category is equivalent to a promonoidal one. Indeed, Bénabou notes that there is an 1-1 correspondence (up to isomorphism) between partial functors with left leg a cosieve and profunctors which factorise through the representable and empty presheaves [25]. In this light the following proposition is not surprising but there is a little effort required in checking that everything works out:

Proposition 28. *A partially monoidal category $(\mathcal{C}, \boxtimes, J)$ whose left leg of the tensor product partial functor is a cosieve is a promonoidal category with representable unit and a tensor $\otimes(-, b, c)$ which is either representable or empty for each $(b, c) \in \mathcal{C} \times \mathcal{C}$.*

Proof. In a slight abuse of notation write $\mathcal{C} \times \mathcal{C} \overset{i}{\leftarrow} \mathcal{S} \overset{\boxtimes}{\to} \mathcal{C}$ for the underlying span of the partial functor $\boxtimes : \mathcal{C} \times \mathcal{C} \rightharpoonup \mathcal{C}$, and note that $J : 1 \rightharpoonup \mathcal{C}$ is simply a normal functor $J : 1 \to \mathcal{C}$, in other words an object J of $\mathcal{C}$. Just like for monoidal categories we can define a promonoidal structure on $\mathcal{C}$ by taking $(X \otimes Y)(Z) := \mathcal{C}(Z, X \boxtimes Y)$ whenever $(X, Y) \in \mathcal{S}$ and $(X \otimes Y)(Z) := \varnothing$ otherwise. The unit is the representable presheaf at J, よ_J.

The associativity isomorphism of a partially monoidal category induces the following arrows:

$$\begin{tikzcd}
 & (\mathcal{S} \times \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}} \mathcal{S} \arrow[ddl, bend right, "(i\times 1)\pi_0"'] \arrow[d, "\phi"] \arrow[ddr, bend left, "\boxtimes\pi_1"] & \\
 & (\mathcal{C} \times \mathcal{S}) \times_{\mathcal{C}\times\mathcal{C}} \mathcal{S} \arrow[dl, "(1\times i)\pi_0"'] \arrow[dr, "\boxtimes\pi_1"] \arrow[r, phantom, "\overset{\alpha}{\Longleftarrow}"] & \\
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & & \mathcal{C}
\end{tikzcd} \tag{4.9}$$

where π_0 and π_1 are the canonical projections from the pullback and α is a natural isomorphism.

Given a cospan of functors $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$, the pullback $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is the category consisting of pairs of objects (c, d) with $Fc = Gd$ and pairs of morphisms (f, g) with $Ff = Gg$. We can think of $(\mathcal{S} \times \mathcal{C}) \times_{\mathcal{C}\times\mathcal{C}} \mathcal{S}$ as the category with objects $(\,((a, b), c), (a\boxtimes b, c)\,)$ where $(a, b) \in \mathcal{S}$ and $c \in \mathcal{C}$ with $(a\boxtimes b, c) \in \mathcal{S}$, while $(\mathcal{C}\times\mathcal{S})\times_{\mathcal{C}\times\mathcal{C}}\mathcal{S}$ has objects $(\,(a, (b, c)), (a, b \boxtimes c)\,)$ where $(b, c) \in \mathcal{S}$ and $a \in \mathcal{C}$ with $(a, b \boxtimes c) \in \mathcal{S}$. The left triangle of (4.9) ensures that ϕ must act to send $(\,((a, b), c), (a \boxtimes b, c)\,) \mapsto (\,(a, (b, c)), (a, b \boxtimes c)\,)$. The right triangle of (4.9) then implies that the components of α have type $\alpha_{a,b,c} : (a \boxtimes b) \boxtimes c \to a \boxtimes (b \boxtimes c)$. This induces the necessary isomorphism $\otimes^a_{xd}\otimes^x_{bc} \to \otimes^a_{bx}\otimes^x_{cd}$ and checking the pentagon coherence equation now follows the same standard proof as Theorem 4.

The right unit isomorphism induces the following arrows:

$$\begin{tikzcd}
 & (\mathcal{C} \times 1) \times_{\mathcal{C}\times\mathcal{C}} \mathcal{S} \arrow[dl, bend right, "\pi_0"'] \arrow[d, "\psi"] \arrow[ddr, bend left, "\boxtimes\pi_1"] & \\
\mathcal{C} \times 1 \arrow[d, "\sim"'] & \mathcal{C} \arrow[dl, "1"] \arrow[dr, "1"'] \arrow[r, phantom, "\overset{\rho}{\Longleftarrow}"] & \\
\mathcal{C} & & \mathcal{C}
\end{tikzcd}$$

the components of ρ have type $\rho_a : a \boxtimes J \to a$ as expected. A similar diagram is induced by λ and in turn one sees that this has components $\lambda_a : J \boxtimes a \to a$. Checking the triangle coherence equation follows like Theorem 4.

Now suppose we begin with a promonoidal category $\mathcal{C}$ where the unit is representable $J(-) \cong \mathcal{C}(-, I)$ and for each $(b, c) \in \mathcal{C} \times \mathcal{C}$, either $\otimes(-, b, c) \cong \mathcal{C}(-, x_{bc})$ is representable, or $\otimes(-, b, c) \cong \varnothing(-)$ is empty. Define a full subcategory $\mathcal{S}$ of $\mathcal{C} \times \mathcal{C}$ spanned by objects (b, c) where $\otimes(-, b, c)$ is representable. Suppose for a contradiction that $(b, c) \in \mathcal{S}$ and there exists a $(f, g) : (b, c) \to (b', c')$ in $\mathcal{C} \times \mathcal{C}$ but with $(b', c') \notin \mathcal{S}$. Then we would have a natural transformation $\mathcal{C}(-, x_{bc}) \to \varnothing(-)$, a contradiction. Thus (f, g) cannot exist and as a result the canonical inclusion functor $\mathcal{S} \hookrightarrow \mathcal{C} \times \mathcal{C}$ is a discrete opfibration. $\square$

There are many examples of partially monoidal categories which are not equivalent to promonoidal ones and vice-versa. For instance, we require that the unit presheaf $J(-)$ of a promonoidal category is representable to have any hope that it is a partially monoidal category. Furthermore, given a general partially monoidal category $\mathcal{C}$ we do not have enough data to define a profunctor $\mathcal{C}\times\mathcal{C} \nrightarrow \mathcal{C}$ (e.g. by taking the representable presheaves at the defined points of the partial tensor). Indeed, a promonoidal category still has a total tensor, just into the presheaf category,

It is possible though to derive partially monoidal structures from a promonoidal one with representable unit presheaf $J(-)$, by pulling back the promonoidal tensor along the Yoneda embedding whenever it is representable. There is of course a canonical "maximal" such partially monoidal structure induced by defining it everywhere it is possible to do so, i.e. everywhere the promonoidal tensor is representable.

One may wonder if there are any further connections between partial functors and profunctors - is there a category that unites them? This would allow us to place the two on equal footing and compare arbitrary partially monoidal and promonoidal categories. The key to this unification is the following result:

Theorem 6 ([25, 122])**.** *There is an equivalence of categories between profunctors $\mathcal{C} \nrightarrow \mathcal{D}$ and two-sided discrete fibrations* $\mathsf{DFib}(\mathcal{C}, \mathcal{D})$.

A two-sided discrete fibration is a span of functors $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$ where:

- each $F(e) \to c'$ in $\mathcal{C}$ has unique lift $f : e \to e'$ in $\mathcal{E}$ such that $G(f) = 1_{G(e)}$,
- each $d \to G(e)$ in $\mathcal{D}$ has unique lift $g : e' \to e$ in $\mathcal{E}$ such that $F(g) = 1_{F(e)}$,
- for each $f : e \to e'$ in $\mathcal{E}$, the codomain of the lift of Ff equals the domain of the lift of Gf, and their composite is f.

The two-sided discrete fibration corresponding to a profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ is given by the projections out of the category $\mathsf{Sec}(P)$ of sections of the collage of P. The objects of $\mathsf{Sec}(P)$ are the elements of the sets $P(d, c)$ for all c and d. A morphism $x \in P(d, c) \to x' \in P(d', c')$ is given by a pair of arrows f and g such that $P(g, 1)(x') = P(1, f)(x)$.

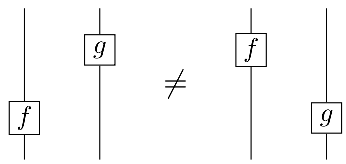
Consequently, each profunctor has a canonical span and by working in the category of spans of functors one can study the partial functors and profunctors side-by-side. For instance, suppose $(\mathcal{C}, \otimes, J)$ is a promonoidal category with $J(-) \cong \mathcal{C}(-, I)$. There is a partial monoidal structure $(\boxtimes, I)$ on $\mathcal{C}$ given by pulling back $\otimes$ along the Yoneda embedding whenever it is representable - that is, whenever $\otimes(-, b, c) \cong \mathcal{C}(-, x_{bc})$ for some objects b and c, we define $b \boxtimes c := x_{bc}$. Write $\overline{\mathcal{C} \times \mathcal{C}}$ for the subcategory of $\mathcal{C} \times \mathcal{C}$ where the promonoidal tensor is representable. Then there is a 2-cell in $\mathsf{Span}(\mathsf{Cat})$ capturing the extension of the partially monoidal structure on $\mathcal{C}$ to the promonoidal structure:

$$\begin{tikzcd}
& \overline{\mathcal{C}\times\mathcal{C}} \arrow[d, "\phi"] \arrow[ddl, bend right, "i"'] \arrow[ddr, bend left, "\boxtimes"] & \\
& \mathsf{Sec}(\otimes) \arrow[dl, "p_1"] \arrow[dr, "p_0"'] & \\
\mathcal{C}\times\mathcal{C} & & \mathcal{C}
\end{tikzcd}$$

where ϕ sends (b, c) to $1_{b\boxtimes c, b\boxtimes c} \in \otimes(b \boxtimes c, b, c)$ and (g, f) to $(g \boxtimes f, g, f)$.

4.6 Premonoidal Categories

Alongside promonoidal categories, the other monoidal-like structures we will be interested in are premonoidal categories. Premonoidal categories are a weakening of monoidal categories to allow for situations when one can join objects together but each half of the tensor is only individually functorial, that is, while it is the case that $(g' \otimes 1)(g \otimes 1) = (g'g \otimes 1)$ and $(1 \otimes f')(1 \otimes f) = (1 \otimes f'f)$ we no longer have the interchange law and thus have the following inequality:



These categories were originally introduced to model computational semantics with side-effects [132] but we expect categories of causal curves to have similar structure. If f and g act on slices which are timelike separated or have a non-trivial intersection, then their causal ordering can be vitally important; f could change the state space in ways that later influence g or vice-versa. These “hidden” influences between maps can be seen to be somewhat akin to the side-effects in the computational semantics for which premonoidal categories were originally intended.

Let us now take formally define enriched premonoidal categories. We take some space to spell this out as there are some technicalities in the enriched case which do not appear to have been explicitly discussed elsewhere.

Definition 34 (Binoidal Category)**.** A category $\mathcal{C}$ is binoidal when, for each object X, it is equipped with a pair of functors $X \ltimes - : \mathcal{C} \to \mathcal{C}$ and $- \rtimes X : \mathcal{C} \to \mathcal{C}$ such that for all X and Y, $X \ltimes Y = X \rtimes Y$.

There is no compatibility condition between the left and right parts of the tensor on morphisms, so in general it will be the case that $(f \ltimes Y')(X \rtimes g) \neq (Y \rtimes g)(f \ltimes X')$

for $f : X \to Y$, $g : X' \to Y'$. In the case $\mathcal{V} = \mathsf{Set}$ the morphisms which interchange with all others are known as *central*.

Definition 35 (Central Morphism [132])**.** A morphism $f : X \to Y$ is central if and only if for all $g : X' \to Y'$, the following two diagrams commute:

$$
\begin{tikzcd}
X \otimes X' \arrow[r, "X \rtimes g"] \arrow[d, "f \ltimes X'"'] & X \otimes Y' \arrow[d, "f \ltimes Y'"] \\
Y \otimes X' \arrow[r, "Y \rtimes g"'] & Y \otimes Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \otimes X \arrow[r, "X' \rtimes f"] \arrow[d, "g \ltimes X"'] & X' \otimes Y \arrow[d, "g \ltimes Y"] \\
Y' \otimes X \arrow[r, "Y' \rtimes f"'] & Y' \otimes Y
\end{tikzcd}
$$

In the enriched case there are no "morphisms" so a more careful definition is required.

Definition 36 (Centre Piece)**.** Let $\mathcal{C}$ be a binoidal category. A centre piece at objects (A, B) is an object $U(A, B)$ in $\mathcal{V}$, endowed with an arrow $\iota : U(A, B) \to \mathcal{C}(A, B)$, such that for any objects (C, D) the following diagrams commute.

$$
\begin{tikzcd}
U(A,B) \boxtimes \mathcal{C}(C,D) \arrow[r, "\iota \boxtimes 1"] \arrow[d, "\iota \boxtimes 1"'] & \mathcal{C}(A,B) \boxtimes \mathcal{C}(C,D) \arrow[r, "(- \rtimes C) \boxtimes (B \ltimes -)"] & \mathcal{C}(AC,BC) \boxtimes \mathcal{C}(BC,BD) \arrow[d, "\circ_\sigma"] \\
\mathcal{C}(A,B) \boxtimes \mathcal{C}(C,D) \arrow[r, "(- \rtimes D) \boxtimes (A \ltimes -)"'] & \mathcal{C}(AD,BD) \boxtimes \mathcal{C}(AC,AD) \arrow[r, "\circ"'] & \mathcal{C}(AC,BD)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C}(C,D) \boxtimes U(A,B) \arrow[r, "1 \boxtimes \iota"] \arrow[d, "1 \boxtimes \iota"'] & \mathcal{C}(C,D) \boxtimes \mathcal{C}(A,B) \arrow[r, "(- \rtimes A) \boxtimes (D \ltimes -)"] & \mathcal{C}(CA,DA) \boxtimes \mathcal{C}(DA,DB) \arrow[d, "\circ_\sigma"] \\
\mathcal{C}(C,D) \boxtimes \mathcal{C}(A,B) \arrow[r, "(- \rtimes B) \boxtimes (C \ltimes -)"'] & \mathcal{C}(CB,DB) \boxtimes \mathcal{C}(CA,CB) \arrow[r, "\circ"'] & \mathcal{C}(CA,DB)
\end{tikzcd}
$$

Note that in the previous diagrams the symbols $\ltimes$ and $\rtimes$ have been suppressed for space, and e.g. AB should be understood to mean $A \ltimes B = A \rtimes B$.

A morphism of centre pieces at (A, B) is an arrow $u : V(A, B) \to U(A, B)$ in $\mathcal{V}$ such that $\iota_U \circ u = \iota_V$. Centre pieces and morphisms of centre pieces form a category $CP(A, B)$.

Definition 37 (Centre)**.** Let $\mathcal{C}$ be binoidal. The centre of $\mathcal{C}$ at objects (A, B) is the universal centre piece, $\iota : Z\mathcal{C}(A, B) \to \mathcal{C}(A, B)$, such that all other centre pieces factorise uniquely through it.

Remark. When $\mathcal{V} = \mathsf{Set}$, universal centre pieces always exist. We leave it to future work to investigate precisely when they exist for general $\mathcal{V}$. In all the following propositions we assume the existence of universal centre pieces.

Universal centre pieces assemble to give a category $Z\mathcal{C}$ with the same objects as $\mathcal{C}$ and with hom-objects given by the universal centre pieces. Composition and identities are inherited from $\mathcal{C}$ because (4.10) and j_A are centre pieces and thus factor via the centre.

Proposition 29. *The arrow* (4.10), *representing composition inherited from* $\mathcal{C}$, *is a centre piece at* (A, C).

$$Z\mathcal{C}(B,C) \boxtimes Z\mathcal{C}(A,B) \xrightarrow{\iota \boxtimes \iota} \mathcal{C}(B,C) \boxtimes \mathcal{C}(A,B) \xrightarrow{\circ} \mathcal{C}(A,C) \tag{4.10}$$

Furthermore, for each object A, *the arrow* $j_A : I_{\mathcal{V}} \to \mathcal{C}(A,A)$, *representing the identities, is a centre piece at* (A, A). *As a result,* $Z\mathcal{C}$ *is a category.*

Proof. Ignoring associativity isomorphisms, the following diagram commutes for any

X and Y, showing that the arrow (4.10) is a centre piece at (A, C).

$$
\begin{tikzcd}
& Z\mathcal{C}(B,C)\boxtimes Z\mathcal{C}(A,B)\boxtimes\mathcal{C}(X,Y) \arrow[rr, "\sigma"] \arrow[d, "\iota\boxtimes\iota\boxtimes 1"'] \arrow[dr, "1\boxtimes\sigma"] & & \mathcal{C}(X,Y)\boxtimes Z\mathcal{C}(B,C)\boxtimes Z\mathcal{C}(A,B) \arrow[d, "1\boxtimes\iota\boxtimes\iota"] & \\
& \mathcal{C}(B,C)\boxtimes\mathcal{C}(A,B)\boxtimes\mathcal{C}(X,Y) \arrow[ddl, "\circ\boxtimes 1"'] \arrow[dd, "(-\times Y)\boxtimes(-\times Y)\boxtimes(A\times -)"] & Z\mathcal{C}(B,C)\boxtimes\mathcal{C}(X,Y)\boxtimes Z\mathcal{C}(A,B) \arrow[ur, "\sigma\boxtimes 1"'] \arrow[d, "\iota\boxtimes 1\boxtimes\iota"] & \mathcal{C}(X,Y)\boxtimes\mathcal{C}(B,C)\boxtimes\mathcal{C}(A,B) \arrow[dd, "(C\times -)\boxtimes(-\times X)\boxtimes(-\times X)"] \arrow[ddr, "1\boxtimes\circ"] & \\
& & \mathcal{C}(B,C)\boxtimes\mathcal{C}(X,Y)\boxtimes\mathcal{C}(A,B) \arrow[d, "(-\times Y)\boxtimes(B\times -)\boxtimes(-\times X)"] & & \\
\mathcal{C}(A,C)\boxtimes\mathcal{C}(X,Y) \arrow[d, "(-\times Y)\boxtimes(A\times -)"] & \mathcal{C}(BY,CY)\boxtimes\mathcal{C}(AY,BY)\boxtimes\mathcal{C}(AX,AY) \arrow[dl, "\circ\boxtimes 1"] \arrow[d, "1\boxtimes\circ"] & \mathcal{C}(BY,CY)\boxtimes\mathcal{C}(BX,BY)\boxtimes\mathcal{C}(AX,BX) \arrow[dl, "1\boxtimes\circ"] \arrow[dr, "\circ\boxtimes 1"] & \mathcal{C}(CX,CY)\boxtimes\mathcal{C}(BX,CX)\boxtimes\mathcal{C}(AX,BX) \arrow[d, "\circ\boxtimes 1"] \arrow[dr, "1\boxtimes\circ"] & \mathcal{C}(X,Y)\boxtimes\mathcal{C}(A,C) \arrow[d, "(C\times -)\boxtimes(-\times X)"] \\
\mathcal{C}(AY,CY)\boxtimes\mathcal{C}(AX,AY) \arrow[d, "\circ"] & \mathcal{C}(BY,CY)\boxtimes\mathcal{C}(AX,BY) \arrow[dl, "\circ"] & & \mathcal{C}(BX,CY)\boxtimes\mathcal{C}(AX,BX) \arrow[dr, "\circ"] & \mathcal{C}(CX,CY)\boxtimes\mathcal{C}(AX,CX) \arrow[d, "\circ"] \\
\mathcal{C}(AX,CY) \arrow[rrrr, "1"] & & & & \mathcal{C}(AX,CY)
\end{tikzcd}
$$

An analogous diagram also commutes for the other interchange law. As a result

$\circ(\iota \boxtimes \iota)$ factorises uniquely via the universal centre piece. This gives us a composition operation for $Z\mathcal{C}$, say $\bullet$, such that $\circ(\iota \boxtimes \iota) = \iota\bullet$. Composition in $Z\mathcal{C}$ is associative because the following diagram commutes.

$$\begin{tikzcd}
(Z\mathcal{C}(C,D)\boxtimes Z\mathcal{C}(B,C))\boxtimes Z\mathcal{C}(A,B) \arrow[rrrr, "\alpha"] \arrow[dr, "(\iota\boxtimes\iota)\boxtimes\iota"] \arrow[ddd, "\bullet\boxtimes 1"'] & & & & Z\mathcal{C}(C,D)\boxtimes(Z\mathcal{C}(B,C)\boxtimes Z\mathcal{C}(A,B)) \arrow[dl, "\iota\boxtimes(\iota\boxtimes\iota)"'] \arrow[ddd, "1\boxtimes\bullet"] \\
& (\mathcal{C}(C,D)\boxtimes\mathcal{C}(B,C))\boxtimes\mathcal{C}(A,B) \arrow[rr, "\alpha"] \arrow[d, "\circ\boxtimes 1"'] & & \mathcal{C}(C,D)\boxtimes(\mathcal{C}(B,C)\boxtimes\mathcal{C}(A,B)) \arrow[d, "1\boxtimes\circ"] & \\
& \mathcal{C}(B,C)\boxtimes\mathcal{C}(A,B) \arrow[r, "\circ"] & \mathcal{C}(A,D) & \mathcal{C}(C,D)\boxtimes\mathcal{C}(A,C) \arrow[l, "\circ"'] & \\
Z\mathcal{C}(B,C)\boxtimes Z\mathcal{C}(A,B) \arrow[ur, "\iota\boxtimes\iota"'] \arrow[rr, "\bullet"'] & & Z\mathcal{C}(A,D) \arrow[u, "\iota"] & & Z\mathcal{C}(C,D)\boxtimes Z\mathcal{C}(A,C) \arrow[ul, "\iota\boxtimes\iota"] \arrow[ll, "\bullet"]
\end{tikzcd}$$

The following diagram commutes for all C, D, showing that j_A is a centre piece at (A, A).

$$\begin{tikzcd}
I\boxtimes\mathcal{C}(C,D) \arrow[rr, "j_A\boxtimes 1"] \arrow[d, "j_A\boxtimes 1"'] \arrow[dr, "1\boxtimes(A\ltimes -)"] & & \mathcal{C}(A,A)\boxtimes\mathcal{C}(C,D) \arrow[r, "(-\rtimes C)\boxtimes(A\ltimes -)"] & \mathcal{C}(AC,AC)\boxtimes\mathcal{C}(AC,AD) \arrow[d, "\sigma"] \\
\mathcal{C}(A,A)\boxtimes\mathcal{C}(C,D) \arrow[d, "(-\rtimes D)\boxtimes(A\ltimes -)"'] & I\boxtimes\mathcal{C}(AC,AD) \arrow[r, "\sigma"] \arrow[urr, "j_{A\rtimes C}\boxtimes 1"] \arrow[dl, "j_{A\rtimes D}\boxtimes 1"] \arrow[drr, "\lambda"'] & \mathcal{C}(AC,AD)\boxtimes I \arrow[r, "1\boxtimes j_{A\rtimes C}"] \arrow[dr, "\rho"] & \mathcal{C}(AC,AD)\boxtimes\mathcal{C}(AC,AC) \arrow[d, "\circ"] \\
\mathcal{C}(AD,AD)\boxtimes\mathcal{C}(AC,AD) \arrow[rrr, "\circ"'] & & & \mathcal{C}(AC,AD)
\end{tikzcd}$$

A similar diagram commutes for the other interchange law. As a result j_A factorises uniquely via the universal centre piece as $j_A = \iota j'_A$. Finally, note that the following diagram commutes showing the left unit law holds for composition in $Z\mathcal{C}$.

$$
\begin{tikzcd}
ZC(B,B) \boxtimes ZC(A,B) \arrow[rrr, "\bullet"] \arrow[dr, "\iota\boxtimes\iota"'] & & & ZC(A,B) \arrow[dl, "\iota"'] \\
& \mathcal{C}(B,B) \boxtimes \mathcal{C}(A,B) \arrow[r, "\circ"] & \mathcal{C}(A,B) & \\
& I \boxtimes \mathcal{C}(A,B) \arrow[r, "\lambda"] \arrow[u, "j_b\boxtimes 1"] & \mathcal{C}(A,B) \arrow[u, "1"'] & \\
I \boxtimes ZC(A,B) \arrow[rrr, "\lambda"'] \arrow[uuu, "j'_b\boxtimes 1"] \arrow[ur, "1\boxtimes\iota"] & & & ZC(A,B) \arrow[ul, "\iota"] \arrow[uuu, "1"']
\end{tikzcd}
$$

The right unit law is very similar. $\square$

Finally we note that the ι assemble to give an identity on objects functor $Z\mathcal{C} \to \mathcal{C}$.

Definition 38 (Central Natural Transformation)**.** Let $\mathcal{C}$ be binoidal and $F, G : \mathcal{D} \to \mathcal{C}$ be two functors. A natural transformation $\eta : F \to G$ is central when the components are central, so that we have a family of morphisms $\eta_A : I_{\mathcal{V}} \to Z\mathcal{C}(FA, GA)$ of $\mathcal{V}$ satisfying the naturality diagrams.

Binoidal categories give us the necessary machinery to define premonoidal categories.

Definition 39 (Premonoidal Category)**.** A premonoidal category, $\mathcal{C}$, is a binoidal category endowed with an object I and central natural isomorphisms, $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ and $A \otimes I \cong A \cong I \otimes A$, such that the triangle and pentagon equations hold. A premonoidal category is *strict* when the coherence isomorphisms are identities.

In the case that $\mathcal{V} = \mathsf{Set}$, it is possible to combine the left and right tensor functors $X \rtimes -$ and $- \ltimes Y$ into a single functor $\mathcal{C} \mathbin{\square} \mathcal{C} \to \mathcal{C}$ from the *funny tensor product* [80]. A concise definition of the funny tensor is as follows.

Definition 40 (Funny tensor product [157])**.** The funny tensor product $\mathcal{C} \mathbin{\square} \mathcal{D}$ is given by the following pushout

$$
\begin{tikzcd}
\mathcal{C}_0 \times \mathcal{D}_0 \arrow[r, hook, "1\times i_{\mathcal{D}}"] \arrow[d, hook, "i_{\mathcal{C}}\times 1"'] & \mathcal{C}_0 \times \mathcal{D} \arrow[d] \\
\mathcal{C} \times \mathcal{D}_0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathcal{C} \mathbin{\square} \mathcal{D}
\end{tikzcd}
\tag{4.11}
$$

where $\mathcal{C}_0$ and $\mathcal{D}_0$ are the discrete categories of the objects of $\mathcal{C}$ and $\mathcal{D}$ respectively.

Explicitly, the category $\mathcal{C} \mathbin{\square} \mathcal{D}$ has as objects pairs (C, D) of an object C of $\mathcal{C}$ and D of $\mathcal{D}$. The morphisms are generated by freely composing $(f; 1) : (C, D) \to (C', D)$ where $f : C \to C'$ in $\mathcal{C}$ and $(1; g) : (C, D) \to (C, D')$ where $g : D \to D'$ in $\mathcal{D}$ with the rule that compositions exclusively in $\mathcal{C}$ or $\mathcal{D}$ may be contracted: $(f'; 1)(f; 1) = (f'f; 1)$ and $(1; g')(1; g) = (1; g'g)$ but $(f; 1)(1; g) \neq (1; g)(f; 1)$ and thus there is no sensible notion of "$(f; g)$". There is a oplax monoidal functor $\mathcal{C} \mathbin{\square} \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ induced by the universal property of the pushout, which forces the interchange squares to commute.

We must avoid this definition in the enriched case because it relies on the discrete category $\mathcal{C}_0$ of objects of a category $\mathcal{C}$ which is ill-defined over arbitrary $\mathcal{V}$. We will see later how to define a version of this funny tensor that is more well-behaved with enriched categories.

Chapter 5

Spacetime

Categorical approaches to modelling the structures of spacetime have become increasingly rich topics of study leading to both the development of new mathematics and a greater understanding of the underlying structures of our theories of physics. Nevertheless, the precise categorical structures that should be present in a model of spacetime are far from settled. The currently most successful approaches include Categorical Quantum Mechanics (CQM) [3], Topological and Functorial Quantum Field Theory (TQFT and FQFT) [7, 9, 123, 140] and Algebraic Quantum Field Theory (AQFT) [93, 38].

TQFT and AQFT are axiomatised in similar ways, roughly as functors mapping spacetimes to processes (often Vect) in the former and to observables (often Alg_k) in the latter. CQM shares much commonality with TQFT - indeed it can be seen as a particular low-dimensional case. Nevertheless, by taking the process theoretic mentality seriously it has led to both conceptual and technical advances in standard quantum mechanics, for instance in the development of the ZX and other graphical calculi for quantum circuits [51].

Both CQM and TQFT are compositional in nature with monoidal structure a standard requirement: CQM often starts with the assumption of a (usually †-compact) symmetric monoidal category and a TQFT is typically a *monoidal* functor of the form:

$$\mathcal{Z} : \mathsf{Bord}_n \to \mathsf{Vect}_n$$

sending the n-category of cobordisms to the n-category of n-vector spaces. The key physical argument for the assumption of monoidal structure is simple: if one has a pair of systems, then one should be able to put them together and consider the composite as a new system. In doing so we build up larger systems from pre-defined atomistic ones.

AQFT on the other hand, is conceptually "dual" taking an underlying philosophy of *decompositionality* [55]. Monoidal structure is not key in the axiomatics of AQFT, and instead one studies functors of the form

$$\mathcal{A} : \mathsf{Sp} \to \mathsf{Alg}_k$$

where Sp is a category of spacetimes (for instance, the objects might be time-orientable Lorentzian manifolds and the morphisms isometric and time-orientation preserving embeddings) and Alg_k is the category of algebras over a field k. The category Sp is taken to have sufficient structure to encode when morphisms are causally disjoint, e.g. by being an orthogonal category [27, 26], and instead of monoidality, $\mathcal{A}$ is taken to be a functor sending causally disjoint morphisms to commuting algebras of observables.

It is this philosophy of decompositionality in which we will be primarily interested in this section. As first explicitly described in [55], rather than starting with a collection of existent systems and presupposing that it is possible to join them together arbitrarily, we start with a global system - the whole of spacetime - and carve out systems with the hope of recovering some fragment of compositional structure. In such a framework, the tensor becomes problematic, for instance, if we pick a particular system, say a specified qubit A, it is clearly not possible to form the product $A \otimes A$ in the usual sense, for what would it mean to consider the composite of a system with itself? Indeed, the fundamental issue here is trying to tensor two objects that are not independent and that can influence each other in non-trivial ways; we would also have issues taking the tensor of timelike separated systems, or of mixed systems whose environments are not causally disjoint.

There are two main obstructions to hoping for a total tensor product on a category modelling spacetime regions in a decompositional fashion. Firstly, one would like the objects of the category to satisfy some mathematical requirements so as to ensure that they can be interpreted physically and such that we can reasonably consider functorial assignments of fields over the spacetime. It can often be the case though that no such physical system exists for the composite of physically reasonable systems - the mathematical constraints we require of such systems may not be closed under tensor-like compositions. For instance, if we take the objects of our category to represent slices of spacetime - closed spacelike subsets of a Lorentzian manifold - when we try to join two slices together they will not form another slice unless the original slices were causally separated. To see this explicitly, consider Minkowski spacetime $\mathbb{R}^2$ and the two slices given by the intervals $A := \{t = 0\} \times [0, 1]$ and $B := \{t = 1\} \times [0, 1]$. A reasonable choice of composite system might be to consider set-theoretic unions

$A \cup B$, but this new set is not a slice of (1+1)-dimensional Minkowski spacetime as there exists points of B in the future light cones of points of A and thus there are causal curves from A to B. So it may not be possible to assign an object of the category to a tensor $A \otimes B$, and the category would fail to be monoidal.

Even where the tensor does exist, functoriality can fail and one often finds that the interchange law does not hold:

$$(g \otimes 1)(1 \otimes f) \neq (1 \otimes f)(g \otimes 1) \tag{5.1}$$

while functoriality in each side of the tensor still holds $(1 \otimes f')(1 \otimes f) = (1 \otimes f'f)$ and $(g' \otimes 1)(g \otimes 1) = (g'g \otimes 1)$. This occurs because the systems involved in the tensor may not be independent - they might causally influence each other or possess a shared environment. Thus the casual ordering of f and g is vitally important.

One possible route forwards could be to define the tensor only partially. It was noted in [55] that one can recover a partial monoidal structure where the tensor product is only defined on regions of spacetime that are causally separated. A group theoretic approach was taken in [85] where the resulting category has partial monoidal structure defined only on compatible systems, which requires both the causal separation of systems and also their coupled environments. Another approach starting with a poset modelling the causal relationships of spacetime events [91], resulted in partial monoidality, again only defined on causally separated systems. As a result the authors were led to define a Causal Field Theory (CFT) as a partially monoidal functor:

$$\mathcal{F} : \mathsf{Slices}(\mathcal{M}) \to \mathsf{Proc}$$

between the partial monoidal category of slices of some spacetime $\mathcal{M}$ and a monoidal category Proc of processes [91]. CFTs therefore take a sort of intermediate position combining elements of TQFTs and AQFTs: they are decompositional like the latter but in a way that recovers fragments of the compositional structure we would expect of the former.

One potential issue with CFTs is their partial monoidality. Such categories are difficult to work with and mathematically displeasing - for instance, given two slices X and Y it is not clear without additional information about X and Y whether their tensor $X \otimes Y$ even exists. It would be more pleasing if $X \otimes Y$ always existed, say as some "virtual" generalised object, but that some property of the tensor product could tell us whether it is an object of the original CFT.

In this Chapter we propose the usage of weakenings of monoidal categories in the form of the promonoidal [68] and premonoidal [132] categories (see Chapter

4) to model causal curves in spacetime. Premonoidal categories are like monoidal ones but dropping the interchange law (5.1). They were developed for modelling computational semantics with side-effects and have been used previously to model spacetime particularly in relation to AQFTs [60, 28], where it was argued one could use them to model the Einstein causality condition. Here, we reinforce their point and argue that the lack of bifunctoriality seems to be fundamental in a decompositional approach to spacetime.

As we discussed in Section 4.5, promonoidal categories are loosely like monoidal categories into the presheaf category. To our knowledge they have not been directly used in a model of spacetime before. Here, we use them to extend the partial monoidality of spacetime to a total tensor by allowing us to assign useful mathematical objects to otherwise physically problematic ones. For instance, the union of two slices of spacetime is another region of the manifold but not necessarily a slice, thus lacking physical interpretation. We can assign the union a presheaf, with these presheaves being representable whenever the union is another slice. The non-representable presheaves can be thought to act like "virtual systems," they carry useful information but are not physically meaningful. We can also think of the promonoidal structure multicategorically, as an assignment for any slices X and $Y_1, \ldots, Y_n$ of the set of causal paths $X \to Y_1, \ldots, Y_n$.

In Section 5.1 we introduce toy categories Slice and Space of causal curves in spacetime before showing in Section 5.2 that Slice is a promonoidal category under the operation of taking intersections of sets of causal curves. In Section 5.3 we discuss the operation of taking unions of sets of causal curves and demonstrate that this gives a premonoidal structure on Space while Slice combines the structures of promonoidal and premonoidal categories. Under either of the tensor-like structures on Slice we prove that the presheaves assigned to the tensors are representable if and only if the slices are jointly spacelike and in doing so show that we recover a type of partial tensor product on causally separated regions.

5.1 A Category of Spacetime Slices

In this Section we develop a toy category of spacetime slices and causal curves and then demonstrate that it exhibits both premonoidal and promonoidal structures.

5.1.1 Spacetimes and Causal Curves

Spacetime is usually described in terms of a particular class of *Lorentzian manifolds*. Let us work towards this notion in a somewhat informal and conceptual manner.

A *manifold* is a topological space $\mathcal{M}$ that (1) *looks locally like* $\mathbb{R}^n$ - every point has a neighbourhood which is homeomorphic to $\mathbb{R}^n$, and (2) every point can be separated by neighbourhoods - the space is Hausdorff so that for any two points there exist neighbourhoods around each which are disjoint. A *differentiable* manifold has additional structure which allows for sensible notions of calculus to be developed. Each local region U looks like $\mathbb{R}^n$ - we have a *chart* mapping the open set U to an open subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a vector space in which we know how to do calculus, if we can put charts all over our manifold, known as an *atlas*, and these charts behave well together, then it is possible to do calculus anywhere on the manifold in a consistent manner. This is known as a differentiable atlas, and a differentiable manifold is a manifold with a differentiable atlas[1].

In a differentiable manifold $\mathcal{M}$ it is possible to associate to each point p a *tangent space* $T_p\mathcal{M}$ which can be thought of as a vector space of equivalence classes of vectors through p, capturing the "directions" one may pass through p. A *metric* is an infinitely differentiable, symmetric, bilinear map assigning a real number to each pair of tangent vectors $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$. A *pseudo-Riemannian* manifold is a differential manifold with a non-degenerate metric at each point, with the metric varying smoothly as the point p is changed. By evaluating the metric on an orthogonal basis $\{X_i\}_i$ one finds a series of non-zero real values $g_p(X_i, X_i)$. The number of these that are positive and negative, respectively is known as the *signature* (p, q). A *Lorentzian* manifold is a pseudo-Riemannian manifold where the signature is $(n-1, 1)$.

From now on we fix a connected Lorentzian manifold $\mathcal{M}$ with metric g. A tangent vector X is said to be *spacelike*, *timelike* or *null* if $g(X, X) > 0$, $g(X, X) < 0$ or $g(X, X) = 0$, respectively. $\mathcal{M}$ is said to be *time-orientable* if it has a non-vanishing timelike vector field and the timelike tangent vectors at each point can be divided (in a continuous fashion) into two classes: a *future-directed* and a *past-directed* class. We assume that $\mathcal{M}$ is time-orientable and fix a time-orientation. The assumptions we make of our spacetime are fairly weak causality-wise, and are weaker than those of past- and future-distinguishability [125, 112] (which was assumed by [91]) and certainly weaker than the existence of a Cauchy slice (equivalently global hyperbolicity) [84]. As a result we have not ruled out the existence of closed timelike curves in the spacetime.

[1]technically a *maximal* differentiable atlas, and such that the manifold is second countable, but these are issues beyond the discussion required for this work

A simple example of the kinds of manifolds we are interested in is Minkowski space $\mathbb{R}^{n+1}$ equipped the metric $g(X_1, X_2) = |\vec{\mathbf{x}}_1 \cdot \vec{\mathbf{x}}_2|^2 - t_1 t_2$ for $X_i = (t_i, \vec{\mathbf{x}}_i)$. The timelike vectors are those $(t, \vec{\mathbf{x}})$ where $t^2 > |\vec{\mathbf{x}}|^2$, of which there are two classes $t > |\vec{\mathbf{x}}|$ and $t < -|\vec{\mathbf{x}}|$ consisting of vectors which point forwards and backwards in time, respectively; a timelike vector $(t, \vec{\mathbf{x}})$ is future-directed when $t > 0$ and past-directed when $t < 0$. There is no issue with restricting oneself to Minkowski space for the remainder of this Chapter, but we note that the results hold in the fully general case.

A *path* in $\mathcal{M}$ is a continuous map $\mu : \iota \to \mathcal{M}$ where $\iota \subseteq \mathbb{R}$ is a (possibly unbounded) real interval. Such a path is *smooth* if it is infinitely differentiable and *regular* if its first derivative is non-vanishing. A smooth regular path is *causal* when the tangent vector is timelike or null at all points in the path and a causal path is *future-directed* when the tangent at every point is future-directed. For a point $x \in \mathcal{M}$, the set of all points $y \in \mathcal{M}$ with a future-directed path x to y is called the *future light cone* of x, whereas the set of all points with a future-directed path from y to x is called the *past light cone* of x.

Often it is more convenient to work with equivalence classes of paths, up to reparametrisation, i.e. $\mu \sim \mu'$ if and only if there exists a monotone map $r : \iota \to \iota'$ such that $\mu' \circ r = \mu$. An equivalence class of causal paths is called a *causal curve*. Since being future-directed is preserved by $\sim$, we can also say a causal curve is future-directed without ambiguity.

A point $x \in \mathcal{M}$ *causally precedes* another point $y \in \mathcal{M}$, written $x \prec y$, if there exists a future-directed causal curve from x to y, or if $x = y$. The assumption of time-orientability of $\mathcal{M}$ is not enough to ensure that $\prec$ gives a total order on points in a causal curve - for instance there could be closed timelike curves in $\mathcal{M}$ containing points $x \neq y$, for which $x \prec y$ and $y \prec x$.

A *region* is any arbitrary subset $A \subseteq \mathcal{M}$ of the manifold. Regions are too general to be useful for many practical applications, they might contain points which causally precede each other or they might have insufficient topological properties to make them well-behaved. As a result we will be more interested in a restricted class of regions, the *spacelike* regions, where for all $x, y \in \Sigma$, $x \neq y$, x does not causally precede y and thus there are no future-directed causal curves connecting x with y, or y with x. For instance, in Minkowski space the surfaces given by fixed times $t = \tau$ are examples of spacelike sets.

Definition 41 (Spacelike Slice)**.** A spacelike slice (or simply a "slice") is a closed spacelike set.

It is worth noting that slices may still be too weak for many applications, and it may be necessary to demand further properties of them, by working with the Cauchy slices for instance. Whilst we do not make these restrictions in this work, in principle, there is no obstacle to applying many of the same methods to categories of more restrictive classes of slices.

We will be very interested in the causal relationship between slices X and Y, which motivates the following definition.

Definition 42 (Jointly Spacelike Slices)**.** Slices X and Y are jointly spacelike if their union $X \cup Y$ is spacelike.

Given regions $A, B \subseteq \mathcal{M}$, $A \neq B$, we say that a future-directed causal curve γ with representative path $\mu : \iota \to \mathcal{M}$, *passes through A and then B* if there exists a $q \in \iota$ with $\mu(q) \in B$ and for all such q there exists $p \leq q \in \iota$ such that $\mu(p) \in A$. We write $\mathcal{C}[A, B]$ for the set of future-directed causal curves passing through A and then B. We write $\mathcal{C}[A] := \mathcal{C}[A, A]$ for the set of future-directed causal curves which pass through A (with no constraint on other regions through which they must pass). This means that *any* causal curve γ with representative path μ such that there exists a $q \in \iota$ with $\mu(q) \in A$, is an element of the set $\mathcal{C}[A]$. It is also worth noting that a closed timelike curve γ containing both the points $a \in A$ and $b \in B$ will be in the sets $\mathcal{C}[A, B]$ and $\mathcal{C}[B, A]$.

5.1.2 A Category of Causal Curves

With these definitions in place we can define the following categories of slices and regions of spacetime:

Definition 43 ($\mathsf{Slice}, \mathsf{Space}$)**.** The category Slice has as objects slices $X \subset \mathcal{M}$ (closed spacelike sets). For two slices $X, Y \subset \mathcal{M}$, the homset $\mathsf{Slice}(X, Y) := \mathcal{P}(\mathcal{C}[X, Y])$ is the powerset of $\mathcal{C}[X, Y]$, that is, a morphism $X \to Y$ is a set of future-directed causal curves through X then Y. Given two subsets $S : X \to Y$ and $T : Y \to Z$, their composition is given by intersection: $T \circ S := T \cap S \subset \mathcal{C}[X, Z]$. The identity morphism $1_X : X \to X$ is given by the set $\mathcal{C}[X, X]$ of all curves through X.

The category Space has as objects arbitrary regions $A \subseteq \mathcal{M}$. All other data is as Slice.

Proposition 30. Slice *and* Space *are categories.*

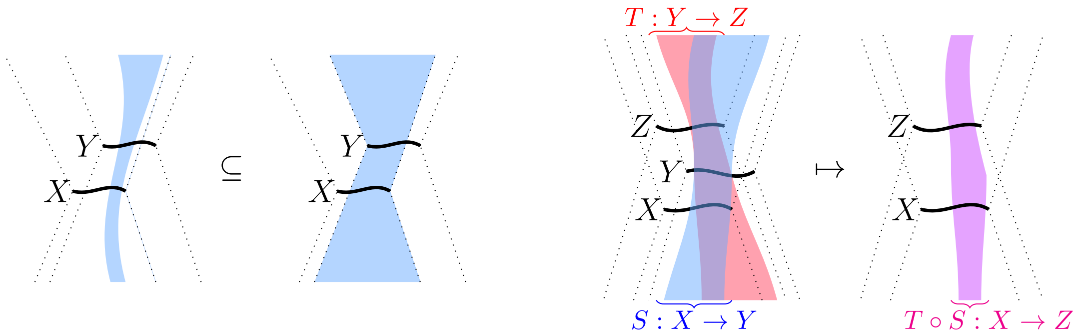


Figure 5.1: *Left:* A morphism in the category Slice is a set of causal curves passing first through X then through Y. *Right:* Composition of two morphisms in Slice via intersection. Note that in both pictures, past as future light cones of slices are depicted as dotted lines, and sets of many causal curves are depicted as filled-in regions.

Proof. Composition is associative because intersection is. Given a set of causal curves $S : X \to Y$, by definition all curves in S pass through X, thus we see $S \circ 1_X = S \cap \mathcal{C}[X, X] = S$. Similarly for the left composition with identity morphisms. $\square$

Now we examine a few basic categorical properties of Slice and Space.

Proposition 31. Slice *and* Space *have equalisers and coequalisers, given by the complement of the symmetric difference.*

Proof. Take $f, g : A \to B$. This pair of parallel arrows is equalised by $(f \triangle g)^c : A \to A$ and coequalised by $(f \triangle g)^c : B \to B$ where $(f \triangle g)^c = \mathcal{C}[A, B] \setminus (f \triangle g) = (f \cup g)^c \cup (f \cap g)$. Any other arrow h making the parallel pair f and g equal factorises uniquely via $(f \triangle g)^c$ because this morphism contains every causal curve that is in both f and g, or neither. Thus h must be a subset of $(f \triangle g)^c$. $\square$

It is interesting that equalisers and coequalisers essentially coincide in Slice - in part this is down to the fact that composition is, up to types, commutative - e.g. for endomorphisms $f \circ g = g \circ f$.

Proposition 32. *Let X and Y be jointly spacelike slices with $X \cap Y = \varnothing$. Then the product and coproduct of X and Y exist in* Slice *and are given by the set theoretic union $X \times Y = X \oplus Y = X \cup Y$.*

Proof. The projections are given by

$$\pi_0 = \mathcal{C}[X] : X \cup Y \to X$$
$$\pi_1 = \mathcal{C}[Y] : X \cup Y \to Y$$

while the coprojections are given by

$$i_0 = \mathcal{C}[X] : X \to X \cup Y$$
$$i_1 = \mathcal{C}[Y] : Y \to X \cup Y$$

Given $f : Z \to X$ and $f' : Z \to Y$, the universal arrow completing the product diagram is $\langle f, f' \rangle = f \cup f' : Z \to X \cup Y$, and given $g : X \to Z$ and $g' : Y \to Z$, the universal arrow completing the coproduct diagram is $[g, g'] = g \cup g' : X \cup Y \to Z$. Indeed, it follows that the diagrams commute because X and Y are jointly spacelike with $X \cap Y = \varnothing$ and thus $f \cap \mathcal{C}[Y] = f' \cap \mathcal{C}[X] = g \cap \mathcal{C}[Y] = g' \cap \mathcal{C}[X] = \varnothing$. $\square$

While we do have products and coproducts of non-intersecting jointly spacelike slices in Slice, the (co)products of other regions e.g. timelike separated regions and of intersecting slices do not exist. These regions are the main issue preventing the set theoretic union from being a monoidal structure on Slice.

Proposition 33. Slice *is **not** a monoidal category under a monoidal product given by taking the union of regions and curves $X \otimes Y := X \cup Y$ and $S \otimes T := S \cup T$.*

Proof. The union of slices is not always a slice so $X \cup Y$ may not be an object of Slice. For the occasions when it is, $\otimes$ cannot in general be bifunctorial. For arbitrary $S : X \to Y$, $S' : Y \to Z$, $T : X' \to Y'$ and $T' : Y' \to Z'$, we have $(S' \otimes T') \circ (S \otimes T) = (S' \cup T') \cap (S \cup T) \supset (S' \cap S) \cup (T' \cap T) = (S' \circ S) \otimes (T' \circ T)$. $\square$

One might hope that by relaxing the sorts of objects we are considering and working instead with the category Space, we could find a monoidal product given by union. Whilst this resolves the issue of the non-existence of the object $X \cup Y$ for arbitrary X and Y, we still find that the union cannot be bifunctorial and thus Space is also not a monoidal category under union.

We also cannot hope that Slice or Space are monoidal categories under intersection because there exist causally connected slices which have an empty intersection:

Proposition 34. Slice *and* Space *are **not** monoidal categories under a monoidal product given by taking the intersection of regions and curves $X \otimes Y := X \cap Y$ and $S \otimes T := S \cap T$.*

Proof. Suppose X and Y are causally connected slices so $\mathcal{C}[X, Y] \neq \varnothing$ but with $X \cap Y = \varnothing$. Then $1_X \otimes 1_Y = \mathcal{C}[X, X] \cap \mathcal{C}[Y, Y] \neq \varnothing$ because there exists a causal curve passing through X and Y. On the other hand we see that $1_{X \cap Y} = 1_\varnothing = \varnothing$. $\square$

In the following sections we will show that while Slice and Space are not monoidal categories in either of these ways, Slice is a promonoidal category under intersection. Under union, Space is premonoidal while Slice combines both promonoidal and premonoidal structures.

5.2 A Promonoidal Structure on Slice

We now aim to show that Slice is a promonoidal category under intersection, that is, it is equipped with a tensor product functor $\mathsf{Slice} \times \mathsf{Slice} \to [\mathsf{Slice}^{\mathrm{op}}, \mathsf{Set}]$ and unit presheaf $\mathsf{Slice}^{\mathrm{op}} \to \mathsf{Set}$ subject to associativity and unit laws.

To each pair of objects X and Y we assign the presheaf $(X \owedge Y)(-) : \mathsf{Slice}^{\mathrm{op}} \to \mathsf{Set}$ which sends a slice Z to the powerset of causal curves which pass through Z and then both X and Y

$$(X \owedge Y)(Z) := \mathcal{P}(\mathcal{C}[Z, X] \cap \mathcal{C}[Z, Y])$$

On morphisms $S : Z' \to Z$ this presheaf acts by intersection:

$$(X \owedge Y)(S) : (X \owedge Y)(Z) \to (X \owedge Y)(Z') :: C \mapsto C \cap S$$

Lemma 1. *$(X \owedge Y)(-)$ is a presheaf.*

Proof. $(X \owedge Y)(1_Z) :: C \mapsto C \cap 1_Z = C$ because every curve in $(X \owedge Y)(Z)$ passes through Z. Thus $(X \owedge Y)(1_Z) = 1_{(X \owedge Y)(Z)}$. Now $(X \owedge Y)(T) \circ (X \owedge Y)(S) :: C \mapsto C \cap S \mapsto (C \cap S) \cap T$ while $(X \owedge Y)(S \circ T) :: C \mapsto C \cap (S \cap T)$ and these are equal by the associativity of intersection. $\square$

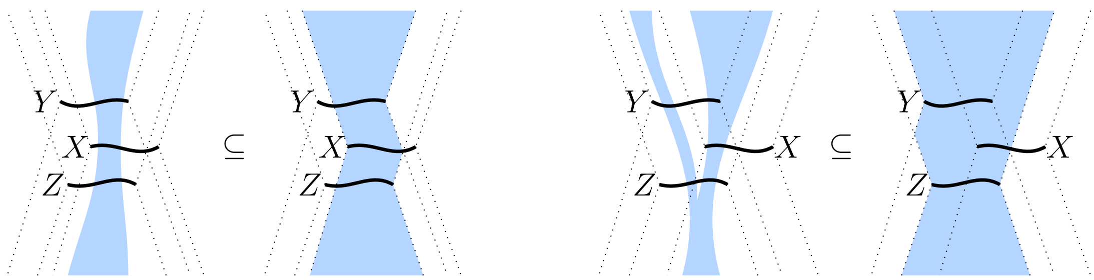


Figure 5.2: *Left:* An element $S \in (X \owedge Y)(Z)$, as defined in Section 5.2. *Right:* An element $T \in (X \ovee Y)(Z)$, as defined in Section 5.3.

To each $(S, T) : (X, Y) \to (X', Y')$ we are required to assign a natural transformation between the presheaves $S \owedge T : (X \owedge Y)(-) \implies (X' \owedge Y')(-)$.

For $S : X \to X'$ there is a natural transformation with components

$$(S \oslash Y)_Z : (X \oslash Y)(Z) \to (X' \oslash Y)(Z) :: C \mapsto C \cap S$$

and for $T : Y \to Y'$ there is a natural transformation with components

$$(X \oslash T)_Z : (X \oslash Y)(Z) \to (X \oslash Y')(Z) :: C \mapsto C \cap T$$

These natural transformations commute, $(S \oslash Y')_Z(X \oslash T)_Z = (X' \oslash T)_Z(S \oslash Y)_Z$ and we can define $(S \oslash T)$ to be given by their composition.

Lemma 2. *$(S \oslash Y)$ and $(X \oslash T)$ are natural transformations with $(S \oslash Y')_Z(X \oslash T)_Z = (X' \oslash T)_Z(S \oslash Y)_Z$.*

Proof. Note that the following diagram commutes for any $U : Z' \to Z$

$$\begin{tikzcd}
(X \oslash Y)(Z) \arrow[r, "(X \oslash Y)(U)"] \arrow[d, "(S \oslash Y)_Z"'] & (X \oslash Y)(Z') \arrow[d, "(S \oslash Y)_{Z'}"] \\
(X' \oslash Y)(Z) \arrow[r, "(X' \oslash Y)(U)"'] & (X' \oslash Y)(Z')
\end{tikzcd}$$

because on the top path we see $C \mapsto C \cap U \mapsto (C \cap U) \cap S$ while on the bottom path $C \mapsto C \cap S \mapsto (C \cap S) \cap U$. Naturality of $(X \oslash T)$ follows similarly and checking the commutativity condition is straightforward. □

Lemma 3. *The assignment $(X, Y) \mapsto (X \oslash Y)(-)$ and $(S, T) \mapsto (S \oslash T)$ gives a functor $\mathsf{Slice} \times \mathsf{Slice} \to [\mathsf{Slice}^{op}, \mathsf{Set}]$.*

Proof. Firstly note that each component of $1_X \oslash 1_Y : (X \oslash Y)(-) \implies (X \oslash Y)(-)$ is just the identity. Thus it is the identity natural transformation and we conclude $1_X \oslash 1_Y = 1_{(X \oslash Y)(-)}$.

Now take $S : X \to X'$ and $S' : X' \to X''$. The arrow $(S' \oslash Y)_Z \circ (S \oslash Y)_Z$ acts as $C \mapsto (C \cap S) \cap S'$ while the arrow $((S' \circ S) \oslash Y)_Z$ acts as $C \mapsto C \cap (S' \cap S)$. Thus the components of the composite natural transformation $(S' \oslash Y) \circ (S \oslash Y)$ equal those of $((S' \circ S) \oslash Y)$.

A similar argument holds for arrows $T : Y \to Y'$ and because $(S \oslash Y)$ and $(X \oslash T)$ commute we are done. □

We are now in a position to prove the main result of this section:

Theorem 7. Slice *is a symmetric promonoidal category where the tensor is given above and the unit presheaf is given by $I(Z) := \mathcal{P}(\mathcal{C}[Z, Z])$.*

Proof. Let us begin with associativity $\oslash(\oslash \times 1) \cong \oslash(1 \times \oslash)$. Note that by Yoneda we have

$$\begin{aligned}\oslash(\oslash \times 1)(W, X, Y, Z) &= \int^{A,B} \oslash(W, A, B) \times \oslash(A, X, Y) \times \mathsf{Slice}(B, Z) \\ &\cong \int^{A} \oslash(W, A, Z) \times \oslash(A, X, Y)\end{aligned}$$

While

$$\oslash(1 \times \oslash)(W, X, Y, Z) \cong \int^{A} \oslash(W, X, A) \times \oslash(A, Y, Z)$$

Let us show there is a canonical identification $\oslash(\oslash \times 1)(W, X, Y, Z) \cong \mathcal{P}(\mathcal{C}[W, X] \cap \mathcal{C}[W, Y] \cap \mathcal{C}[W, Z]) =: \Lambda$. There are functions

$$\oslash(W, A, Z) \times \oslash(A, X, Y) \to \Lambda :: (S, T) \mapsto S \cap T$$

which form a cowedge with apex Λ. By the universal property of the coend this induces a unique function $g : \int^A \oslash(W, A, Z) \times \oslash(A, X, Y) \to \Lambda$ making the obvious cowedge diagrams commute.

We can also construct a function f by composing

$$\Lambda \xrightarrow{f'} \oslash(W, W, Z) \times \oslash(W, X, Y) \xrightarrow{\mathrm{copr}_W} \int^{A} \oslash(W, A, Z) \times \oslash(A, X, Y)$$

where f' acts as $S \mapsto (S, S)$.

The universal property of the coend implies that the composition $fg = 1$, or we can check explicitly:

$$(S, T) \mapsto S \cap T \mapsto (S \cap T, S \cap T)$$

upon which we simply need to note that we have $(S, T) = (S \cap S, T \cap T) \sim (S \cap T, S \cap T)$.

Similarly, it is straightforward to show that $gf = 1$: $S \mapsto (S, S) \mapsto S \cap S = S$. Thus $\Lambda \cong \int^A \oslash(W, A, Z) \times \oslash(A, X, Y)$ as sets.

Now note that this isomorphism is in fact natural in W, X, Y and Z. Let $w : W' \to W, x : X \to X', y : Y \to Y', z : Z \to Z'$, then we have

$$\begin{tikzcd}
(S, T) \arrow[r, mapsto] \arrow[d, mapsto, "g_{WXYZ}"'] & (S \cap w \cap z, T \cap x \cap y) \arrow[d, mapsto, "g_{W'X'Y'Z'}"] \\
S \cap T \arrow[r, mapsto] & S \cap T \cap w \cap x \cap y \cap z
\end{tikzcd}$$

Thus exhibiting the desired natural isomorphism.

A similar argument shows that $\oslash(1 \times \oslash)(W, X, Y, Z) \cong \Lambda$, and thus we have established the associativity natural isomorphism.

The pentagon equation is given by (writing i for the interchange and ignoring the associativity isomorphisms of profunctor composition):

$$\begin{tikzcd}
& \oslash^a_{xe} \oslash^x_{yd} \oslash^y_{bc} \arrow[rr, "1\circ\alpha"] \arrow[dl, "\alpha\circ 1"'] & & \oslash^a_{xe} \oslash^x_{by} \oslash^y_{cd} \arrow[dr, "\alpha\circ 1"] & \\
\oslash^a_{yx} \oslash^x_{de} \oslash^y_{bc} \arrow[drr, "(\alpha\circ 1)i"'] & & & & \oslash^a_{bx} \oslash^x_{ye} \oslash^y_{cd} \arrow[dll, "1\circ\alpha"] \\
& & \oslash^a_{bx} \oslash^x_{cy} \oslash^y_{de} & &
\end{tikzcd}$$

Clockwise we have the following mapping:

$$\begin{aligned}
(S,T,V) \mapsto (S, T\cap V, T\cap V) &\mapsto (S\cap T\cap V, S\cap T\cap V, T\cap V) \\
&\mapsto (S\cap T\cap V, S\cap T\cap V, S\cap T\cap V)
\end{aligned}$$

while anticlockwise we have

$$(S,T,V) \mapsto (S\cap T, S\cap T, V) \mapsto (S\cap T\cap V, S\cap T, S\cap T\cap V)$$

and it clear that $(S\cap T\cap V, S\cap T, S\cap T\cap V) \sim (S\cap T\cap V, S\cap T\cap V, S\cap T\cap V)$ under the coend equivalence relation. Thus the pentagon commutes.

Now we show the existence of the unit isomorphisms $\oslash(I\times 1) \cong 1 \cong \oslash(1\times I)$. Much of the construction is similar to the previous argument, so we leave the reader to fill in some of the details. There exist functions $\oslash(-,=,B)\times\mathcal{P}(\mathcal{C}[B,B]) \to \mathsf{Slice}(-,=)$ for each B given by sending $(S,T)\mapsto S\cap T$. These functions form a cowedge and therefore induce a unique function $\int^B \oslash(-,=,B)\times\mathcal{P}(\mathcal{C}[B,B]) \to \mathsf{Slice}(-,=)$.

The inverse of this function is given by the function $S\mapsto (S,S)$ which factorises via copr. It is straightforward to check that these give the left unit natural isomorphism, and the construction of the right unit is similar.

Writing よ for an application of the Yoneda lemma, the triangle equation is given by

$$\begin{tikzcd}
& \oslash^a_{bc} & \\
\oslash^a_{xc} \oslash^x_{by} I^y \arrow[ur, "\text{よ}\rho"] \arrow[rr, "\alpha\circ 1"'] & & \oslash^a_{bx} I^y \oslash^x_{yc} \arrow[ul, "\text{よ}\lambda"']
\end{tikzcd}$$

and it is little work to check that this commutes.

The symmetry $(X\oslash Y)(Z) \to (Y\oslash X)(Z)$ is given by the identity map for all X, Y and Z. $\square$

Now we know that Slice is promonoidal under intersection, we will study when the presheaves assigned by this tensor are representable. This allows us to ascertain where $\owedge$ acts like a standard monoidal product on Slice and where it is possible for us to consider the tensor of slices to be another slice.

Theorem 8. *When X and Y are jointly spacelike slices, the presheaf $(X \owedge Y)(-)$ is representable.*

Proof. Suppose X and Y are jointly spacelike. Note $\mathcal{C}[Z,X] \cap \mathcal{C}[Z,Y] \supseteq \mathcal{C}[Z, X \cap Y]$. Suppose there exists $\gamma \in \mathcal{C}[Z,X] \cap \mathcal{C}[Z,Y]$ with $\gamma \notin \mathcal{C}[Z, X \cap Y]$. Then γ must pass through some $x \in X \backslash Y$ and some $y \in Y \backslash X$ but this would imply that X and Y are not jointly spacelike. Thus γ cannot exist and it follows that $(X \owedge Y)(Z) = \mathcal{P}(\mathcal{C}[Z, X \cap Y]) = \mathsf{Slice}(Z, X \cap Y) = よ_{X \cap Y}(Z)$, noting that $X \cap Y$ is a slice because $X \cap Y \subseteq X$ and thus is an object of Slice. $\square$

In particular, the previous theorem shows that on jointly spacelike slices $\owedge$ acts like intersection and we can make the identification $(X \owedge Y)(-) \simeq X \cap Y$. On the other hand, when the slices are not jointly spacelike there is no representative for $(X \owedge Y)(-)$. To show this we need the following lemma:

Lemma 4. *Let $A \subseteq \mathcal{M}$ be a closed subset of $\mathcal{M}$. Then for any $x \in \mathcal{M}$, $x \notin A$, there exists a causal curve through x which does not intersect A.*

Proof. The timelike vector field is non-vanishing on $\mathcal{M}$ and as a result there must be a causal curve γ through x. In a sufficiently small neighbourhood U of x, γ must restrict to a causal curve which is contained entirely within U. Since A is closed and $\mathcal{M}$ is Hausdorff, this neighbourhood can be made sufficiently small such that $U \cap A = \varnothing$. $\square$

Theorem 9. *When X and Y are not jointly spacelike, the presheaf $(X \owedge Y)(-)$ is not representable.*

Proof. We make much use of Lemma 4. Suppose X and Y are not jointly spacelike and suppose for a contradiction that $(X \owedge Y)(-) = \mathsf{Slice}(-, Z)$ for some slice Z.

Now suppose there exists a $z \in Z$ such that $z \notin X \cup Y$. We can find a causal curve γ through z that does not also pass through $X \cup Y$. It follows that $\gamma \in \mathsf{Slice}(Z, Z)$, but $\gamma \notin (X \owedge Y)(Z)$. So Z cannot represent the presheaf and we conclude $Z \subseteq X \cup Y$.

Now take a $x \in X \backslash Y$. There exists a causal curve γ passing through x but not Y. Suppose that $x \in Z$, then $\gamma \in \mathsf{Slice}(Z, Z)$, but $\gamma \notin (X \owedge Y)(Z)$. So $x \notin Z$.

A similar argument shows that any $y \in Y \backslash X$ cannot be in Z and thus $Z \subseteq X \cap Y$.

Since X and Y are not jointly spacelike, $X \cup Y$ is not spacelike and there exists a causal curve γ from $X \cup Y$ to itself. In particular γ must pass through a point of X and a point of Y, and not, say, through two points of X, since X and Y are slices. Then we would have $\gamma \in (X \owedge Y)(X)$ but $\gamma \notin \mathsf{Slice}(X, Z)$ because if $\gamma \in \mathsf{Slice}(X, Z)$ it would pass through X and $X \cap Y \subseteq X$, a contradiction with X being a slice. □

So we have shown that $(X \owedge Y)(-)$ is representable if and only if X and Y are jointly spacelike. Note that one **cannot** define a partially monoidal category by just working with $\owedge$ where it is representable because the unit presheaf is not representable and therefore there is no unit object available in Slice.

5.3 The Structure of Slice and Space under Union

Let us now consider the structure of Slice and Space under union of slices and sets of curves. The larger category Space where the objects are arbitrary subsets of the manifold $\mathcal{M}$ and the homsets are powersets of causal curves is a premonoidal category:

Proposition 35. Space *is a strict premonoidal category under the operation of taking the union of regions and curves.*

Proof. For objects X and Y assign them the object $X \otimes Y := X \cup Y$. The assignment $(T : Y \to Y') \mapsto (\mathcal{C}[X] \cup T : X \cup Y \to X \cup Y')$ gives a functor $X \rtimes - : \mathcal{C} \to \mathcal{C}$ because

$$X \rtimes 1_Y = \mathcal{C}[X] \cup \mathcal{C}[Y] = \mathcal{C}[X \cup Y] = 1_{X \cup Y}$$

$$(X \rtimes f')(X \rtimes f) = (\mathcal{C}[X] \cup f') \cap (\mathcal{C}[X] \cup f) = \mathcal{C}[X] \cup (f' \cap f) = X \rtimes f'f$$

Similarly the assignment $(S : X \to X') \mapsto (S \cup \mathcal{C}[Y])$ extends to a functor $- \ltimes Y : \mathcal{C} \to \mathcal{C}$. The unit object is $I := \varnothing$ and the unit and associativity isomorphisms are identities, which it is straightforward to check are central. □

The above has a clear issue - $X \cup Y$ is generally not another slice and thus not an object of Slice. This means Slice cannot form a premonoidal category under union and we need to search for something that combines both premonoidal and promonoidal structures together.

There is no obstacle to defining presheaves $(X \ovee Y)(-) : \mathsf{Slice}^{\mathrm{op}} \to \mathsf{Set}$ which send a slice Z to the powerset of causal curves through Z and either X or Y:

$$(X \ovee Y)(Z) := \mathcal{P}(\mathcal{C}[Z, X] \cup \mathcal{C}[Z, Y])$$

On morphisms $S : Z' \to Z$ this presheaf acts by intersection:

$$(X \circledvee Y)(S) : (X \circledvee Y)(Z) \to (X \circledvee Y)(Z') :: C \mapsto C \cap S$$

Lemma 5. $(X \circledvee Y)(-)$ *is a presheaf.*

Similarly, there is no obstacle to defining natural transformations acting on either the left or right of $\circledvee$. For $S : X \to X'$ there is a natural transformation with components

$$(S \circledvee Y)_Z : (X \circledvee Y)(Z) \to (X' \circledvee Y)(Z) :: C \mapsto C \cap (S \cup \mathcal{C}[Y])$$

and for $T : Y \to Y'$ there is a natural transformation with components

$$(X \circledvee T)_Z : (X \circledvee Y)(Z) \to (X \circledvee Y')(Z) :: C \mapsto C \cap (\mathcal{C}[X] \cup T)$$

Lemma 6. $(S \circledvee Y)$ *and* $(X \circledvee T)$ *are natural transformations.*

What fails in comparison to $\circledwedge$ is that, in general, the components of these natural transformations do not obey the interchange law, so we cannot hope that these data give a functor $\mathsf{Slice} \times \mathsf{Slice} \to [\mathsf{Slice}^{\mathrm{op}}, \mathsf{Set}]$. Nevertheless, the natural transformations are functorial on each side of the tensor and it is easy to verify that the assignment does give a functor $\circledvee : \mathsf{Slice} \,\square\, \mathsf{Slice} \to [\mathsf{Slice}^{\mathrm{op}}, \mathsf{Set}]$ where $\square$ is the funny tensor product of categories.

Lemma 7. *The data of Lemmas 5 and 6 specify a functor* $\mathsf{Slice} \,\square\, \mathsf{Slice} \to [\mathsf{Slice}^{op}, \mathsf{Set}]$

Proof. Take $S : X \to X'$ and $S' : X' \to X''$. Then $(S' \circledvee Y)_Z(S \circledvee Y)_Z$ acts as $C \mapsto C \cap (S \cup \mathcal{C}[Y]) \cap (S' \cup \mathcal{C}[Y]) = C \cap ((S \cap S') \cup \mathcal{C}[Y])$ which is precisely the same as the action of $(S'S \circledvee Y)_Z$. We conclude $(S' \circledvee Y)_Z(S \circledvee Y)_Z = (S'S \circledvee Y)_Z$.

A similar argument shows that $(X \circledvee T')_Z(X \circledvee T)_Z = (X \circledvee T'T)_Z$ and thus we have functoriality of $(- \circledvee =)$ in each component. This is enough to extend to functoriality from the funny tensor. $\square$

In this way Slice seems to combine both the structures of premonoidal and promonoidal categories. We leave it as future work to make rigorous the associativity and unitality of this structure but we note that the representable presheaf at the empty slice よ$_\varnothing$ is likely the unit of a suitably defined structure.

Similarly to the intersection case we can study when the presheaves $(X \circledvee Y)(-)$ are representable:

Theorem 10. *When X and Y are jointly spacelike, the presheaf $(X \ovee Y)(-)$ is representable.*

Proof. Suppose X and Y are jointly spacelike. Then $(X \ovee Y)(Z) = \mathcal{P}(\mathcal{C}[Z, X] \cup \mathcal{C}[Z, Y]) = \mathcal{P}(\mathcal{C}[Z, X \cup Y]) = \text{よ}_{X \cup Y}(Z)$ where we have used the fact that $X \cup Y$ is spacelike and thus an object of Slice. □

Theorem 11. *When X and Y are not jointly spacelike, the presheaf $(X \ovee Y)(-)$ is not representable.*

Proof. We make use of Lemma 4. Suppose X and Y are not jointly spacelike and suppose for a contradiction that $(X \ovee Y)(-) = \mathsf{Slice}(-, Z)$ for some slice Z. By the same argument made in the proof of Theorem 9 we must have $Z \subseteq X \cup Y$.

Since X and Y are not jointly spacelike, $X \cup Y$ is not spacelike and thus there exists a causal curve γ connecting two points of $X \cup Y$. It must be the case that one of these points is in $X \backslash Y$ and the other in $Y \backslash X$ else X or Y could not be slices. Write $x \in X \backslash Y$ and $y \in Y \backslash X$ for these points that γ passes through and note that they can be the only points of $X \cup Y$ that γ intersects else X or Y could not be slices.

Now note that γ restricts to a causal curve γ_x which passes through x but not y and similarly a causal curve γ_y which passes through y but not x.

Suppose that $x \notin Z$, then $\gamma_x \in (X \ovee Y)(X)$ but $\gamma_x \notin \mathsf{Slice}(X, Z)$, noting that $Z \subseteq X \cup Y$ so that γ_x intersects Z at only x. So we conclude that $x \in Z$.

Similarly, suppose that $y \notin Z$, then $\gamma_y \in (X \ovee Y)(Y)$ but $\gamma_y \notin \mathsf{Slice}(Y, Z)$. So we conclude that $y \in Z$.

We see that γ is a causal curve connecting two distinct points of Z and consequently Z cannot be a slice. □

So we have shown that the presheaf $(X \ovee Y)(-)$ is representable if and only if X and Y are jointly spacelike. By restricting $\ovee$ to these slices we can recover a partial premonoidal structure on Slice by defining the tensor to be given by the representative. The unit of this partial premonoidal category is the empty slice $\varnothing$.

Now that we have two tensor-like structures on Slice we would like to know how they interact. Given that $\ovee$ behaves like union and $\owedge$ like intersection, it seems reasonable to expect some sort of distributivity between them. To understand this at the level of the profunctors we require the following definition:

Definition 44 (Multiplicative Kernel [66])**.** Let $(\mathcal{C}, P, I)$ and $(\mathcal{D}, Q, J)$ be promonoidal categories. A multiplicative kernel is a profunctor $K : \mathcal{C} \nrightarrow \mathcal{D}$ such that

$$Q(K \times K) \cong KP \qquad\qquad KI \cong J$$

where concatenation is profunctor composition.

Remark. Viewing $\mathcal{C}$ and $\mathcal{D}$ as pseudomonoids in Prof, a multiplicative kernel is a homomorphism of these monoids.

Each slice X determines an endoprofunctor $(X \ovee -)(-) : \mathsf{Slice} \nrightarrow \mathsf{Slice}$ and it is the case that each of these is a multiplicative kernel for Slice equipped with $\owedge$.

Theorem 12. *For every slice X, $(X \ovee -)(-)$ is a multiplicative kernel for $(\mathsf{Slice}, \owedge)$.*

Proof. (Sketch). The proof is similar and uses the same methods as Theorem 7 so we only sketch the idea.

Fix a slice A. We will show that $(A \ovee -)(-)$ is a kernel.

Starting with the units we need to show that $\int^X \ovee^Z_{AX} J^X \cong J^Z$. There are functions $\ovee^Z_{AX} J^X \to J^Z$ sending $(S,T) \mapsto S \cap (T \cup \mathcal{C}[A])$. These are dinatural in X and thus form a cowedge factorising uniquely via the coend. As a result we have a function $\int^X \ovee^Z_{AX} J^X \to J^Z$. This function is an isomorphism with inverse given by $S \mapsto (S,S)$ which factorises via copr. Indeed,

$$S \mapsto (S,S) \mapsto S \cap (S \cup \mathcal{C}[A]) = S$$

and

$$\begin{aligned}(S,T) \mapsto S \cap (T \cup \mathcal{C}[A]) &\mapsto (S \cap (T \cup \mathcal{C}[A]), S \cap (T \cup \mathcal{C}[A])) \\ &\sim (S \cap (S \cup \mathcal{C}[A]), T \cap T) \\ &= (S,T)\end{aligned}$$

As for the multiplications we want to show $\int^Z \ovee^W_{AZ} \owedge^Z_{XY} \cong \int^{ZZ'} \ovee^Z_{AX} \ovee^{Z'}_{AY} \owedge^W_{ZZ'}$ which it is easiest to do by showing each is naturally isomorphic to $\Lambda := \mathcal{P}(\mathcal{C}[W,A] \cup (\mathcal{C}[W,X] \cap \mathcal{C}[W,Y]))$. For the former, there is a cowedge with components $(S,T) \mapsto S \cap (T \cup \mathcal{C}[A])$, with the inverse to the induced map given by $S \mapsto (S,S)$, as in the case of the units. For the latter, there is a cowedge with components $(S,T,V) \mapsto S \cap T \cap V$, with the inverse to the induced map given by $S \mapsto (S,S,S)$.

To show that all the isomorphisms are natural is little work. $\square$

Chapter 6

Pre-promonoidal and Pro-effectful Categories

In this Chapter we consider a route to making formal the "pre-promonoidal" structure suggested in the previous section by the union of slices. The idea is to combine together elements of both premonoidal and promonoidal structure by equipping a category $\mathcal{C}$ with a profunctor $P : \mathcal{C} \,\square\, \mathcal{C} \nrightarrow \mathcal{C}$ that behaves like the tensor of a promonoidal category, but that lacks the interchange law in its domain. One would hope that this "tensor" should be associative and unital like in monoidal, premonoidal and promonoidal categories but one runs into an immediate problem trying to define this - it is not clear how to make sense of the coends such as $\int^{C} P(-,-,C) \times P(C,-,-)$ because the domain of P is the funny tensor $\mathcal{C} \,\square\, \mathcal{C}$ not $\mathcal{C} \times \mathcal{C}$.

One possibility is to only take coends over the premonoidal centre $Z(\mathcal{C})$ of $\mathcal{C}$ so that we weaken the (co)domains of the associativity and unit natural isomorphisms to expressions like $\int^{C \in Z(\mathcal{C})} P(-,-,C) \times P(C,-,-)$, which we can now make sense of since we have changed the problematic funny tensor $\mathcal{C} \,\square\, \mathcal{C}$ for the subcategory $\mathcal{C} \times Z(\mathcal{C})$.

In order to make this formal it is necessary to equip $\mathcal{C}$ with a specified premonoidal centre from the outset. We call these premonoidal categories with specified centre *effectful* categories as in [138], and they have been also known elsewhere as non-cartesian Freyd categories [131, 130]. Effectful categories have a very close relationship with strong promonads (monads in the category of profunctors), and it is well-understood in the functional programming community (where they are known as *arrows*) that there is an equivalence between the two [102, 5]. Given a strong promonad $T : \mathcal{C} \nrightarrow \mathcal{C}$ there is a corresponding effectful category given by the canonical free functor $F : \mathcal{C} \to \mathsf{Kl}_T$ into the Kleisli category of T. Conversely, given an effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$ there is a strong promonad given by $\mathcal{C}_1(J-, J-) : \mathcal{C}_0 \nrightarrow \mathcal{C}_0$.

Here, we will consider "pro-effectful" categories which weaken the structure of an effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$ such that $\mathcal{C}_0$ is only promonoidal and $\mathcal{C}_1$ is "pre-promonoidal". In particular, we will show that pro-effectful categories are equivalently:

- prostrong promonads,
- biproactegories (two-sided actions in the category of profunctors) which suitably extend a canonical action on the centre of the category,
- pseudomonoids in the bicategory of tight $\mathcal{V}^2$-profunctors.

Each of these gives a different perspective on pro-effectful categories, connecting them, respectively, with monads; the action definition of Freyd categories given by Levy [120]; the pseudomonoid definition of effectful categories given by Román [138]; and the work on closed effectful categories due to Power [131, 130]. In particular, this final perspective demonstrates that pro-effectful categories are equivalent to closed effectful categories on the tight cocompletion, where the effectful structure is given by a version of Day convolution.

6.1 Effectful Categories

When we defined premonoidal categories in Section 4.6, one may have been surprised that there was not a swift definition. Since the coherence isomorphisms (α, λ, ρ) need to be central we were required to define binoidal categories before it was even possible to define what it means for a morphism to be central. This meant a three-step definition: make $\mathcal{C}$ binoidal, define centrality, and ask for the existence of central natural isomorphisms satisfying the coherence conditions. Part of the reason for this long-winded route is that even in the case $\mathcal{V} = \mathsf{Set}$, Cat fails to be a monoidal 2-category under the funny tensor product because the funny tensor of natural transformations is not well-defined unless the components are all central. This prevents the swift and elegant definition "*a premonoidal category is a pseudomonoid in* $\mathsf{Cat}_\square$."

Power realised that premonoidal categories are more algebraically well-defined when one shifts to working with premonoidal categories *with specified centre* [131], which we call *effectful* categories as suggested in [138].

Definition 45 (Premonoidal Functor)**.** A premonoidal functor $F : \mathcal{C} \to \mathcal{D}$ between premonoidal categories is a functor which maps central morphisms to central morphisms and which preserves the premonoidal structure up to natural transformations $FA \otimes$

$FB \to F(A \otimes B)$ and $I \to FI$ subject to coherence conditions like those for a monoidal functor.

Definition 46 (Effectful Category)**.** An effectful category consists of a monoidal category $\mathcal{C}_0$, a premonoidal category $\mathcal{C}_1$ with the same objects as $\mathcal{C}_0$ and a strict, identity on objects, premonoidal functor $J : \mathcal{C}_0 \to \mathcal{C}_1$.

Remark. If $\mathcal{C}_0$ is cartesian then an effectful category is known as a Freyd category.

Example 13. Any premonoidal category $\mathcal{C}$ gives rise to an effectful category given by the embedding of the centre $Z(\mathcal{C}) \to \mathcal{C}$.

The next subsections describe three different equivalent presentations of effectful categories: as actegories due to Levy [120]; as strong promonads due to Jacobs, Heunen and Hasuo [102]; and as pseudomonoids which builds upon the work of Román [138]. For the latter, we show that effectful categories are pseudomonoids in the category $\mathcal{V}^2\text{-}\mathsf{Cat}_\square$ of $\mathcal{V}^2$-categories equipped with the funny tensor product and explain that this is equivalent to the result of Román due to an equivalence $\mathcal{V}^2\text{-}\mathsf{Cat}_\square \cong \mathsf{Promonad}$ with the category of promonads.

Effectful Categories as Actegories

Effectful categories can be seen as particular instances of actegories - that is, a category with an action by a monoidal category.

Example 14. [120]. An effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$ specifies a left and right $\mathcal{C}_0$-action on $\mathcal{C}_1$, making $\mathcal{C}_1$ into a $\mathcal{C}_0$-$\mathcal{C}_0$-biactegory. An effectful category is equivalently the following data:

- a monoidal category $(\mathcal{C}_0, \otimes)$,
- a category $\mathcal{C}_1$ with the same objects as $\mathcal{C}_0$ and an identity on objects functor $J : \mathcal{C}_0 \to \mathcal{C}_1$,
- a left $\mathcal{C}_0$-action on $\mathcal{C}_1$, $\ltimes : \mathcal{C}_0 \boxtimes \mathcal{C}_1 \to \mathcal{C}_1$, which preserves the canonical left $\mathcal{C}_0$-action on $\mathcal{C}_0$, i.e. J extends the action $J\otimes = \ltimes(1 \boxtimes J)$ and preserves the coherence isomorphisms,
- a right $\mathcal{C}_0$-action on $\mathcal{C}_1$, $\rtimes : \mathcal{C}_1 \boxtimes \mathcal{C}_0 \to \mathcal{C}_1$, which preserves the canonical right $\mathcal{C}_0$-action on $\mathcal{C}_0$,

- a natural isomorphism $C \ltimes (D \rtimes C') \cong (C \ltimes D) \rtimes C'$ making the actions $\ltimes$ and $\rtimes$ into a biaction.

As a result an effectful category is a (particular) representation of $\mathfrak{M}$ in $\mathcal{V}$-Cat. This definition is often the easiest to explicitly work with, particularly when performing calculations, as it spells out all the required data and removes any difficulty around the funny tensor product. It will be useful for us later in a number of proofs.

Effectful Categories as Strong Promonads

It turns out that effectful categories are precisely the same thing as strong promonads. While a monad is a monoid in the category of endofunctors, a promonad is a monoid in the category of endoprofunctors. Unpacking this yields the following definition:

Definition 47 (Promonad)**.** A promonad is a triple (T, μ, η) of a profunctor $T : \mathcal{C} \nrightarrow \mathcal{C}$, a natural transformation $\mu : T^2 = T \circ T \to T$ and a natural transformation $\eta : 1 \to T$ such that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

Example 15. Let $T : \mathcal{C} \to \mathcal{C}$ be a monad. Then the contravariant Yoneda embedding $\mathcal{C}(-, T=) : \mathcal{C} \nrightarrow \mathcal{C}$ yields a promonad on $\mathcal{C}$.

Conceptually it may be useful to think of a promonad T as a generalised hom-functor: for each pair of objects A, B there is an object $T(A, B)$ of "arrows" from A to B together with a composition rule given by μ and a unit rule given by η. In terms of our spacetime models, we can think of $T(A, B)$ of assigning a generalised set of paths from A to B, a set of paths that is not necessarily a part of our original category but which is functorially and compositionally well-behaved over it. We will return to this idea later after introducing the remainder of the necessary theory.

Any promonad can also be viewed as a cocontinuous monad because of the equivalence $\mathcal{V}$-$\mathsf{Prof} \cong \mathcal{V}$-$\mathsf{Cocont}$. Given a promonad T, we can view T instead as a cocontinuous functor $\hat{T} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$. The promonad laws of T can then be shown to make $\hat{T}$ into a monad. Conversely, any cocontinuous monad $\hat{T} : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ is equivalently a promonad $T : \mathcal{C} \nrightarrow \mathcal{C}$.

The bicategory $\mathcal{V}$-Prof permits the Kleisli construction for monads [153] so that we can assign any promonad a Kleisli category.

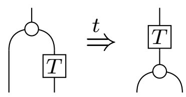
Definition 48 (Kleisli category)**.** Let $T : \mathcal{C} \nrightarrow \mathcal{C}$ be a promonad. The Kleisli category Kl_T has the same objects as $\mathcal{C}$ and hom-objects $\mathsf{Kl}_T(A, B) = T(A, B)$. Composition is induced by the multiplication μ of the promonad, and units by η.

The unit η of the promonad T induces an identity on objects functor $F : \mathcal{C} \to \mathsf{Kl}_T$. Moreover, given any identity on objects functor $J : \mathcal{C} \to \mathcal{D}$ one can construct a promonad $\mathcal{D}(J-, J=) : \mathcal{C} \nrightarrow \mathcal{C}$. These constructions are mutually inverse so that $\mathsf{Kl}_T(F-, F=) = T(-, =)$.

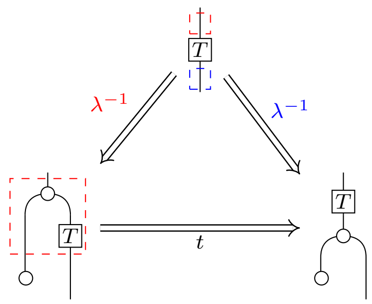
Example 16. For simplicity take $\mathcal{V} = \mathsf{Set}$. Let $T : \mathcal{C} \to \mathcal{C}$ be a monad and write $T' = \mathcal{C}(-, T=) : \mathcal{C} \nrightarrow \mathcal{C}$ for the induced promonad. The Kleisli category of T' has the same objects as $\mathcal{C}$ while an arrow $A \to B$ is an element of $\mathcal{C}(A, TB)$, that is an arrow $A \to TB$. We can see that $\mathsf{Kl}_{T'}$ coincides with the standard Kleisli category Kl_T of the monad T.

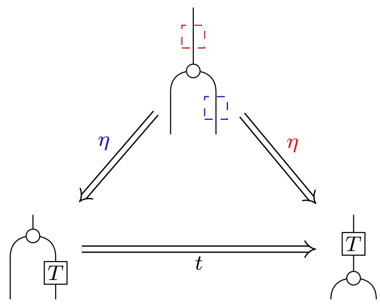
Now when $\mathcal{C}$ is promonoidal one can define a prostrength for a promonad on $\mathcal{C}$ so as to ensure that the promonad behaves compatibly with the promonoidal structure.

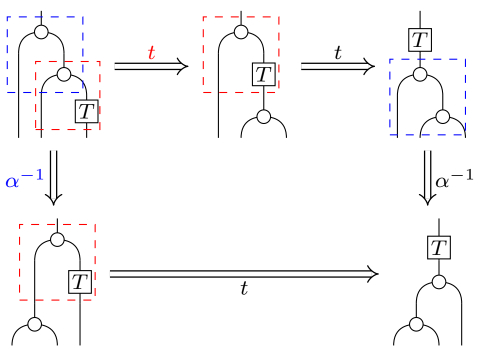
Definition 49 (Prostrong Promonad)**.** Suppose $\mathcal{C}$ is a promonoidal category and suppose (T, μ, η) is a promonad on $\mathcal{C}$. A left prostrength for T is a natural transformation

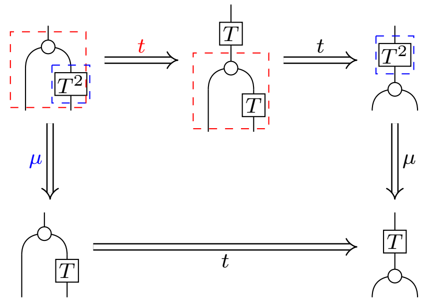


such that the following diagrams commute (ignoring interchange isomorphisms in $\mathcal{V}$-Prof).

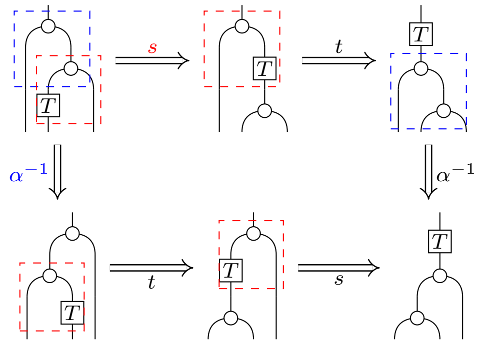






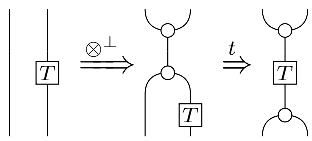


A right prostrength s is defined analogously and we say that a promonad is prostrong if it is equipped with left and right prostrengths such that the two evident maps agree:



When $\mathcal{C}$ is monoidal (so that the promonoidal structure is representable) we recover the notion of a *strong* promonad. These are also known as "arrows" because they axiomatise such objects in functional programming [5, 102].

Given a strong promonad $T : \mathcal{C} \nrightarrow \mathcal{C}$ there is a canonical premonoidal structure on the Kleisli category Kl_T which on objects acts like the tensor of $\mathcal{C}$. On morphisms we can use the left and right strengths to give left and right actions of $\mathcal{C}$ on Kl_T. In particular we have a natural transformation:



which gives the left action $\mathcal{C} \boxtimes \mathsf{Kl}_T \to \mathsf{Kl}_T$. Similarly one can construct the right action. Furthermore, one can check that the free functor $F : \mathcal{C} \to \mathsf{Kl}_T$ is identity on objects and premonoidal, thus F is an effectful category.

Conversely, given an effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$, there is a promonad $\mathcal{C}_1(J-, J-) : \mathcal{C}_0 \nrightarrow \mathcal{C}_0$. This promonad can be shown to be strong with strengths induced by the action of $\mathcal{C}_0$ on $\mathcal{C}_1$. In summary,

Theorem 13 ([102, 83])**.** *To give a strong promonad $T : \mathcal{C} \nrightarrow \mathcal{C}$ is to give an effectful category $F : \mathcal{C} \to \mathsf{Kl}_T$.*

Effectful Categories as Pseudomonoids

In this section, we show that effectful categories are pseudomonoids in the category of $\mathcal{V}^2$-enriched categories equipped with a modified version of the funny tensor product, where $\mathcal{V}^2 = [\to, \mathcal{V}]$ is the category of arrows and commutative squares in $\mathcal{V}$. In doing so we place effectful categories on the same footing as monoidal and promonoidal categories, showing that they are representations of the same algebraic data $\mathfrak{P}$. This builds upon the work of Power who first studied the algebraicity of effectful categories in $\mathcal{V}^2$-Cat [131, 130].

Proposition 36. *Let $\mathcal{V}$ be a complete, cocomplete, closed symmetric monoidal category. Then $\mathcal{V}^2$ is also a complete, cocomplete, closed symmetric monoidal category and therefore constitutes a cosmos.*

Proof. The category $\mathcal{V}^2$ inherits a symmetric monoidal structure from $\mathcal{V}$. On objects, which are arrows of $\mathcal{V}$, this monoidal structure acts by

$$(a_0 \xrightarrow{f} a_1) \boxtimes (b_0 \xrightarrow{g} b_1) := a_0 \boxtimes b_0 \xrightarrow{f \boxtimes g} a_1 \boxtimes b_1.$$

On morphisms, which are squares of $\mathcal{V}$, it acts in an analogous way.

Now, consider three objects of $\mathcal{V}^2$, say $a_0 \xrightarrow{f} a_1$, $b_0 \xrightarrow{g} b_1$ and $c_0 \xrightarrow{h} c_1$. We aim to construct the internal-hom and demonstrate the natural isomorphism $\mathcal{V}^2(f \boxtimes g, h) \cong \mathcal{V}^2(f, [g,h])$. Consider the following pullback in $\mathcal{V}$, which exists because $\mathcal{V}$ is complete.

$$\begin{tikzcd}
\mathcal{V}(b_0,c_0) \times_{\mathcal{V}(b_0,c_1)} \mathcal{V}(b_1,c_1) \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{V}(b_0,c_0) \arrow[d, "\mathcal{V}(1,h)"] \\
\mathcal{V}(b_1,c_1) \arrow[r, "\mathcal{V}(g,1)"'] & \mathcal{V}(b_0,c_1)
\end{tikzcd}$$

Let us now demonstrate that the internal-hom is given by the projection out of the pullback, $[g,h] = p_1$. To give an arrow $z : f \to p_1$ of $\mathcal{V}^2$ is to give a pair of arrows such that the following square commutes,

$$\begin{tikzcd}
a_0 \arrow[r, "f"] \arrow[d, "z_0"'] & a_1 \arrow[d, "z_1"] \\
\mathcal{V}(b_0,c_0) \times_{\mathcal{V}(b_0,c_1)} \mathcal{V}(b_1,c_1) \arrow[r, "p_1"'] & \mathcal{V}(b_1,c_1)
\end{tikzcd} \tag{6.1}$$

To give z_0 is to give $z_{00} := p_0 z_0 : a_0 \to \mathcal{V}(b_0,c_0)$ and $z_{01} := p_1 z_0 : a_0 \to \mathcal{V}(b_1,c_1)$ such that $\mathcal{V}(1,h)z_{00} = \mathcal{V}(g,1)z_{01}$. To make (6.1) commute is to ask $z_{01} = z_1 f$.

Under the adjunction due to the closure of $\mathcal{V}$, we now have $z_{00}^* : a_0 \boxtimes b_0 \to c_0$ and $z_{01}^* : a_0 \boxtimes b_1 \to c_1$ such that $h z_{00}^* = z_{01}^*(1 \boxtimes g)$. Also from z_1 we get a $z_1^* : a_1 \boxtimes b_1 \to c_1$ such that $z_1^*(f \boxtimes 1) = z_{01}^*$. As a result we find $h z_{00}^* = z_1^*(f \boxtimes g)$, so that z is equivalent to giving $z^* : f \boxtimes g \to h$.

Finally, the completeness and cocompleteness of $\mathcal{V}^2$ are inherited from $\mathcal{V}$, pointwise. $\square$

Since $\mathcal{V}^2$ is a cosmos, we can consider categories enriched in $\mathcal{V}^2$ [130]. A $\mathcal{V}^2$-category $\mathcal{C}$ consists of a pair of categories $\mathcal{C}_0$ and $\mathcal{C}_1$ with the same objects and an identity on objects functor $J : \mathcal{C}_0 \to \mathcal{C}_1$. A $\mathcal{V}^2$-functor $F : \mathcal{C} \to \mathcal{D}$ consists of a pair of functors $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that the following square commutes:

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "J_{\mathcal{C}}"] \arrow[d, "F_0"'] & \mathcal{C}_1 \arrow[d, "F_1"] \\
\mathcal{D}_0 \arrow[r, "J_{\mathcal{D}}"'] & \mathcal{D}_1
\end{tikzcd}$$

A $\mathcal{V}^2$-natural transformation $\eta : F \Rightarrow G$ between $\mathcal{V}^2$-functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of natural transformations $\eta^0 : F_0 \Rightarrow G_0$ and $\eta^1 : F_1 \Rightarrow G_1$ with components that satisfy $J_{\mathcal{D}}(\eta^0_c) = \eta^1_c$. If $J_{\mathcal{D}}$ is an embedding then we can think of this transformation as having components in the centre $\mathcal{D}_0$.

There is a 2-category $\mathcal{V}^2$-Cat of $\mathcal{V}^2$-categories, $\mathcal{V}^2$-functors and $\mathcal{V}^2$-natural transformations. This 2-category has an interesting tensor that arises as a slight modification of the funny tensor product.

Definition 50 (Funny Tensor of $\mathcal{V}^2$-Categories)**.** Given two $\mathcal{V}^2$-categories $J_{\mathcal{C}}$ and $J_{\mathcal{D}}$, their funny tensor $J_{\mathcal{C}\square\mathcal{D}} : \mathcal{C}_0 \boxtimes \mathcal{D}_0 \to \mathcal{C}_1 \square \mathcal{D}_1$ is the identity on objects functor given by the diagonal of the following pushout in $\mathcal{V}$-Cat.

$$\begin{tikzcd}
\mathcal{C}_0 \boxtimes \mathcal{D}_0 \arrow[r, "J_{\mathcal{C}}\boxtimes 1"] \arrow[d, "1\boxtimes J_{\mathcal{D}}"'] & \mathcal{C}_1 \boxtimes \mathcal{D}_0 \arrow[d, "i_0"] \\
\mathcal{C}_0 \boxtimes \mathcal{D}_1 \arrow[r, "i_1"'] \arrow[ur, phantom, "\ulcorner" very near end] & \mathcal{C}_1 \square \mathcal{D}_1
\end{tikzcd} \tag{6.2}$$

The pushout exists because $\mathcal{V}$ is cocomplete and thus $\mathcal{V}$-Cat is also cocomplete [162]. Given $\mathcal{V}^2$-functors $F : J_{\mathcal{A}} \to J_{\mathcal{B}}$ and $G : J_{\mathcal{C}} \to J_{\mathcal{D}}$ their funny tensor $F \square G$ has components $(F \square G)_0 = F_0 \boxtimes G_0$ and $(F \square G)_1 = F_1 \square G_1$ given by the unique arrow induced by the pushout. The funny tensor is also well-behaved on $\mathcal{V}^2$-natural transformations because their components $J_{\mathcal{D}}(\eta^0_c) = \eta^1_c$ are central and thus interchange with all other morphisms in $\mathcal{C} \square \mathcal{D}$.

As a result we find that

Theorem 14. $\mathcal{V}^2$-Cat *is a monoidal 2-category under the funny tensor* $\square$*.*

Proof sketch. The behaviour of the funny tensor on functors is encapsulated by the following cube.

$$
\begin{tikzcd}
 & \mathcal{B}_0 \boxtimes \mathcal{D}_0 \arrow[rr] \arrow[dd] & & \mathcal{B}_1 \boxtimes \mathcal{D}_0 \arrow[dd] \\
\mathcal{A}_0 \boxtimes \mathcal{C}_0 \arrow[ur, "F_0 \boxtimes G_0"] \arrow[rr] \arrow[dd] & & \mathcal{A}_1 \boxtimes \mathcal{C}_0 \arrow[ur, "F_1 \boxtimes G_0"'] \arrow[dd] & \\
 & \mathcal{B}_0 \boxtimes \mathcal{D}_1 \arrow[rr] & & \mathcal{B}_1 \,\square\, \mathcal{D}_1 \arrow[ul, phantom, very near start, "\ulcorner"] \\
\mathcal{A}_0 \boxtimes \mathcal{C}_1 \arrow[ur, "F_0 \boxtimes G_1"] \arrow[rr] & & \mathcal{A}_1 \,\square\, \mathcal{C}_1 \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[ur, dashed, "F_1 \square G_1"'] &
\end{tikzcd}
$$

Functoriality of $\square$ on 1-cells follows by pasting of cubes and the uniqueness of the arrows induced by the pushout.

Explicitly, we have $(\alpha : F \Rightarrow F') \,\square\, (\beta : G \Rightarrow G')$ has components $(\alpha \,\square\, \beta)^0_{cd} = (\alpha^0_c, \beta^0_d)$ and $(\alpha \,\square\, \beta)^1_{cd} = (\alpha^1_c, \beta^1_d) = (J\alpha^0_c, J\beta^0_d)$. Naturality of this transformation follows from naturality of α and β and from the centrality of the components. $\square$

In fact, as a 1-category $\mathcal{V}^2$-Cat is closed monoidal.

Proposition 37. *$\mathcal{V}^2$-Cat is a closed monoidal category where the internal-hom is given by the inclusion $[\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]_u$ of the category of $\mathcal{V}^2$-functors and $\mathcal{V}^2$-natural transformations into the category of $\mathcal{V}^2$-functors and $\mathcal{V}^2$-unnatural transformations.*

This leads to the main theorem of this section.

Theorem 15. *An effectful category is a pseudomonoid (a representation of $\mathfrak{P}$) in $\mathcal{V}^2$-$\mathsf{Cat}_\square$.*

Proof. A pseudomonoid in $\mathcal{V}^2$-$\mathsf{Cat}_\square$ consists of a $\mathcal{V}^2$-category $J : \mathcal{C}_0 \to \mathcal{C}_1$ equipped with $\mathcal{V}^2$-functors $\otimes : J \,\square\, J \to J$ and $I : 1 \to J$, such that there are $\mathcal{V}^2$-natural isomorphisms

$$\otimes(\otimes \boxtimes 1) \overset{\alpha}{\cong} \otimes(1 \boxtimes \otimes) \text{ and } \otimes(I \boxtimes 1) \overset{\lambda}{\cong} 1 \overset{\rho}{\cong} \otimes(1 \boxtimes I).$$

Note that $\otimes$ consists of two functors $\otimes_0 : \mathcal{C}_0 \boxtimes \mathcal{C}_0 \to \mathcal{C}_0$ and $\otimes_1 : \mathcal{C}_1 \,\square\, \mathcal{C}_1 \to \mathcal{C}_1$ such that $J\otimes_0 = \otimes_1 J_{\mathcal{C}\square\mathcal{C}}$. $\otimes_0$ together with I_0 and the natural isomorphisms α_0, ρ_0 and λ_0, give a monoidal structure on $\mathcal{C}_0$.

The $\mathcal{C}_0$-biaction on $\mathcal{C}_1$ is given by the compositions $\ltimes := \otimes i_1$ and $\rtimes := \otimes i_0$. That J preserves the canonical actions given by $\otimes_0$ on $\mathcal{C}_0$ follows by the diagram (6.2) and the equality $J\otimes_0 = \otimes_1 J_{\mathcal{C}\square\mathcal{C}}$, together with the fact that α_1, ρ_1 and λ_1 have components in the image of J. The coherence equations of the biaction are a consequence of those of α_1, ρ_1 and λ_1: for instance α_1 is a natural isomorphism between functors with type $\mathcal{C}_1 \square \mathcal{C}_1 \square \mathcal{C}_1 \to \mathcal{C}_1$. This amounts to "separate" naturality in each $\mathcal{C}_1$ of the domain which in turns induces the left, bimodule and right coherences for the biaction. $\square$

Theorem 15 is equivalent to the result of [138] where it is shown that effectful categories are pseudomonoids in the 2-category of promonads, promonad homomorphisms and promonad modifications. In fact we have:

Theorem 16. *There is an equivalence of 2-categories* $\mathcal{V}^2\text{-}\mathsf{Cat}_\square \cong \mathcal{V}\text{-}\mathsf{Promonad}$ *between the 2-category of* $\mathcal{V}^2$*-categories under the funny tensor product and the 2-category of promonads.*

Proof sketch. The result follows upon unwinding the definitions in [138] and comparing with those of the present section. $\square$

6.2 Closed Effectful Categories

Now that we have a thorough understanding of effectful categories, we can start to work towards their "pro-" analogue. To start, recall that a promonoidal category is equivalently a *closed* monoidal presheaf category. This suggests we should turn our attention to the closure of effectful categories, which will be the focus of this Section.

Power gave the following definition of closure for effectful categories, where there is still an adjunction between tensoring and the internal-hom, but only for the centre [131].

Definition 51 (Closed Effectful Category)**.** An effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$ is right-closed when for each object X, $J(-) \otimes X : \mathcal{C}_0 \to \mathcal{C}_1$ has a right adjoint $[X, -] : \mathcal{C}_1 \to \mathcal{C}_0$. An effectful category is left-closed when for each X, $X \otimes J(-) : \mathcal{C}_0 \to \mathcal{C}_1$ has a right adjoint. We say an effectful category is closed if it is both left and right-closed.

Power proved the following result which generalises Day's result that every monoidal category embeds into a closed monoidal category [68].

Theorem 17 ([131])**.** *Every (small) effectful category embeds into a closed effectful category.*

We say that an effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$ is *small* when both $\mathcal{C}_0$ and $\mathcal{C}_1$ are small. Given small J, we can take the strong promonad $T(-,-) := \mathcal{C}_1(J-, J-) : \mathcal{C}_0 \nrightarrow \mathcal{C}_0$ and lift it to a strong monad on the presheaf category $\widehat{T} : \widehat{\mathcal{C}_0} \to \widehat{\mathcal{C}_0}$. The Kleisli category $\mathsf{Kl}_{\widehat{T}}$ has as objects presheaves $F : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{V}$ and homs $\mathsf{Kl}_{\widehat{T}}(F, G) = \widehat{\mathcal{C}_0}(F, \widehat{T}G)$.

Moreover $\widehat{\mathcal{C}_0}$ is monoidal under Day convolution while $\mathsf{Kl}_{\widehat{T}}$ is premonoidal. As a result there is an effectful category given by the identity on objects functor $\widehat{\mathcal{C}_0} \to \mathsf{Kl}_{\widehat{T}}$.

Power gave another characterisation of the effectful category $\widehat{\mathcal{C}_0} \to \mathsf{Kl}_{\widehat{T}}$ as the free *tight* cocompletion of the $\mathcal{V}^2$-category $J : \mathcal{C}_0 \to \mathcal{C}_1$ - that is, the cocompletion in only $\mathcal{V}$-colimits, not all $\mathcal{V}^2$-colimits. In the case of $\mathcal{V} = \mathsf{Set}$ these are precisely the "conical" colimits. The name "tight" was first suggested in [114] where the theory of categories enriched in Cat^2 is studied in some detail.

Theorem 18 ([130])**.** *The free tight cocompletion of a small $\mathcal{V}^2$-category $J : \mathcal{C}_0 \to \mathcal{C}_1$ is the bijective on objects functor $Lan^L_{J^{op}} : \widehat{\mathcal{C}_0} \to \overline{\mathcal{C}_1}$ induced by the functor of $Lan_{J^{op}} : \widehat{\mathcal{C}_0} \to \widehat{\mathcal{C}_1}$, via its canonical factorisation into a bijective on objects functor followed by a fully faithful functor (its bo-ff factorisation).*

Note that the bo-ff factorisation of any functor can be constructed in the following fashion.

Proposition 38. *Any $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ factorises as the composition of a bijective-on-objects functor and a fully-faithful functor.*

Proof. Define $\operatorname{Im} F$ to be the $\mathcal{V}$-category with objects given by those of $\mathcal{C}$ and hom-objects given by $\operatorname{Im} F(A,B) := \mathcal{D}(FA, FB)$. Its composition and identities are inherited from $\mathcal{D}$. The $\mathcal{V}$-functor F now factorises as the composition of $F^L \colon \mathcal{C} \to \operatorname{Im} F$ and $F^R \colon \operatorname{Im} F \to \mathcal{D}$, where the $\mathcal{V}$-functor F^L is identity on objects and on homs, $F^L_{AB} : \mathcal{C}(A,B) \to \operatorname{Im} F(A,B)$ is given by F_{AB}; and the $\mathcal{V}$-functor F^R acts as F on objects, and on homs, $F^R_{AB} : \operatorname{Im} F(A,B) \to \mathcal{D}(FA,FB)$ is given by $1_{\mathcal{D}(FA,FB)}$. □

As a result, the category $\overline{\mathcal{C}_1} := \operatorname{Im}(\operatorname{Lan}_{J^{\mathrm{op}}})$ of Theorem 18 has as objects presheaves $F : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{V}$ and homs $\overline{\mathcal{C}_1}(F,G) = \widehat{\mathcal{C}_1}(\operatorname{Lan}_{J^{\mathrm{op}}} F, \operatorname{Lan}_{J^{\mathrm{op}}} G)$. By the adjunction between extension and restriction of presheaves along J there is a natural isomorphism

$$\mathsf{Kl}_{\widehat{T}}(F,G) = \widehat{\mathcal{C}_0}(F, \widehat{T}G) \cong \widehat{\mathcal{C}_1}(\operatorname{Lan}_{J^{\mathrm{op}}} F, \operatorname{Lan}_{J^{\mathrm{op}}} G) = \overline{\mathcal{C}_1}(F,G)$$

To see this explicitly, firstly note the following.

$$(\widehat{T}G)(-) \cong \int^X T(-,X) \boxtimes GX = \int^X \mathcal{C}_1(J-, JX) \boxtimes GX \cong (\operatorname{Lan}_{J^{\mathrm{op}}} G)(J-)$$

Then we can demonstrate the adjunction between extension and restriction of presheaves by some coend calculus.

$$\begin{aligned}
\mathrm{Nat}(F, \widehat{T}G) &\cong \mathrm{Nat}(F, (\mathrm{Lan}_{J^{\mathrm{op}}}G)(J-)) \\
&\cong \int_X \mathcal{V}\left(FX, (\mathrm{Lan}_{J^{\mathrm{op}}}G)(JX)\right) \\
&\cong \int_X \mathcal{V}\left(FX, \int_Y \mathcal{V}\big(\mathcal{C}_1(Y, JX), (\mathrm{Lan}_{J^{\mathrm{op}}}G)(Y)\big)\right) \\
&\cong \int_{XY} \mathcal{V}\big(FX \boxtimes \mathcal{C}_1(Y, JX), (\mathrm{Lan}_{J^{\mathrm{op}}}G)(Y)\big) \\
&\cong \int_Y \mathcal{V}\left(\int^X \mathcal{C}_1(Y, JX) \boxtimes FX, (\mathrm{Lan}_{J^{\mathrm{op}}}G)(Y)\right) \\
&\cong \mathrm{Nat}(\mathrm{Lan}_{J^{\mathrm{op}}}F, \mathrm{Lan}_{J^{\mathrm{op}}}G)
\end{aligned}$$

Thus to give a natural transformation $F \Rightarrow \widehat{T}G$ is equivalent to giving one $\mathrm{Lan}_{J^{\mathrm{op}}}F \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}}}G$. This demonstrates an isomorphism $\overline{\mathcal{C}_1} \cong \mathsf{Kl}_{\widehat{T}}$

As a consequence of Theorem 18, the following diagram commutes, giving a factorisation (y^L, y^R) of the $\mathcal{V}^2$-enriched Yoneda embedding $y : J \to [J^{\mathrm{op}}, \mathcal{V}^2]$, via the free $\mathcal{V}$-cocompletion.

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "J"] \arrow[d, "y_0^L"] \arrow[dd, bend right=60, "y_0"'] & \mathcal{C}_1 \arrow[d, "y_1^L"] \arrow[dd, bend left=60, "y_1"] \\
\widehat{\mathcal{C}}_0 \arrow[r, "\mathrm{Lan}^L_{J^{\mathrm{op}}}"] \arrow[d, "y_0^R"] & \overline{\mathcal{C}_1} \arrow[d, "y_1^R"] \\
{[J^{\mathrm{op}}, \mathcal{V}^2]_0} \arrow[r] & {[J^{\mathrm{op}}, \mathcal{V}^2]_1}
\end{tikzcd}$$

So we now have an effectful category $\mathrm{Lan}^L_{J^{\mathrm{op}}}$ into which J embeds. The last thing to do is to check that it is closed, which follows by noting that $\mathrm{Lan}_{J^{\mathrm{op}}}$ is left adjoint to the functor which restricts presheaves along J, and taking bo-ff factorisations ensures that $\mathrm{Lan}^L_{J^{\mathrm{op}}}$ is also a left adjoint [132].

6.3 $\mathcal{V}^2$-Profunctors

In the previous Section we studied the notion of closure for effectful categories. At this point we could stop and define "pro-effectful" categories as "closed effectful presheaf categories" in analogy to promonoidal categories. In fact, this definition is more subtle than it might first appear and requires a little care. In particular, given that the closed effectful embedding of any effectful category J is given by the free *tight* cocompletion $\mathrm{Lan}^L_{J^{\mathrm{op}}}$ and not the free cocompletion $[J^{\mathrm{op}}, \mathcal{V}^2]$, we must take care of what we mean by "presheaf" category here. Furthermore, we would like to place pro-effectful categories on the same footing as promonoidal categories - as pseudomonoids in some form of bicategory of profunctors.

This will be the aim of this Section; to study the structure of $\mathcal{V}^2$-profunctors $P : J_{\mathcal{D}}^{\mathrm{op}} \boxtimes J_{\mathcal{C}} \to \mathcal{V}^2$. By the following result we are able to unpack P into a pair of $\mathcal{V}$-profunctors together with a natural transformation between them. The $\mathcal{V}^2$-natural transformations $\phi : P \Rightarrow Q$ can also be similarly unpacked.

Proposition 39. *Let $P : J_{\mathcal{D}}^{op} \boxtimes J_{\mathcal{C}} \to \mathcal{V}^2$ be a $\mathcal{V}^2$-profunctor. Then P is a triple of:*

1. *a $\mathcal{V}$-profunctor $P_0 : \mathcal{D}_0^{op} \boxtimes \mathcal{C}_0 \to \mathcal{V}$,*
2. *a $\mathcal{V}$-profunctor $P_1 : \mathcal{D}_1^{op} \boxtimes \mathcal{C}_1 \to \mathcal{V}$,*
3. *a $\mathcal{V}$-natural transformation $\eta : P_0 \Rightarrow P_1(J^{op} \boxtimes J)$.*

A $\mathcal{V}^2$-natural transformation $\phi : P \Rightarrow Q$ consists of $\mathcal{V}$-natural transformations $\phi_0 : P_0 \Rightarrow Q_0$ and $\phi_1 : P_1 \Rightarrow Q_1$ such that $(\phi_1(J^{op} \boxtimes J))\eta^P = \eta^Q \phi_0$.

Proof. This follows by applying a $\mathcal{V}$-enriched version of a result by Power [130, Prop. 24] to the functor category $[J_{\mathcal{D}}^{\mathrm{op}} \boxtimes J_{\mathcal{C}}, \mathcal{V}^2] \cong \mathsf{Prof}(J_{\mathcal{C}}, J_{\mathcal{D}})$. □

The next proposition demonstrates that the coend of a $\mathcal{V}^2$-profunctor P is given by the coends of P_0 and P_1 together with a canonical arrow between them.

Proposition 40. *Let $P : J^{op} \boxtimes J \to \mathcal{V}^2$ be a $\mathcal{V}^2$-endoprofunctor. Then the coend $\int^C P(C, C)$ is given by the arrow $\int^C P_0(C, C) \to \int^c P_1(C, C)$ induced by η and the adjunction $y_J \dashv y^J$ in $\mathcal{V}$-Prof.*

Proof. Suppose we have a $\mathcal{V}^2$-extranatural family $w_C : P(C, C) \to D$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C}_1(C,C') \boxtimes P_1(C',C) \arrow[rrr] \arrow[dddd] & & & P_1(C',C') \arrow[dddd, "w^1_{C'}"] \\
& & & \\
& \mathcal{C}_0(C,C') \boxtimes P_0(C',C) \arrow[r] \arrow[d] \arrow[uul, "J\boxtimes\eta_{C'C}"'] & P_0(C',C') \arrow[d, "w^0_{C'}"] \arrow[uur, "\eta_{C'C'}"] & \\
& P_0(C,C) \arrow[r, "w^0_C"'] \arrow[dl, "\eta_{CC}"] & D_0 \arrow[dr, "D"] & \\
P_1(C,C) \arrow[rrr, "w^1_C"'] & & & D_1
\end{tikzcd}$$

In particular, the families $w^0_C : P_0(C,C) \to D_0$ and $w^1_C : P_1(C,C) \to D_1$ are $\mathcal{V}$-extranatural and thus factorise via their respective coends giving arrows $\int^C P_0(C,C) \to D_0$ and $\int^C P_1(C,C) \to D_1$ making the obvious diagrams commute. Now note that the arrows $P_0(C,C) \xrightarrow{\eta_{CC}} P_1(C,C) \xrightarrow{\mathrm{copr}_C} \int^c P_1(C,C)$ are $\mathcal{V}$-extranatural, this induces a arrow $\int^C P_0(C,C) \to \int^C P_1(C,C)$. $\square$

$\mathcal{V}^2$-endoprofunctors and the $\mathcal{V}^2$-natural transformations assemble into a $\mathcal{V}^2$-category $[J^{\mathrm{op}} \boxtimes J, \mathcal{V}^2] \cong \mathsf{Prof}(J,J)$. The category $\mathsf{Prof}(J,J)_0$ consists of the $\mathcal{V}^2$-profunctors and $\mathcal{V}^2$-natural transformations as outlined in Proposition 39, while $\mathsf{Prof}(J,J)_1$ has homs consisting of only the components ϕ_1 of the natural transformations. The identity on objects functor $\mathsf{Prof}(J,J)_0 \to \mathsf{Prof}(J,J)_1$ forgets the ϕ_0 components.

As with any other category of endoprofunctors $\mathsf{Prof}(J,J)$ has a closed monoidal structure given by composition of the profunctors. Given $P = (P_0, P_1, \eta_P)$ and $Q = (Q_0, Q_1, \eta_Q)$, their composition is given by $QP = (Q_0P_0, Q_1P_1, \eta_{QP})$ - we compose the underlying profunctors and take η_{QP} to be given by

$$\int^C Q(-,C) \boxtimes P(C,-) \xRightarrow{\int \eta_Q \boxtimes \eta_P} \int^{C \in \mathcal{C}_0} Q(J-,JC) \boxtimes P(JC,J-)$$

$$\xRightarrow{y_J \dashv y^J} \int^{C \in \mathcal{C}_1} Q(J-,C) \boxtimes P(C,J-)$$

6.3.1 Tight Profunctors

In Section 6.2 we saw that effectful structure on $J : \mathcal{C}_0 \to \mathcal{C}_1$ induced a closed effectful structure on the free tight cocompletion of J. It turns out that this effectful structure on J is only a sufficient and not necessary condition for closed effectful structure on the free tight cocompletion of J. Analogously to the case of monoidal categories where, in order for the presheaf category $\widehat{\mathcal{C}}$ to be closed monoidal it is only necessary that the category $\mathcal{C}$ is promonoidal [68, 65], we only require J to be a "pro-effectful" category. To define these categories we need firstly to study the class of profunctors which factor through the tight cocompletion. This will be the aim of this section.

To define pro-effectful categories we would like to replace the functors of a effectful category with profunctors, but we have a problem: we cannot consider arbitrary $\mathcal{V}^2$-profunctors $P : J^{\mathrm{op}}_{\mathcal{D}} \otimes J_{\mathcal{C}} \to \mathcal{V}^2$ because these assign arbitrary presheaves $J^{\mathrm{op}}_{\mathcal{D}} \to \mathcal{V}^2$ to objects of $J_{\mathcal{C}}$. These presheaves will not in general be contained in the free tight cocompletion. Thus, we need a restricted class of profunctors, those that we call the *tight* profunctors.

Definition 52 (Tight $\mathcal{V}^2$-Profunctor)**.** A tight $\mathcal{V}^2$-profunctor $P : J_\mathcal{C} \nrightarrow J_\mathcal{D}$ is a $\mathcal{V}^2$-functor $P : J_\mathcal{C} \to \overline{J_\mathcal{D}}$, where $\overline{J_\mathcal{D}} \cong \mathrm{Lan}^L_{J^{\mathrm{op}}_\mathcal{D}}$ is the free tight cocompletion of $J_\mathcal{D}$.

Tight $\mathcal{V}^2$-profunctors can be unpacked component-wise analogously to Proposition 39, to see that they are precisely the $\mathcal{V}^2$-profunctors where η is a natural *isomorphism.*

Proposition 41. *Let $P : J^{op}_\mathcal{D} \otimes J_\mathcal{C} \to \mathcal{V}^2$ be a tight $\mathcal{V}^2$-profunctor. Then P is a triple of:*

1. *a $\mathcal{V}$-profunctor $P_0 : \mathcal{D}^{op}_0 \boxtimes \mathcal{C}_0 \to \mathcal{V}$,*
2. *a $\mathcal{V}$-profunctor $P_1 : \mathcal{D}^{op}_1 \boxtimes \mathcal{C}_1 \to \mathcal{V}$,*
3. *a $\mathcal{V}$-natural isomorphism $\eta : P_0 \Rightarrow P_1(J^{op} \boxtimes J)$.*

Similarly to how a profunctor $P : \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently a cocontinuous functor between free cocompletions $\widehat{P} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$, tight $\mathcal{V}^2$-profunctors are *tightly* cocontinuous functors between free tight cocompletions.

Definition 53 (Tightly Cocontinuous Functor)**.** A $\mathcal{V}^2$-functor $F : J_\mathcal{C} \to J_\mathcal{D}$ between tightly cocomplete categories is tightly cocontinuous if it preserves all tight colimits.

Theorem 19 ([108])**.** *Let $\overline{J_\mathcal{C}}$ be the closure of $J_\mathcal{C}$ in $[J^{op}_\mathcal{C}, \mathcal{V}^2]$ under tight colimits and write $y^L : J_\mathcal{C} \to \overline{J_\mathcal{C}}$ for the inclusion. Then for tightly cocomplete $J_\mathcal{D}$, there is an equivalence*

$$Lan_{y^L} : [J_\mathcal{C}, J_\mathcal{D}] \cong \mathsf{Cocont}_{Tight}(\overline{J_\mathcal{C}}, J_\mathcal{D})$$

where the right-hand is the category of tightly cocontinuous functors. This exhibits $\overline{J_\mathcal{C}}$ as the free tight cocompletion of $J_\mathcal{C}$.

Indeed, y^L is fully faithful, so that there is a natural isomorphism $F \cong (\mathrm{Lan}_{y^L} F)y^L$. Consequently, we can think of a tight $\mathcal{V}^2$-profunctor $P : J_\mathcal{C} \to \overline{J_\mathcal{D}}$ as a tightly cocontinuous functor $\overline{P} : \overline{J_\mathcal{C}} \to \overline{J_\mathcal{D}}$. We can now define the following bicategory of tight $\mathcal{V}^2$-profunctors.

Definition 54. Denote by $\mathcal{V}^2$-$\mathsf{Prof}^{\mathsf{Tight}}$ the bicategory that has

- 0-cells the $\mathcal{V}^2$-categories $J : \mathcal{C}_0 \to \mathcal{C}_1$,
- 1-cells, $P : J_\mathcal{C} \nrightarrow J_\mathcal{D}$, the tight $\mathcal{V}^2$-profunctors $P : J_\mathcal{C} \to \overline{J_\mathcal{D}}$,
- 2-cells the $\mathcal{V}^2$-natural transformations.

Composition of 1-cells is given by taking the left Kan extension along y^L and composing the functors we obtain $Q \circ P = (\mathrm{Lan}_{y^L} Q)P$.

Remark. We could also have defined tight $\mathcal{V}^2$-profunctors $J_\mathcal{C} \to \overline{J_\mathcal{D}}$ as usual $\mathcal{V}^2$-profunctors $J_\mathcal{C} \to [J_\mathcal{D}^{\mathrm{op}}, \mathcal{V}^2]$ that factorise via the embedding $y^R : \overline{J_\mathcal{D}} \to [J_\mathcal{D}^{\mathrm{op}}, \mathcal{V}^2]$. Their usual composition as profunctors coincides (up to natural isomorphism) with the composition defined previously because y^R is fully faithful and thus the unit of the Kan extension along y^R is an isomorphism, $F \cong (\mathrm{Lan}_{y^R} F)y^R$. It follows that

$$\begin{aligned} Q \circ P = (\mathrm{Lan}_y Q)P = (\mathrm{Lan}_{y^R y^L} Q)P \cong (\mathrm{Lan}_{y^R}\mathrm{Lan}_{y^L} Q)P &= (\mathrm{Lan}_{y^R}\mathrm{Lan}_{y^L} Q)y^R P' \\ &\cong (\mathrm{Lan}_{y^L} Q)P'. \end{aligned}$$

There is a more abstract but cleaner way to define the bicategory $\mathcal{V}^2\text{-}\mathsf{Prof}^{\mathsf{Tight}}$, by noting that it is the Kleisli bicategory of a certain relative pseudomonad on $\mathcal{V}^2\text{-}\mathsf{Cat}$. Relative pseudomonads were introduced in [79] were it also demonstrated that Prof is the Kleilsi bicategory of the relative pseudomonad $\widehat{(\cdot)}$ of presheaves, which freely adds colimits by acting on 0-cells as $\mathcal{C} \mapsto \widehat{\mathcal{C}}$. Due to size issues, $\widehat{(\cdot)}$ is a relative pseudomonad and not just a plain pseudomonad: $\widehat{(\cdot)}$ sends small categories to locally small categories and so it is only a relative pseudomonad over the inclusion $\mathsf{Cat} \to \mathsf{CAT}$ of the 2-category of small categories into the 2-category of locally small categories.

In the same fashion there is a relative pseudomonad $\overline{(\cdot)}$ over the inclusion $\mathcal{V}^2\text{-}\mathsf{Cat} \to \mathcal{V}^2\text{-}\mathsf{CAT}$ which sends a small $\mathcal{V}^2$-category to its free tight cocompletion. It is then fairly straightforward to check that $\mathcal{V}^2\text{-}\mathsf{Prof}^{\mathsf{Tight}}$ is the Kleisli bicategory of this relative pseudomonad and therefore also check that it is indeed a bicategory.

Proposition 42 (External Tensor Product)**.** *Let $J_\mathcal{C}$ and $J_\mathcal{D}$ be $\mathcal{V}^2$-categories and write $\overline{J_\mathcal{C}}$ and $\overline{J_\mathcal{D}}$ be their free tight cocompletions. Then there is a $\mathcal{V}^2$-functor*

$$\hat{\otimes} : \overline{J_\mathcal{C}} \,\square\, \overline{J_\mathcal{D}} \to \overline{J_{\mathcal{C}\square\mathcal{D}}} \tag{6.3}$$

with components that act on objects as $(F \hat{\otimes} G)(C, D) := FC \otimes GD$.

Proof. To give (6.3) is to give a pair of functors such that the following square commutes:

$$\begin{tikzcd} \widehat{\mathcal{C}_0} \boxtimes \widehat{\mathcal{D}_0} \arrow[r] \arrow[d, "\hat{\otimes}_0"'] & \overline{\mathcal{C}_1} \,\square\, \overline{\mathcal{D}_1} \arrow[d, "\hat{\otimes}_1"] \\ \widehat{\mathcal{C}_0 \boxtimes \mathcal{D}_0} \arrow[r, "\mathrm{Lan}^L_{J^{\mathrm{op}}_{\mathcal{C}\square\mathcal{D}}}"'] & \overline{\mathcal{C}_1 \,\square\, \mathcal{D}_1} \end{tikzcd}$$

The tensor $\hat{\otimes}_0$ acts on objects as $(F\hat{\otimes}G)(c,d) := FC \boxtimes GD$ and on morphisms in the obvious way. The tensor $\hat{\otimes}_1$ also acts following the same formula, note that a morphism of $\overline{\mathcal{C}_1} \,\square\, \overline{\mathcal{D}_1}$ is a free composition of natural transformations $\alpha : \mathrm{Lan}_{J_{\mathcal{C}}^{\mathrm{op}}} F \Rightarrow \mathrm{Lan}_{J_{\mathcal{C}}^{\mathrm{op}}} F'$ and $\beta : \mathrm{Lan}_{J_{\mathcal{D}}^{\mathrm{op}}} G \Rightarrow \mathrm{Lan}_{J_{\mathcal{D}}^{\mathrm{op}}} G'$ with $(1;\beta)(\alpha;1) \neq (\alpha;1)(1;\beta)$ in general. Each such arrow induces a natural transformation $\mathrm{Lan}_{J_{\mathcal{C}\square\mathcal{D}}^{\mathrm{op}}}(F \otimes G) \Rightarrow \mathrm{Lan}_{J_{\mathcal{C}\square\mathcal{D}}^{\mathrm{op}}}(F' \otimes G')$, for instance:

$$
\begin{aligned}
\mathrm{Lan}_{J_{\mathcal{C}\square\mathcal{D}}^{\mathrm{op}}}(F \otimes G) &\cong \mathrm{Lan}_{i_1^{\mathrm{op}}}(\mathrm{Lan}_{J_{\mathcal{C}}^{\mathrm{op}}} F \otimes G) \\
&\xRightarrow{\mathrm{Lan}_{i_1^{\mathrm{op}}}(\alpha\otimes 1)} \mathrm{Lan}_{i_1^{\mathrm{op}}}(\mathrm{Lan}_{J_{\mathcal{C}}^{\mathrm{op}}} F' \otimes G) \\
&\cong \mathrm{Lan}_{i_0^{\mathrm{op}}}(F' \otimes \mathrm{Lan}_{J_{\mathcal{D}}^{\mathrm{op}}} G) \\
&\xRightarrow{\mathrm{Lan}_{i_0^{\mathrm{op}}}(1\otimes \beta)} \mathrm{Lan}_{i_0^{\mathrm{op}}}(F' \otimes \mathrm{Lan}_{J_{\mathcal{D}}^{\mathrm{op}}} G') \\
&\cong \mathrm{Lan}_{J_{\mathcal{C}\square\mathcal{D}}^{\mathrm{op}}}(F' \otimes G')
\end{aligned}
$$

□

$\mathcal{V}^2$-$\mathsf{Prof}^{\mathsf{Tight}}$ has an interesting tensor product given by generalising the funny tensor product.

Definition 55 (Funny Tensor Product of Tight $\mathcal{V}^2$-Profunctors)**.** On categories the funny tensor acts like in $\mathcal{V}^2$-Cat. On tight $\mathcal{V}^2$-profunctors $P : J_{\mathcal{A}} \to \overline{J_{\mathcal{B}}}$ and $Q : J_{\mathcal{C}} \to \overline{J_{\mathcal{D}}}$ we define their funny tensor to be given by their funny tensor in $\mathcal{V}^2$-Cat composed with the external tensor of free tight cocompletions (6.3):

$$J_{\mathcal{A}} \,\square\, J_{\mathcal{C}} \xrightarrow{P\square Q} \overline{J_{\mathcal{B}}} \,\square\, \overline{J_{\mathcal{D}}} \xrightarrow{\hat{\otimes}} \overline{J_{\mathcal{B}\square\mathcal{D}}}$$

Theorem 20. $\mathcal{V}^2$-$\mathsf{Prof}^{\mathsf{Tight}}$ *is a monoidal bicategory under the funny tensor product.*

Proof sketch. $\mathcal{V}^2$-$\mathsf{Prof}^{\mathsf{Tight}}$ is the Kleilsi bicategory of the relative pseudomonad $\overline{(\cdot)}$ that adds tight colimits. Under the funny tensor product on $\mathcal{V}^2$-Cat, this pseudomonad is monoidal, and therefore its Kleilsi bicategory is also monoidal. □

6.4 Pro-effectful Categories

Finally in this Section we are in a position to define pro-effectful categories: as pseudomonoids in $\mathcal{V}^2$-$\mathsf{Prof}_{\square}^{\mathsf{Tight}}$, placing them on equal footing algebraically with monoidal, promonoidal and effectful categories.

Definition 56. A pro-effectful category is a pseudomonoid (a representation of $\mathfrak{P}$) in $\mathcal{V}^2$-$\mathsf{Prof}_{\square}^{\mathsf{Tight}}$. Explicitly, a pro-effectful category $J_{\mathcal{C}}$ is a $\mathcal{V}^2$-category equipped with

- a tensor product tight $\mathcal{V}^2$-profunctor $P : J_{\mathcal{C}\square\mathcal{C}} \nrightarrow J_{\mathcal{C}}$,
- and a unit tight $\mathcal{V}^2$-profunctor $I : 1 \nrightarrow J_{\mathcal{C}}$,

together with $\mathcal{V}^2$-natural isomorphisms $P(P \square 1) \overset{\alpha}{\cong} P(1 \square P)$ and $P(I \square 1) \overset{\lambda}{\cong} 1 \overset{\rho}{\cong} P(1 \square I)$ such that the triangle and pentagon equations hold.

Unpacking this definition further we see that a pro-effectful category is an identity on objects functor $J : \mathcal{C}_0 \to \mathcal{C}_1$ equipped with functors $P_0 : \mathcal{C}_0 \boxtimes \mathcal{C}_0 \to \widehat{\mathcal{C}_0}$, $P_1 : \mathcal{C}_1 \square \mathcal{C}_1 \to \overline{\mathcal{C}_1}$, $I_0 : 1 \to \widehat{\mathcal{C}_0}$ and $I_1 : 1 \to \overline{\mathcal{C}_1}$ such that the following squares commute:

$$
\begin{tikzcd}
\mathcal{C}_0 \boxtimes \mathcal{C}_0 \arrow[r, "J_{\mathcal{C}\square\mathcal{C}}"] \arrow[d, "P_0"'] & \mathcal{C}_1 \square \mathcal{C}_1 \arrow[d, "P_1"] \\
\widehat{\mathcal{C}_0} \arrow[r, "\mathrm{Lan}^L_{J^{\mathrm{op}}}"'] & \overline{\mathcal{C}_1}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "1"] \arrow[d, "I_0"'] & 1 \arrow[d, "I_1"] \\
\widehat{\mathcal{C}_0} \arrow[r, "\mathrm{Lan}^L_{J^{\mathrm{op}}}"'] & \overline{\mathcal{C}_1}
\end{tikzcd}
$$

So we see that a pro-effectful category has a promonoidal centre $(\mathcal{C}_0, P_0, I_0)$ (together with the obvious components of the coherence isomorphisms α, λ, ρ).

Pro-effectful Categories as Pro-actegories

In this subsection we unpack the definition of a pro-effectful category to show that they are a particular instance of a category equipped with an action by a promonoidal category.

We will be interested in representations of $\mathfrak{M}$ in $\mathcal{V}$-Prof which are equivalent to the proactegories of [39] (in the non-skew case).

Definition 57 (Proactegory)**.** A left proactegory is a representation of $\mathfrak{M}^L$ in $\mathcal{V}$-Prof. Explicitly this means we have a promonoidal category $(\mathcal{C}_0, P, I)$ and a category $\mathcal{C}_1$ equipped with a left proaction by $\mathcal{C}_0$, that is, a profunctor $L : \mathcal{C}_0 \boxtimes \mathcal{C}_1 \nrightarrow \mathcal{C}_1$ and natural isomorphisms

$$\int^{X \in \mathcal{C}_1} L(A, B, X) \boxtimes L(X, C, D) \overset{a}{\cong} \int^{X \in \mathcal{C}_0} L(A, X, D) \boxtimes P(X, B, C)$$

$$\int^{X \in \mathcal{C}_0} L(A, X, B) \boxtimes I(X) \overset{l}{\cong} \mathcal{C}_1(A, B)$$

satisfying the coherence diagrams. Right proactegories are representations of $\mathfrak{M}^R$ in $\mathcal{V}$-Prof.

Biproactegories are representations of $\mathfrak{M}$ in $\mathcal{V}$-Prof and thus in addition to the data of a left and right proactegory there is a natural isomorphism

$$\int^X R(D,X,C) \boxtimes L(X,A,B) \overset{b}{\cong} \int^X L(D,A,X) \boxtimes R(X,B,C)$$

satisfying the coherences.

Example 17. A promonoidal category $\mathcal{C}$ is canonically a left (and right) $\mathcal{C}$-proactegory where the proaction is given by the promonoidal tensor.

Proposition 43. *A pro-effectful category is equivalently the following data:*

- *a promonoidal category* $(\mathcal{C}_0, P_0, I_0)$*,*
- *a category* $\mathcal{C}_1$ *with the same objects as* $\mathcal{C}_0$ *and an identity on objects functor* $J : \mathcal{C}_0 \to \mathcal{C}_1$*,*
- *A left* $\mathcal{C}_0$*-proaction on* $\mathcal{C}_1$*,* $P_1^L : \mathcal{C}_0 \boxtimes \mathcal{C}_1 \nrightarrow \mathcal{C}_1$*, which extends the canonical left* $\mathcal{C}_0$*-proaction on* $\mathcal{C}_0$*:*

$$\begin{tikzcd} \mathcal{C}_0 \boxtimes \mathcal{C}_0 \arrow[r, "P_0", "\shortmid"{marking}] \arrow[d, "1\times yJ"', "\shortmid"{marking}] & \mathcal{C}_0 \arrow[d, "yJ", "\shortmid"{marking}] \\ \mathcal{C}_0 \boxtimes \mathcal{C}_1 \arrow[r, "P_1"', "\shortmid"{marking}] & \mathcal{C}_1 \end{tikzcd} \tag{6.4}$$

- *A right* $\mathcal{C}_0$*-proaction on* $\mathcal{C}_1$*,* $P_1^R : \mathcal{C}_1 \boxtimes \mathcal{C}_0 \nrightarrow \mathcal{C}_1$*, which extends the canonical right* $\mathcal{C}_0$*-proaction on* $\mathcal{C}_0$*:*

$$\begin{tikzcd} \mathcal{C}_0 \boxtimes \mathcal{C}_0 \arrow[r, "P_0", "\shortmid"{marking}] \arrow[d, "yJ\times 1"', "\shortmid"{marking}] & \mathcal{C}_0 \arrow[d, "yJ", "\shortmid"{marking}] \\ \mathcal{C}_1 \boxtimes \mathcal{C}_0 \arrow[r, "P_1"', "\shortmid"{marking}] & \mathcal{C}_1 \end{tikzcd} \tag{6.5}$$

- *A natural isomorphism* $P_1^R(P_1^L \boxtimes 1) \cong P_1^L(1 \boxtimes P_1^R)$ *making* $\mathcal{C}_1$ *into a* $\mathcal{C}_0$*-*$\mathcal{C}_0$*-biproactegory.*

Proof. Fix a pro-effectful category (J, P, I). J is a $\mathcal{V}^2$-category so we have two categories $\mathcal{C}_0$ and $\mathcal{C}_1$ with the same objects and an identity on objects functor $J : \mathcal{C}_0 \to \mathcal{C}_1$.

The tight $\mathcal{V}^2$-profunctor $P : J_{\mathcal{C}\square\mathcal{C}} \nrightarrow J_{\mathcal{C}}$ consists of a profunctor $P_0 : \mathcal{C}_0 \boxtimes \mathcal{C}_0 \nrightarrow \mathcal{C}_0$ and a functor $P_1 : \mathcal{C}_1 \square \mathcal{C}_1 \to \overline{\mathcal{C}_1}$. Similarly, the tight $\mathcal{V}^2$-profunctor $I : 1 \nrightarrow J_{\mathcal{C}}$ consists of presheaves $I_0 : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{V}$ and $I_1 = \mathrm{Lan}_{J^{\mathrm{op}}} I_0 : \mathcal{C}_1^{\mathrm{op}} \to \mathcal{V}$. (P_0, I_0) induce a promonoidal structure on $\mathcal{C}_0$.

P_1 induces the left and right proactions of $\mathcal{C}_0$ on $\mathcal{C}_1$. Starting with the left proaction, P_1 induces a functor $y_1^R P_1 i_1 =: P_1^L : \mathcal{C}_0 \boxtimes \mathcal{C}_1 \to \widehat{\mathcal{C}_1}$. It follows that:

$$P_1^L(1 \boxtimes J) = y_1^R P_1 i_1 (1 \boxtimes J) = y_1^R P_1 J_{\mathcal{C}\square\mathcal{C}} = \mathrm{Lan}_{J^{\mathrm{op}}\boxtimes 1 \boxtimes 1} P_0$$

showing that (6.4) commutes and that the left proaction extends the canonical one on $\mathcal{C}_0$. A similar argument holds for the right proaction.

Suppose now that we start with the data specified in the proposition. The equalities (6.4) together with the universal property of the pushout induce a functor $P_1 : \mathcal{C}_1 \,\square\, \mathcal{C}_1 \to \overline{\mathcal{C}_1}$

$$\begin{tikzcd}
\mathcal{C}_0 \boxtimes \mathcal{C}_0 \arrow[r, "J\boxtimes 1"] \arrow[d, "1\boxtimes J"'] & \mathcal{C}_1 \boxtimes \mathcal{C}_0 \arrow[d, "i_0"] \arrow[ddr, bend left, "P_1^R"] & \\
\mathcal{C}_0 \boxtimes \mathcal{C}_1 \arrow[r, "i_1"'] \arrow[drr, bend right, "P_1^L"'] & \mathcal{C}_1 \,\square\, \mathcal{C}_1 \arrow[dr, dashed, "P_1"] & \\
 & & \overline{\mathcal{C}_1}
\end{tikzcd}$$

and it follows that $P_1 J_{\mathcal{C}\square\mathcal{C}} = \mathrm{Lan}^L_{J^{\mathrm{op}}} P_0$ making (P_0, P_1) the components of a tight $\mathcal{V}^2$-profunctor $P : J_{\mathcal{C}\square\mathcal{C}} \twoheadrightarrow J_{\mathcal{C}}$. The presheaf $I_0 : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{V}$ together with its Kan extension $I_1 := \mathrm{Lan}_{J^{\mathrm{op}}} I_0$ give the components of a $\mathcal{V}^2$-profunctor $I : 1 \twoheadrightarrow J$. Checking all the coherences is a long but ultimately routine calculation. $\square$

Pro-effectful Categories as Prostrong Promonads

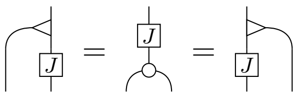
The next proposition generalises the equivalence between effectful categories and strong promonads [102] to the pro-effectful case. The proof methods are related to those for promonoidal monads in [67].

Proposition 44. *A pro-effectful category is equivalently a prostrong promonad (see Definition 49).*

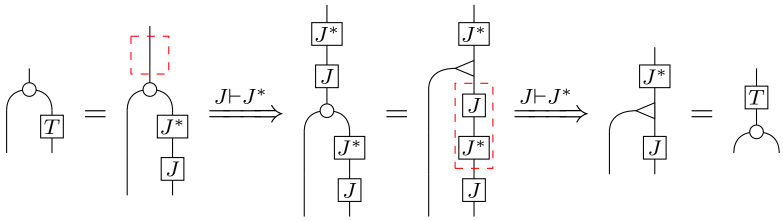
Proof. Take a prostrong promonad $T : \mathcal{C} \twoheadrightarrow \mathcal{C}$. We will show we have the data of Proposition 43.

T has a Kleisli category in $\mathcal{V}$-Prof and there is an identity on objects free functor $F : \mathcal{C} \to \mathsf{Kl}_T$. By assumption $\mathcal{C}$ has a promonoidal structure (P_0, I_0) and we can use the left and right prostrengths to define left and right proactions of $\mathcal{C}$ on Kl_T. On objects the left proaction acts as $P_1^L(-, C, FC') := \int^X \mathsf{Kl}_T(-, FX) \boxtimes P_0(X, C, C')$ extending the canonical proaction on the centre, so that (6.4) commutes. Its action on homs is induced by the strength $\int^C P_0(-,-,C) \boxtimes T(C,-) \Rightarrow \int^C T(-,c) \boxtimes P_0(C,-,-)$.

Conversely, suppose we are given a pro-effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$. Then $T(-,=) := \mathcal{C}_1(J-, J=)$ a promonad on $\mathcal{C}_0$ where the promonad multiplication and units are given by composition in $\mathcal{C}_1$. Moreover, $\mathcal{C}_1$ is precisely the Kleisli category of T. Now, since J is pro-effectful, $\mathcal{C}_0$ is promonoidal and we are left to show that T is prostrong over this structure. By Proposition 43, we have left and right proactions of $\mathcal{C}_1$ on $\mathcal{C}_0$ which preserve the canonical proaction on the centre:



From these we can construct the prostrength of T, for instance the left prostrength is given as follows.



$\square$

Closed Embeddings of Pro-effectful Categories

Pro-effectful categories are also exactly what is required to place a closed effectful structure on the free tight cocompletion of a $\mathcal{V}^2$-category. This generalises Day's theorem [68, 65] from monoidal to effectful categories, thus also generalising the result of Power on closed effectful embeddings of effectful categories [131, 130]. The result follows by generalising the methods of Day's original proof, and from the folklore results regarding Day convolution for actegories, see [104, 39].

Theorem 21. *There is an equivalence between pro-effectful structures on J and closed effectful structures on the free tight cocompletion $\overline{J} = Lan^L_{J^{op}}$.*

Proof. Suppose $J : \mathcal{C}_0 \to \mathcal{C}_1$ is a pro-effectful category. We will show that $\mathrm{Lan}^L_{J^{op}} : \widehat{\mathcal{C}_0} \to \overline{\mathcal{C}_1}$ is a closed premonoidal category. Since $\mathcal{C}_0$ is promonoidal, $\widehat{\mathcal{C}_0}$ is closed monoidal under Day convolution.

As for the premonoidal structure on $\overline{\mathcal{C}_1}$: on objects it is the same as on $\widehat{\mathcal{C}_0}$. On morphisms, suppose we are given a $\eta : F \Rightarrow G$ in $\overline{\mathcal{C}_1}$. Then we have a $\eta : \mathrm{Lan}_{J^{op}} F \Rightarrow$

$\mathrm{Lan}_{J^{\mathrm{op}}} G$ and we can describe the left hand part of the premonoidal structure by

$$\begin{aligned}
\mathrm{Lan}_{J^{\mathrm{op}}}(F \star F')(-) &\cong \int^{ABC} \mathcal{C}_1(-, JC) \boxtimes P_0(C, A, B) \boxtimes FA \boxtimes F'B \\
&\cong \int^{AB} P_1^R(-, JA, B) \boxtimes FA \boxtimes F'B \\
&\cong \int^{BC} P_1^R(-, C, B) \boxtimes (\mathrm{Lan}_{J^{\mathrm{op}}} F)(C) \boxtimes F'B \\
&\xrightarrow{\int \eta} \int^{BC} P_1^R(-, C, B) \boxtimes (\mathrm{Lan}_{J^{\mathrm{op}}} G)(C) \boxtimes F'B \\
&\cong \mathrm{Lan}_{J^{\mathrm{op}}}(G \star F')(-)
\end{aligned}$$

and similarly for the right hand part. It is easily seen that $\mathrm{Lan}^L_{J^{\mathrm{op}}}$ factorises through the centre of this premonoidal structure.

The internal-hom of the left-closed premonoidal structure, $[G, -] : \overline{\mathcal{C}_1} \to \widehat{\mathcal{C}_0}$ is given by

$$[G, H](A) \cong \int_{CD} \mathcal{V}\Big(P_1^L(C, A, D), \mathcal{V}\big((\mathrm{Lan}_{J^{\mathrm{op}}} G)(D), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(C)\big)\Big)$$

while the right-closed structure is similar, replacing P_1^L with P_1^R. In both cases, checking we have the required adjunction is a matter of standard coend calculus e.g:

$$\begin{aligned}
\widehat{\mathcal{C}_0}(F, [G, H]) &\cong \int_A \mathcal{V}\Big(FA, \int_{CD} \mathcal{V}\big(P_1^L(C, A, D) \boxtimes (\mathrm{Lan}_{J^{\mathrm{op}}} G)(D), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(C)\big)\Big) \\
&\cong \int_{ACD} \mathcal{V}\Big(FA \boxtimes P_1^L(C, A, D) \boxtimes (\mathrm{Lan}_{J^{\mathrm{op}}} G)(D), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(C)\Big) \\
&\cong \int_C \mathcal{V}\Big(\int^{AB} FA \boxtimes P_1^L(C, A, JB) \boxtimes GB, (\mathrm{Lan}_{J^{\mathrm{op}}} H)(C)\Big) \\
&\cong \int_C \mathcal{V}\Big(\mathrm{Lan}_{J^{\mathrm{op}}}(F \star G)(C), (\mathrm{Lan}_{J^{\mathrm{op}}} H)(C)\Big) \\
&\cong \overline{\mathcal{C}_1}(F \boxtimes G, H)
\end{aligned}$$

Suppose now that $\mathrm{Lan}^L_{J^{\mathrm{op}}}$ is a closed effectful category. Then it follows that $\widehat{\mathcal{C}_0}$ is a closed monoidal category because:

$$\begin{aligned}
\widehat{\mathcal{C}_0}\left(-, [G, \mathrm{Lan}^L_{J^{\mathrm{op}}}(=)]\right) &\cong \overline{\mathcal{C}_1}\left(\mathrm{Lan}^L_{J^{\mathrm{op}}}(-) \boxtimes G, \mathrm{Lan}^L_{J^{\mathrm{op}}}(=)\right) \\
&= \overline{\mathcal{C}_1}\left(\mathrm{Lan}^L_{J^{\mathrm{op}}}(- \otimes G), \mathrm{Lan}^L_{J^{\mathrm{op}}}(=)\right) \\
&\cong \widehat{\mathcal{C}_0}\left(- \otimes G, J^{\mathrm{op}*}(\mathrm{Lan}^L_{J^{\mathrm{op}}}(=))\right) \\
&\cong \widehat{\mathcal{C}_0}\left(- \otimes G, =\right)
\end{aligned}$$

where $J^{\mathrm{op}*}$ is the right adjoint to $\mathrm{Lan}^L_{J^{\mathrm{op}}}$, both of which are ioo. Therefore $\mathcal{C}_0$ is a promonoidal category.

The left $\mathcal{C}_0$-proaction on $\mathcal{C}_1$ is given by $P_1^L(-, A, B) := y_0^L(A) \boxtimes y_1^L(B) = \boxtimes i_1(y_0^L(A), y_1^L(B))$ and similarly for the right. These extend the canonical proaction because:

$$\begin{aligned} P_1^L(-, A, JB) = \boxtimes i_1(y_0^L(A), y_1^L(JB)) &= \boxtimes i_1(y_0^L(A), \mathrm{Lan}^L_{J^{\mathrm{op}}} y_0^L(B)) \\ &= \boxtimes i_1(1 \otimes_{\mathcal{V}} \mathrm{Lan}^L_{J^{\mathrm{op}}})(y_0^L(A), y_0^L(B)) \\ &= \mathrm{Lan}^L_{J^{\mathrm{op}}} \otimes (y_0^L(A), y_0^L(B)) \\ &= \mathrm{Lan}^L_{J^{\mathrm{op}}} P_0(-, A, B) \end{aligned}$$

where we have written the monoidal operation $\otimes$ on $\widehat{\mathcal{C}_0}$ and the premonoidal operation $\boxtimes$ on $\overline{\mathcal{C}_1}$ with prefix notation. □

Pro-effectful Categories as Premulticategories

For simplicity, in this section we take $\mathcal{V} = \mathsf{Set}$, but there is no true obstruction to applying the following discussion to enriched multicategories.

Premulticategories were defined by Staton and Levy by dropping the interchange law from the definition of a multicategory [151].

Definition 58 (Premulticategory)**.** A premulticategory $\mathfrak{C}$ consists of

- a class $\mathfrak{C}_0$ of objects,
- for objects $A_1, \ldots, A_n, A \in \mathfrak{C}_0$, a class $\mathfrak{C}(A_1, \ldots, A_n; A)$ of arrows, an element of which is written $f : A_1, \ldots, A_n \to A$,
- for any pair of arrows $f : A_1, \ldots, A_l \to A$ and $g : B_1, \ldots, B_m, A, B'_1, \ldots, B'_n \to B$ where the codomain of f matches one those of the domain of g, a composite arrow $g \circ_A f : B_1, \ldots, B_m, A_1, \ldots, A_l, B'_1, \ldots, B'_n \to B$,
- for each object $A \in \mathfrak{C}$, an identity arrow $1_A \in \mathfrak{C}(A; A)$

such that the following two conditions are satisfied:

- Associativity: $h \circ_B (g \circ_A f) = (h \circ_B g) \circ_A f$ whenever this is well-typed,
- Unitality: $1_A \circ_A f = f \circ_{A_i} 1_{A_i}$ whenever this is well-typed.

Just as promonoidal categories are examples of (co)multicategories, pro-effectful categories are examples of co-premulticategories. Given a pro-effectful category $J : \mathcal{C}_0 \to \mathcal{C}_1$, there is a co-premulticategory $\mathfrak{C}$ with objects given by those of $\mathcal{C}_1$. For $A, A_1 \in \mathfrak{C}$ the class of arrows is given by $\mathfrak{C}(A; A_1) := \mathcal{C}_1(A, A_1)$ and for $A, A_1, A_2 \in \mathfrak{C}$

the class of arrows is given by $\mathfrak{C}(A; A_1, A_2) := P_1(A, A_1, A_2)$. The rest of the classes of arrows are defined inductively.

It is worth noting that pro-effectful categories provide non-degenerate examples of co-premulticategories where the interchange law does not hold (in contrast to promonoidal and monoidal categories which are multicategories) and where the "tensor" is not representable (in contrast to monoidal and premonoidal categories).

Chapter 7

Supermaps

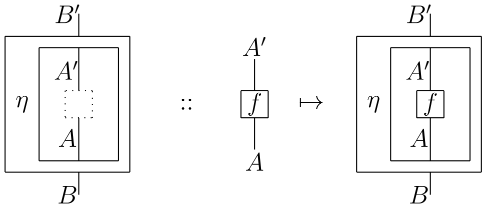
7.1 Introduction

The traditional way in which physical systems are modelled is by considering a state space which evolves according to processes which act on that space. For example, a quantum circuit is traditionally viewed in terms of linear operators being applied to a Hilbert space; electrical circuits in terms of certain operators acting on phase space; probabilistic theories in terms of stochastic maps acting on probability spaces.

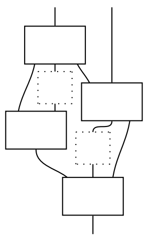
This approach has proven to be amenable to categorical analysis. For example, the ZX-calculus [51, 159], graphical affine algebra [32, 30, 61] and Markov categories [81] have all been successful in formally modelling these respective classes of systems using the theory of monoidal categories. Moreover, categorical quantum mechanics [3, 49, 89] and the framework of generalised/operational probabilistic theories [16, 42] provide semantics for modelling more general quantum-like theories.

However, the approach of modelling systems merely in terms of the action of operators on the state space may not fully capture the behaviour of the system. When the collection of operators is itself regarded as the state space, this traditional approach gives little insight into the evolution of this new, "higher order" state space. What is missing is a theory of second order processes, a theory of processes which themselves act on (first order) processes. Or indeed a theory of n^{th} order processes which act on $(n-1)^{\text{th}}$ order processes.

In the theory of quantum circuits, these higher order processes are known as quantum *supermaps* [45, 21, 22, 110, 126, 158]. We may think of a quantum supermap η as a process whose input is itself a process.



The simplest quantum supermaps are the *circuits with holes* also known as *combs* [43, 44]. Here, one has a quantum circuit with slots that can be filled with first-order maps.



It turns out that all second-order deterministic single-party supermaps on quantum channels possess a factorisation as a circuit [45, 73, 110]. Nevertheless, it is also known that there exist multi-party quantum supermaps, such as the quantum `switch`, which go beyond the standard quantum circuit model by not possessing a factorisation as a circuit with definite causal ordering of gates and no time-loops [46, 40]. This makes the study of quantum supermaps very deep and rich with investigations from perspectives including computational advantage and causality.

In this chapter we will consider the problem of developing categorical semantics for supermaps, with minimal assumptions on the category of first-order processes. This contrasts many other approaches to supermaps which either do not consider categorical aspects or where they do, rely on significant structure on the category of first-order processes. For instance, in the original [45], the setting is taken to be completely positive maps between Hilbert spaces and supermaps are defined in terms of this, with requirements on positivity and linearity. We do not restrict ourselves to only quantum theory here, meaning our approach is applicable to subclasses of quantum theory (e.g. only unitaries, or isometries) and also to other process theories.

One explicitly categorical approach is the $\mathsf{Caus}(-)$ construction of [110]. There, it is assumed that the first-order processes form a compact closed category and therefore, the $\mathsf{Caus}(-)$ construction assumes that all higher-order processes are already present as processes in the underlying "raw materials" process theory $\mathcal{C}$: this is precisely because we can bend wires around to turn higher-order maps into states, and thus consider the maps on them as lower-order. Causality is the guiding principle of the framework, which equips the processes of $\mathcal{C}$ with new types capturing certain causality constraints on the processes; indeed there is a functor $\mathsf{Caus}(\mathcal{C}) \to \mathcal{C}$ which forgets this typing data by sending all the "copies" of a process f with different causality conditions to the same underlying f in $\mathcal{C}$.

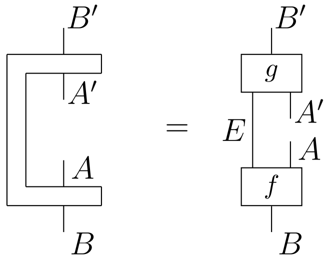
In contrast, we do not make the assumption of compact closure here, with the framework applicable to any process theory. Rather than starting with processes of all orders, the constructions here may generate new processes, so that the higher-order processes may *not* be part of the original process theory. This philosophy is very similar to that taken in [161, 160].

In Section 7.2 we will consider the simplest case and compare two constructions which take an arbitrary symmetric monoidal category and produce a symmetric monoidal category of 1-combs. In Section 7.3 we generalise this category to produce a polycategory of n-combs. We then discuss how this polycategory is a fragment of the duoidal category of Tambara modules, thus giving us a way of embedding the polycategorical semantics into a (doubly) monoidal one. We also discuss the monoidal-like structure of the category of combs without the assumption of symmetry. In Section 7.4 we generalise the category of combs even further to allow for a premonoidal category of first-order processes. This provides us with our first example of a pro-effectful category and we anticipate possible applications of this category to premonoidal models of spacetime, and also to effectful programming semantics.

In the final Section 7.5 we consider the laws for supermaps in terms of *locally-applicable transformations* suggested in [161, 160] and we demonstrate that these can be formalised categorically as the homomorphisms of Tambara modules. This connects the work of this chapter with the approach of [161, 160] and suggests a fruitful direction forward for further research.

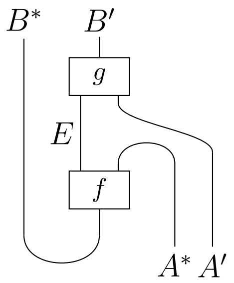
7.2 1-Combs

In this Section we restrict our attention to 1-combs [43, 44], the single-party supermaps which possess a factorisation as a circuit. These maps are often drawn suggestively as diagrams of the following form.



(7.1)

Some care is needed to make these drawings rigorous and to demonstrate that a suitable (possibly symmetric monoidal) category of combs can be defined. In much of the quantum literature it is assumed that the base category of first order processes is compact closed, or at least embeds into one. In this case it is possible to bend input and output wires to express combs as maps without holes and use the drawing (7.1) in an unambiguous way [43, 110]. For example, we could *define* the drawing (7.1) to mean:

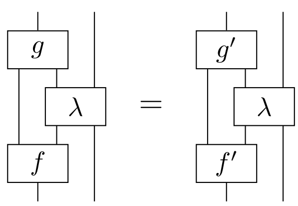


and so we are able to reduce higher-order maps to lower-order ones and interpret the diagrams in the original symmetric monoidal category.

Outside of the quantum literature there are approaches to defining comb diagrams without the assumption of closure [136, 135], but it is not clear when this coincides with the quantum definition. Let us start by comparing two constructions which represent a comb as a pair of morphisms (f, g) from the theory of first order processes, quotiented by their behaviour on first-order processes.

The first construction, which we define in Subsection 7.2.1, $\mathsf{Comb} : \mathsf{SymMonCat} \to \mathsf{SymMonCat}$, quotients combs by their extensional behaviour: two combs are equal when they produce the same output on all first-order inputs. In other words this identifies two combs when they appear to be the same when probed with all first order processes λ:

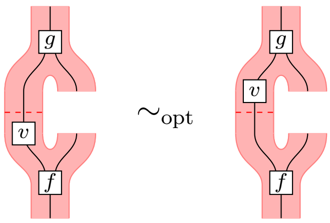
$(f, g) \sim_{\text{comb}} (f', g')$ when



$\forall \lambda$

This equivalence relation has been discussed before [52] [1] and is perhaps the one that would be most immediate to those studying quantum theory.

The second construction, which we review in Subsection 7.2.2, is that of the category of coend optics, $\mathsf{Optic} : \mathsf{SymMonCat} \to \mathsf{SymMonCat}$ (which we shall henceforth just call optics) [47, 134, 137, 127]. Optics are used to encompass bidirectional data accessors familiar to the computer science community such as lenses, prisms and grates, amongst many others. Their usage to model combs and more general "circuits with holes" has been described in [136, 135]. In contrast to the previous construction this quotients the combs by their intensional behaviour, allowing first-order maps to slide along the shared environment connecting the two factors together:



In Subsection 7.2.3 we show that there is always a full and identity on objects monoidal functor from optics to the extensional definition, $\mathsf{Optic}(\mathcal{C}) \to \mathsf{Comb}(\mathcal{C})$. We then give some sufficient conditions for this functor to exhibit an isomorphism of symmetric monoidal categories. In particular we show that when the category of first-order processes is cartesian and there exists a state for every type or when it is compact closed, the two definitions coincide. We also show that in the case of the category of unitaries between Hilbert spaces, the definitions again coincide. This case (alongside compact closed categories) is particularly important for quantum theory. We leave it as future work to fully characterise when $\mathsf{Optic}(\mathcal{C}) \cong \mathsf{Comb}(\mathcal{C})$ and note that there are important cases of combs not covered by the sufficient conditions proven in this work.

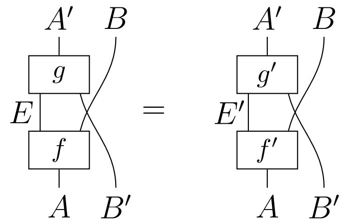
7.2.1 Extensional Combs

Let us begin by considering possible extensional definitions of combs. Firstly, one could ask that the combs are equal as morphisms in the original category when we

[1] we note that our category of combs is distinct from that developed there: the objects of their category being different than those studied in this document

extend their inputs:

$$(f, g) \sim_\sigma (f', g') \quad \iff$$

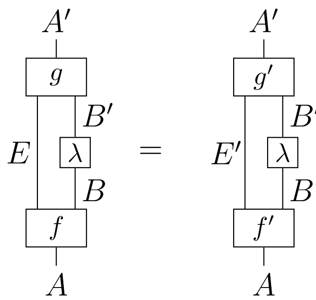


(7.2)

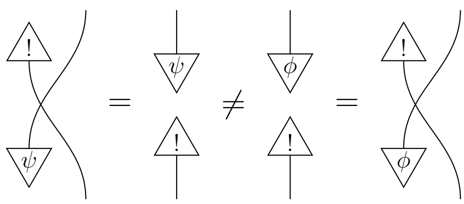
While this is an equivalence relation on pairs of morphisms, it is not a congruence with respect to composition. Suppose $(f, g) \sim_\sigma (f', g')$ and $(h, k) \sim_\sigma (h', k')$. Then $(h, k) \circ (f, g) = ((1 \otimes h)f, g(1 \otimes k)) \sim_\sigma ((1 \otimes h')f, g(1 \otimes k')) = (h', k') \circ (f, g)$ which is not in general equivalent to $(h', k') \circ (f', g')$.

We could instead ask that two combs are equivalent if they are equal on all inputs to the comb:

$$(f, g) \sim_\tau (f', g') \quad \iff \quad \forall \lambda : B \to B'$$



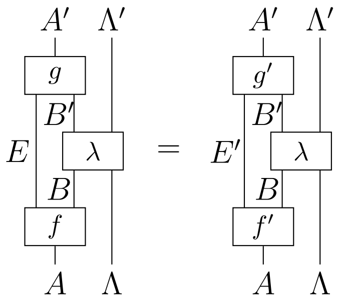
This also forms an equivalence relation on pairs of morphisms, although it is too coarse. Consider the free symmetric monoidal category generated by one object A, two states $\phi, \psi : I \to A$ and an effect $! : A \to I$ such that $! \circ \phi = ! \circ \psi = 1_I$. Then $(1_I \otimes \psi, 1_I \otimes !) \sim_\tau (1_I \otimes \phi, 1_I \otimes !)$; however evaluating these combs on the braid one finds,



Here, the monoidal structure of our category is allowing us to probe the combs in ways the naive equivalence relation $\sim_\tau$ does not, and in doing so access additional information about the behaviour of the combs. So if we want combs to behave compatibly with the monoidal structure of the category, we need something stronger than equality on all inputs.

Definition 59 (Extensional Comb Equivalence)**.** We say that two combs are equivalent if they are equal on all extended inputs:

$(f,g)_E \sim_{\text{comb}} (f',g')_{E'} \iff \begin{array}{c}\forall \Lambda, \Lambda' \\ \forall \lambda : B \otimes \Lambda \to B' \otimes \Lambda'\end{array}$

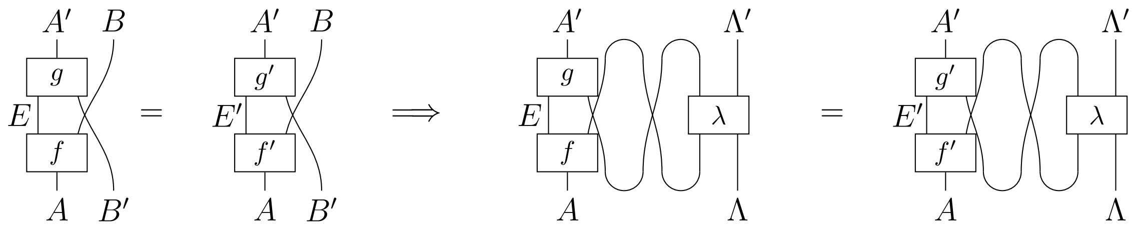


(7.3)

This definition subsumes both of the previous definitions, but in the compact closed case (7.2) is sufficient to recover the full extensional equivalence.

Proposition 45. *When $\mathcal{C}$ is compact closed* $(f,g) \sim_{comb} (f',g') \iff (f,g) \sim_\sigma (f',g')$.

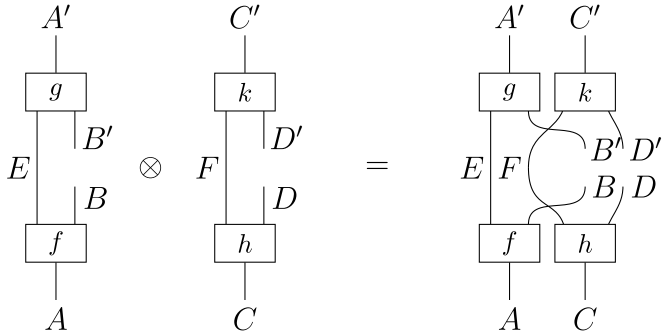
Proof. The forwards direction is immediate. The backwards direction follows by graphical manipulation:



□

Definition 60. Given a symmetric monoidal category $\mathcal{C}$, the symmetric monoidal category of extensional combs $\mathsf{Comb}(\mathcal{C})$ has objects given by pairs (A, A') of objects of $\mathcal{C}$. A morphism $(f,g) : (A, A') \to (B, B')$ is an equivalence classes of pairs of morphisms $f : A \to E \otimes B$ and $g : E \otimes B' \to A'$ of $\mathcal{C}$ under the comb equivalence relation $\sim_{\text{comb}}$. Composition of morphisms is given by $(f',g') \circ (f,g) = ((1 \otimes f')f, g(1 \otimes g'))$.

The monoidal structure acts on objects as $(A, A') \otimes (B, B') = (A \otimes B, A' \otimes B')$ and on morphisms:



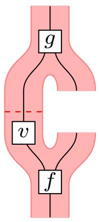
The unit object is (I, I) with structural isomorphisms given by $(\lambda, \lambda^{-1}) : (A, A') \otimes (I, I) = (A \otimes I, A' \otimes I) \to (A, A')$ and (ρ, ρ^{-1}). The symmetry σ is defined similarly.

Lemma 8. Comb *defines a functor* SymMonCat $\to$ SymMonCat.

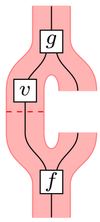
7.2.2 Intensional Combs: Optics

Optics provide another potential definition of combs; albeit an intensional one, as opposed to the extensional one described in the previous subsection.

Definition 61 (Category of optics [127, 47])**.** Given a symmetric monoidal category $\mathcal{C}$, the category of optics $\mathsf{Optic}(\mathcal{C})$, has objects given by pairs (A, A') of objects of $\mathcal{C}$. Morphisms are pairs $(f, g)_E$ like in $\mathsf{Comb}(\mathcal{C})$ however, instead of quotienting the morphisms by the equivalence relation $\sim_{\text{comb}}$, we quotient morphisms by the equivalence relation $\sim_{\text{opt}}$ imposed by embedding the combs inside the cobordisms:



$\sim_{\text{opt}}$



(7.4)

The string diagrams can be freely moved around the interior of the cobordism, but can not pass through the surface: as a result we are able to slide maps on the environment wire between the two halves with the equivalence relation generated by $((v \otimes 1)f, g)_{E'} \sim (f, g(v \otimes 1))_E$. Explicitly the hom-sets of $\mathsf{Optic}(\mathcal{C})$ are given by the following coend:

$$\int^{E} \mathcal{C}(A, E \otimes B) \times \mathcal{C}(E \otimes B', A')$$

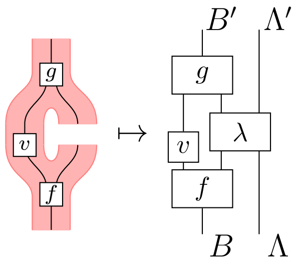
Composition, identities, and symmetric monoidal structure is as in $\mathsf{Comb}(\mathcal{C})$. That $\sim_{\text{opt}}$ is a congruence and that the composite of two optics is another optic (i.e. that the composite of the comb-shaped cobordisms in (7.4) can be manipulated to give another comb-shaped cobordism) follows by a composition of the 2-cells in (4.7), see e.g. [134] for more details.

7.2.3 Equivalence of Extensional and Intensional Combs

In this section we consider the question of when $\mathsf{Optic}(\mathcal{C})$ and $\mathsf{Comb}(\mathcal{C})$ are equivalent. It is fairly straightforward to show that there is always a functor $\mathsf{Optic}(\mathcal{C}) \to \mathsf{Comb}(\mathcal{C})$ turning the intensional combs into extensional combs.

Proposition 46. *Given a symmetric monoidal category* $\mathcal{C}$*, there is a bijective on objects, full symmetric monoidal functor* $\mathsf{Optic}(\mathcal{C}) \to \mathsf{Comb}(\mathcal{C})$.

Proof. For each λ there is a mapping:



This preserves the sliding of morphisms v along the ancillary wire. $\square$

Remark. Formally, the mapping above gives a cowedge for $\mathcal{C}(A, - \otimes B) \times \mathcal{C}(= \otimes B', A')$ and must therefore factor uniquely via the coend.

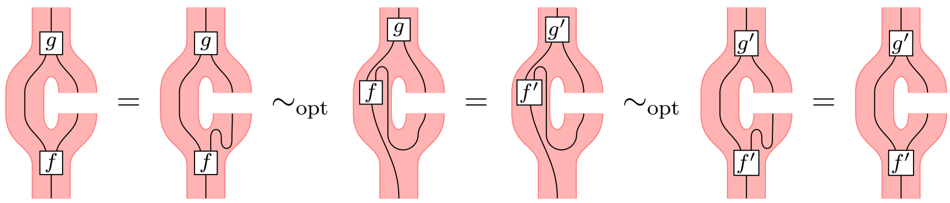
It is not immediately obvious whether the functor of the previous proposition is faithful and thus witnesses an equivalence of categories.

Counterexample 1. Consider the free commutative monoidal category generated by one object A and a single idempotent $f : A \to A$. Then $(1_A, f)_I \nsim_{\text{opt}} (f, 1_A)_I$ but $(1_A, f)_I \sim_{\text{comb}} (f, 1_A)_I$ and thus $\mathsf{Optic}(\mathcal{C}) \ncong \mathsf{Comb}(\mathcal{C})$ in this case.

We now explore some classes of categories where there is an equivalence $\mathsf{Optic}(\mathcal{C}) \cong \mathsf{Comb}(\mathcal{C})$.

Proposition 47. *Given a compact closed category $\mathcal{C}$, there is a symmetric monoidal isomorphism of categories* $\mathsf{Optic}(\mathcal{C}) \cong \mathsf{Comb}(\mathcal{C})$*.*

Proof.



So we have established that comb equivalence implies optic equivalence. This is sufficient to show that the functor of proposition 46 is also faithful. $\square$

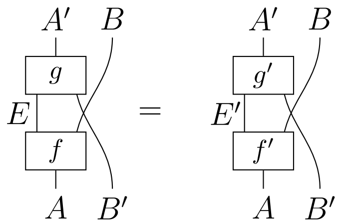
Remark. The previous result could also be established by Yoneda reduction (see e.g. [135, Sec. 4.2]) as follows:

$$\int^{E} \mathcal{C}(A, E \otimes B) \times \mathcal{C}(E \otimes B', A') \cong \int^{E} \mathcal{C}(A, E \otimes B) \times \mathcal{C}(E, B'^{*} \otimes A')$$
$$\cong \mathcal{C}(A, B'^{*} \otimes A' \otimes B) \cong \mathcal{C}(A \otimes B', A' \otimes B)$$

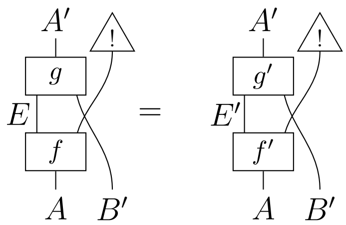
Note that $(f, g)_E \sim_{\text{comb}} (f', g')_{E'}$ implies $(f, g)_E \sim_{\sigma} (f', g')_{E'}$ which ensures they are the same element of the set $\mathcal{C}(A \otimes B', A' \otimes B)$.

Proposition 48. *Given a cartesian category $\mathcal{C}$ where each type is inhabited, so that there exists a state $I \to A$ for any A, there is a symmetric monoidal isomorphism of categories* $\mathsf{Optic}(\mathcal{C}) \cong \mathsf{Comb}(\mathcal{C})$.

Proof. Suppose $(f, g)_E \sim_{\text{comb}} (f', g')_{E'}$. We know that these combs are equal on the braid:

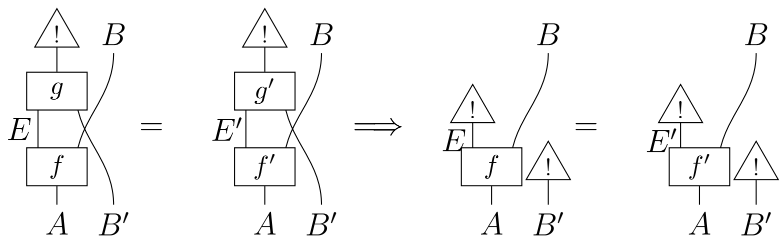


By the universal property of the product, this map is completely determined by its projections into A' and B. The former gives:

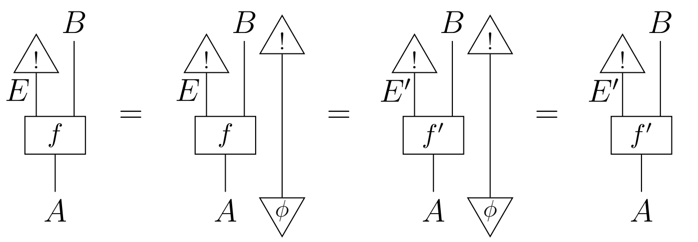


$\in \mathcal{C}(A \times B', A')$ (7.5)

while the latter gives

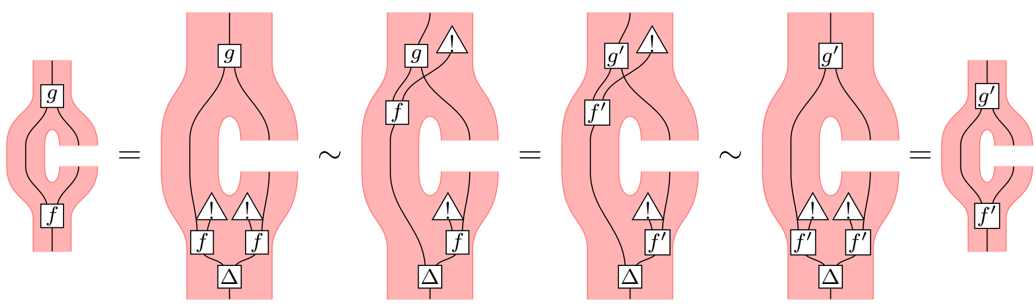


Pick a map $\phi : I \to B'$, then



(7.6)

Thus:



□

Remark. The final part of the proof can also be derived by Yoneda reduction (see e.g. [47, Sec. 3.1]):

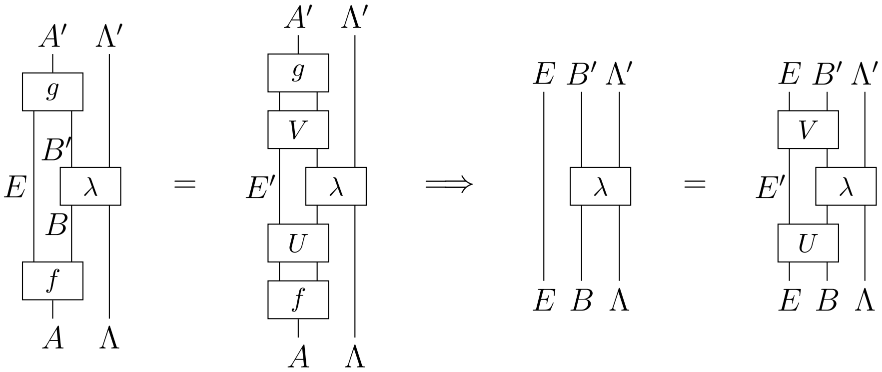
$$\begin{aligned}\int^{E} \mathcal{C}(A, E \times B) \times \mathcal{C}(E \times B', A') &\cong \int^{E} \mathcal{C}(A, E) \times \mathcal{C}(A, B) \times \mathcal{C}(E \times B', A') \\ &\cong \mathcal{C}(A, B) \times \mathcal{C}(A \times B', A')\end{aligned}$$

and then noting that the projections (7.5) and (7.6) precisely determine an element of $\mathcal{C}(A, B) \times \mathcal{C}(A \times B', A')$.

Proposition 49. *There is a symmetric monoidal isomorphism* $\mathsf{Optic}(\mathsf{Unitary}) \cong \mathsf{Comb}(\mathsf{Unitary})$*, where* $\mathsf{Unitary}$ *is the category of unitary maps between (not necessarily finite dimensional) Hilbert spaces.*

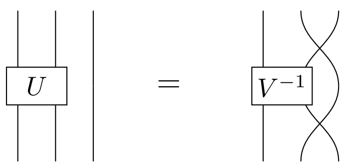
Proof. $f : A \to E \otimes B$ is a unitary and thus $A \cong E \otimes B$ are isomorphic as Hilbert spaces. Similarly from f' we see $A \cong E' \otimes B$ and from g and g', $A' \cong E \otimes B' \cong E' \otimes B'$. This means there must exist a unitary $U : E \otimes B \to E' \otimes B$ such that $f' = Uf$ and a unitary $V : E' \otimes B' \to E \otimes B'$ such that $g' = gV$.

Using the fact that $(f, g)_E \sim_{\text{comb}} (f', g')_{E'}$ and that f and g have two-sided inverses, we see that for all λ:

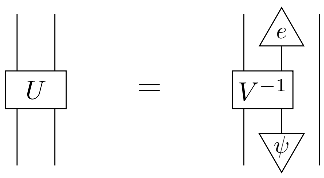


(7.7)

Taking $\lambda = \sigma$ we arrive at the following equality:

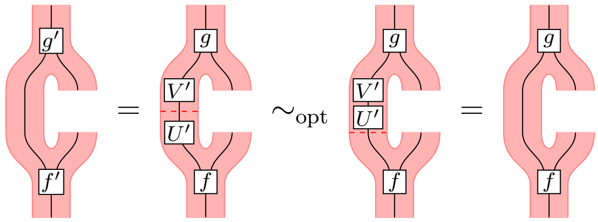


There exists a faithful embedding of $\mathsf{Unitaries}$ into Hilb where we can pick any state $|\psi\rangle$ and effect $\langle e|$ with $\langle e|\psi\rangle = 1$ to see that:



As a result U can be seen to $\otimes$-separate as $U = U' \otimes 1$ where $U' := (1 \otimes e)V^{-1}(1 \otimes \psi)$ must be a unitary else U could not be unitary and we would have a contradiction. Analogously one can show that V $\otimes$-separates as $V' \otimes 1$. Inserting these factorisations into the right hand side of (7.7) one can see that $V'U' = 1$.

Therefore:



□

7.3 n-Combs

In this section we consider generalisations of the Optic and Comb constructions to encompass n-combs. There are several categorical structures that could provide an adequate semantics for dealing with the many inputs and outputs that a generalised n-comb could have. Here we will use polycategories to handle n-combs.

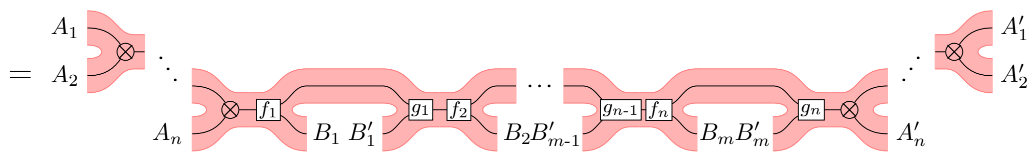
A candidate definition of such an n-comb was suggested in [135] as a generalisation of the Optic construction. We generalise this even further, obtaining a polycategory. Our definition of the combs themselves is similar, but crucially our notion of composition is very different and coincides more closely with that of [52].

Definition 62. Given a symmetric monoidal category $\mathcal{C}$, the polycategory of n-combs $\mathsf{OPTIC}(\mathcal{C})$ has objects given by pairs of objects in $\mathcal{C}$. The polymorphisms of type $[(A_1, A_1'), \ldots, (A_n, A_n')] \to [(B_1, B_1'), \ldots, (B_m, B_m')]$ are elements of the set (where the zero-fold tensor in $\mathcal{C}$ is the tensor unit):

$$\int^{X_0,\ldots,X_{m+1}} \mathcal{C}\left(\bigotimes_{i=1}^{n} A_i, X_0\right) \times \prod_{j=1}^{m} \mathcal{C}(X_{j-1}, X_j \otimes B_j) \times \mathcal{C}(X_j \otimes B_j', X_{j+1}) \times \mathcal{C}\left(X_{m+1}, \bigotimes_{i=1}^{n} A_i'\right)$$

For example, the following is an internal string diagram for a polymorphism of this type (drawn from left to right to conserve space).

$$\langle f_1, \ldots, f_n | g_1, \ldots, g_n \rangle_{X_1,\ldots,X_n} : [(A_1, A_1'), \ldots, (A_n, A_n')] \to [(B_1, B_1'), \ldots, (B_m, B_m')]$$



(7.8)

The identities are the same as in optics. Given a map as above and another map

$$\langle h_0, \ldots, h_\ell | k_0, \ldots, k_n \rangle_{Y_1,\ldots,Y_\ell} : [(C_1, C_1'), \ldots, (C_\ell, C_\ell')] \to [(D_1, D_1'), \ldots, (D_p, D_p')]$$

where $(C_q, C_q') = (B_j, B_j')$ for some $0 \leq q \leq \ell$, $0 \leq j \leq m$. Then the composite

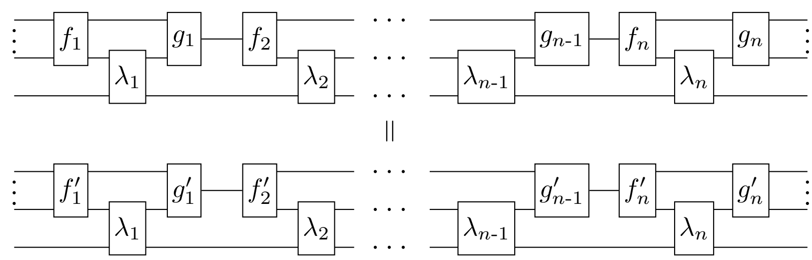
$$\langle f_1, \ldots, f_n | g_1, \ldots, g_n \rangle_{X_1,\ldots,X_n} \circ_{(B_j, B_j')} \langle h_0, \ldots, h_\ell | k_0, \ldots, k_n \rangle_{Y_1,\ldots,Y_\ell}$$

is given by plugging the first comb into the (B_j, B_j') hole and the collapsing the bubble. This can be verified to produce a diagram of the same shape via a lengthy, yet elementary application of the coend calculus, or equivalently a composition of the 2-cells (4.7) and associators.

There is also a polycategory of n-combs that generalises the Comb construction,

Definition 63. Given a symmetric monoidal category $\mathcal{C}$, the polycategory of n-combs $\mathsf{COMB}(\mathcal{C})$ has the same objects as $\mathsf{OPTIC}(\mathcal{C})$. The polymorphisms are given by tuples of maps under a generalisation of the comb equivalence relation where two combs are equivalent if they are equal on all extended inputs:

$\forall \lambda_1, \ldots, \lambda_n$



Composition and identities are the same as in $\mathsf{COMB}(\mathcal{C})$.

As in the case of 1-combs we can always quotient the intensional optics definition to get the extensional comb definition:

Proposition 50. *There is a full and identity on objects polyfunctor* $\mathsf{OPTIC}(\mathcal{C}) \to \mathsf{COMB}(\mathcal{C})$.

Proof. (Sketch) The proof is similar to proposition 46: removing the cobordisms and evaluating the comb on the given $\lambda_1, \ldots, \lambda_n$ gives a cowedge and thus factorises uniquely via the coend. □

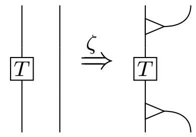
Proposition 51. *When $\mathcal{C}$ is compact closed there is an isomorphism of polycategories* $\mathsf{OPTIC}(\mathcal{C}) \cong \mathsf{COMB}(\mathcal{C})$.

Proof. (Sketch) The isomorphism is shown in a similar way to the proof of Proposition 47, by pulling all of the circuits into the same bubble. □

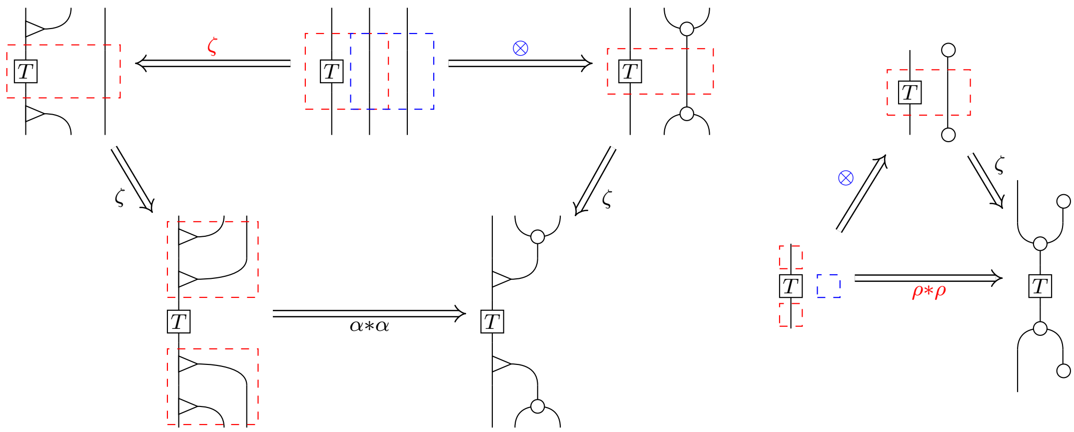
7.3.1 Tambara Modules

Let us now investigate another way to formalise n-combs, this time using the theory of Tambara modules [155, 127]. Suppose we have a category $\mathcal{C}$ with a right action $- \ltimes - : \mathcal{C} \boxtimes \mathcal{M} \to \mathcal{C}$ by a monoidal category $\mathcal{M}$. A Tambara module is an endoprofunctor $T : \mathcal{C} \nrightarrow \mathcal{C}$ equipped with a strength for the action by $\mathcal{M}$.

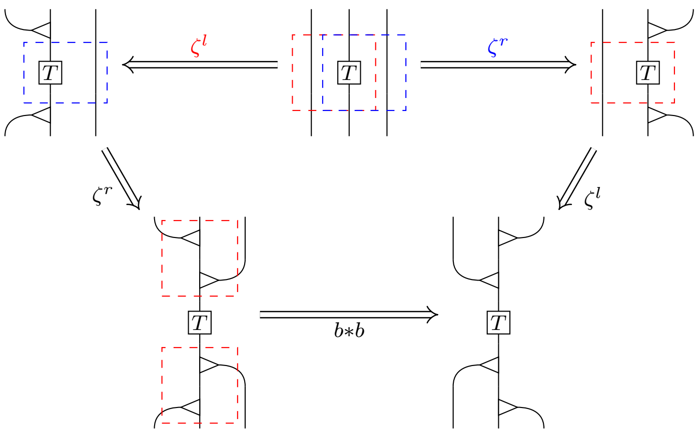
Definition 64 (Tambara Module [155, 127])**.** A right Tambara module (T, ζ) consists of an endoprofunctor $T : \mathcal{C} \nrightarrow \mathcal{C}$ and a natural transformation:



such that the following diagrams commute:

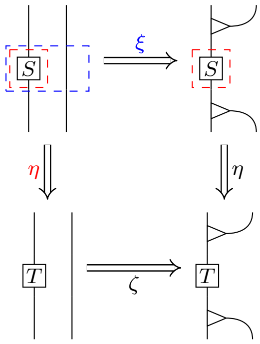


Left Tambara modules for a left actegory and are defined analogously. Given an $\mathcal{M}$-$\mathcal{N}$-biactegory $\mathcal{C}$, we can also define the Tambara bimodules. These are endoprofunctors $T : \mathcal{C} \nrightarrow \mathcal{C}$ which are simultaneously left and right modules with an additional compatibility condition.



Given two Tambara modules we can define the arrows between them.

Definition 65 (Tambara Module Homomorphism [155, 127])**.** A homomorphism $\eta : (S, \zeta) \to (T, \xi)$ of right Tambara modules consists of a natural transformation $\eta : S \to T$ such that the following square commutes.[2]



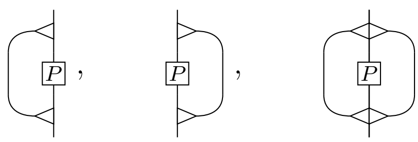
Left, right and bi- Tambara modules and their homomorphisms assemble into categories $\mathsf{LTamb}_{\mathcal{M}}(\mathcal{C})$, $\mathsf{RTamb}_{\mathcal{M}}(\mathcal{C})$ and $\mathsf{Tamb}_{\mathcal{M},\mathcal{N}}(\mathcal{C})$ respectively.

The categories of Tambara modules share a lot in common with the category $\mathsf{Prof}(\mathcal{C})$ of endoprofunctors on $\mathcal{C}$. It is clear that there are forgetful functors U from each which simply forget the strengths. Furthermore given any endoprofunctor there are functors F and G generating free and cofree Tambara modules which are adjoints to the forgetful functor in the expected way [127, 47].

$$\begin{tikzcd} \mathsf{Prof}(\mathcal{C}) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathsf{LTamb}_{\mathcal{M}}(\mathcal{C}) \arrow[l, "U" description] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Prof}(\mathcal{C}) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathsf{RTamb}_{\mathcal{M}}(\mathcal{C}) \arrow[l, "U" description] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Prof}(\mathcal{C}) \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathsf{Tamb}_{\mathcal{M},\mathcal{N}}(\mathcal{C}) \arrow[l, "U" description] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

Given $P : \mathcal{C} \nrightarrow \mathcal{C}$, the free left, right and bi- Tambara modules are given respectively by:



[2]Technically to work in the enriched setting we need to be slightly more careful when defining the homomorphisms of Tambara modules since it is not clear with our presentation that the space of homomorphisms form an object of $\mathcal{V}$. See [127, 47] for the more careful definition. Nevertheless, it is usually okay to think of the homomorphisms as being the natural transformations that commute with the strengths, and when $\mathcal{V} = \mathsf{Set}$ this is true precisely.

The cofree Tambara modules are given by the following formulae involving ends:

$$\int_X P(X \rtimes -, X \rtimes =), \qquad \int_X P(- \ltimes X, = \ltimes X), \qquad \int_{XY} P(X \rtimes - \ltimes Y, X \rtimes = \ltimes Y)$$

Perhaps the most important result of [127] connects Tambara modules with the category of optics. This result was extended in [47] to include Tambara modules for actions as presented here.

Theorem 22 ([127, 47])**.** *There are equivalences of categories:*

$$[\mathsf{LOptic}_{\mathcal{M}}(\mathcal{C})^{op}, \mathcal{V}] \cong \mathsf{LTamb}_{\mathcal{M}}(\mathcal{C})$$
$$[\mathsf{ROptic}_{\mathcal{M}}(\mathcal{C})^{op}, \mathcal{V}] \cong \mathsf{RTamb}_{\mathcal{M}}(\mathcal{C})$$
$$[\mathsf{Optic}_{\mathcal{M},\mathcal{N}}(\mathcal{C})^{op}, \mathcal{V}] \cong \mathsf{Tamb}_{\mathcal{M},\mathcal{N}}(\mathcal{C})$$

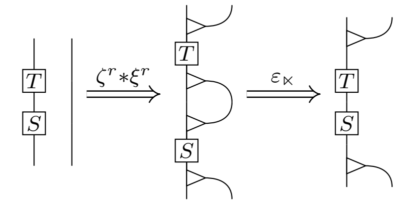
Now, we note that $\mathsf{Prof}(\mathcal{C})$ has a monoidal structure given by composition of the endoprofunctors, $P \otimes_V Q := P \circ Q$. Furthermore this monoidal structure is closed - the internal-hom $[-, =]_V$ can be calculated with some coend calculus.

$$\begin{aligned}
\mathrm{Nat}(P \otimes_V Q, R) &\cong \int_{XZ} \mathcal{V}\left(\int^Y P(X,Y) \boxtimes Q(Y,Z), R(X,Z)\right) \\
&\cong \int_{XYZ} \mathcal{V}\Big(P(X,Y), \mathcal{V}\big(Q(Y,Z), R(X,Z)\big)\Big) \\
&\cong \int_{XY} \mathcal{V}\left(P(X,Y), \int_Z \mathcal{V}\big(Q(Y,Z), R(X,Z)\big)\right) \\
&\cong \mathrm{Nat}\left(P(-,=), \int_Z \mathcal{V}\big(Q(=,Z), R(-,Z)\big)\right)
\end{aligned}$$

One consequence of this is to equip $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ with a promonoidal structure. The category of endoprofunctors is precisely the presheaf category $\mathsf{Prof}(\mathcal{C}) \cong [\mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C}, \mathcal{V}]$. We then note that Day's theorem gives an equivalence between closed monoidal structures on presheaf categories and promonoidal structures on the underlying category. The behaviour of this promonoidal structure on $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$, can be calculated by evaluating the monoidal structure of $\mathsf{Prof}(\mathcal{C})$ on representable presheaves.

$$\begin{aligned}
よ_{\mathbf{A}} \otimes_V よ_{\mathbf{B}} &= (\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}})((-,=),(A,A')) \otimes_V (\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}})((-,=),(B,B')) \\
&= (\mathcal{C}(-,A) \boxtimes \mathcal{C}(A',=)) \otimes_V (\mathcal{C}(-,B) \boxtimes \mathcal{C}(B',=)) \\
&= \int^X \mathcal{C}(-,A) \boxtimes \mathcal{C}(A',X) \boxtimes \mathcal{C}(X,B) \boxtimes \mathcal{C}(B',=) \\
&\cong \mathcal{C}(-,A) \boxtimes \mathcal{C}(A',B) \boxtimes \mathcal{C}(B',=)
\end{aligned}$$

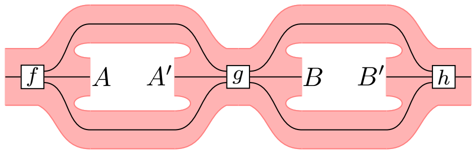
Similarly to $\mathsf{Prof}(\mathcal{C})$, the categories of Tambara modules also have closed monoidal structures given by composition of the endoprofunctors together with the following induced strength.



This in turn induces promonoidal structures on $\mathsf{LOptic}_{\mathcal{M}}(\mathcal{C})$, $\mathsf{ROptic}_{\mathcal{M}}(\mathcal{C})$ and $\mathsf{Optic}_{\mathcal{M},\mathcal{N}}(\mathcal{C})$. On the latter this acts as,

$$(\mathbf{A} \otimes_V \mathbf{B})(\mathbf{C}) = \int^{XX'YY'} \mathcal{C}(C, X \rtimes A \ltimes X') \boxtimes \mathcal{C}(X \rtimes A' \ltimes X', Y \rtimes B \ltimes Y') \boxtimes \mathcal{C}(Y \rtimes B' \ltimes Y', C')$$

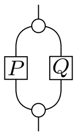
Simplifying to the case when a monoidal $\mathcal{C}$ acts canonically on itself we can draw the tensor $\otimes_V$ in the internal string diagrams as follows.



(7.9)

Now, when $\mathcal{C}$ is monoidal, the category $\mathsf{Prof}(\mathcal{C})$ permits another tensor product given by Day convolution over the monoidal structure of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$.

$$(P \otimes_H Q)(-, =) = \int^{XX'YY'} \mathcal{C}(-, X \otimes Y) \boxtimes P(X, X') \boxtimes Q(Y, Y') \boxtimes \mathcal{C}(X' \otimes Y', =)$$

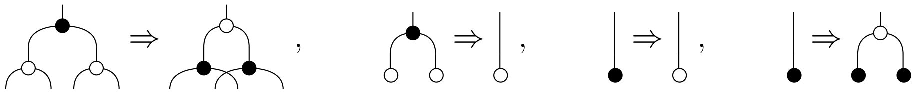
$=$ 

The tensor $\otimes_H$ is also closed, the internal-hom $[-,=]_H$ can be calculated with some coend calculus.

$$\begin{aligned}
&\mathrm{Nat}(P \otimes_H Q, R) \\
&\cong \int_{ZZ'} \mathcal{V}\left(\int^{XX'YY'} \mathcal{C}(Z, X \otimes Y) \boxtimes P(X, X') \boxtimes Q(Y, Y') \boxtimes \mathcal{C}(X' \otimes Y', Z'), R(Z, Z') \right) \\
&\cong \int_{XX'} \mathcal{V}\Big(P(X, X'), \int_{YY'} \mathcal{V}\big(Q(Y, Y'), R(X \otimes Y, X' \otimes Y')\big) \Big) \\
&\cong \mathrm{Nat}\left(P(-, =), \int_{YY'} \mathcal{V}\big(Q(Y, Y'), R(- \otimes Y, = \otimes Y')\big) \right)
\end{aligned}$$

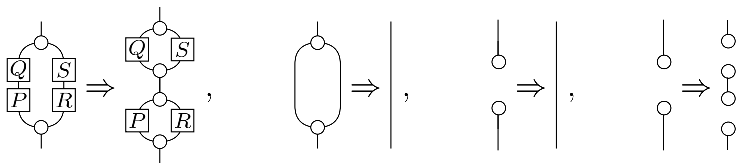
The two tensors $\otimes_V$ and $\otimes_H$ interact to make $\mathsf{Prof}(\mathcal{C})$ into a *duoidal* category [83].

Definition 66 (Duoidal Category)**.** A category $\mathcal{C}$ is duoidal when it is equipped with two monoidal structures (,) and (,) and the following natural transformations witnessing the distributivity of one tensor over another.



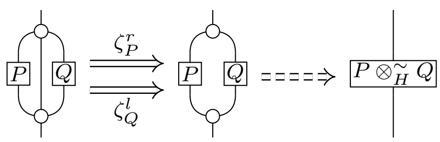
These structural transformations must satisfy a series of coherence conditions which can be found in e.g. [4, 13, 20]. A duoidal category is *normal* when the two units coincide $=$, and *closed* when both tensors are closed.

For $\mathsf{Prof}(\mathcal{C})$ the natural transformations giving the duoidal structure are as follows.



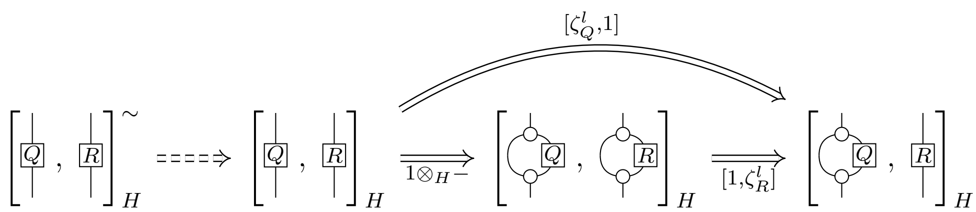
Normal duoidal categories are interesting because they can be interpreted as roughly encoding a notion of dependent and independent composition [144] with one distributing over the other. For $\mathsf{Prof}(\mathcal{C})$, the dependent composition is $\otimes_V$ and the independent is $\otimes_H$, roughly we might think of these as timelike and spacelike composition operations.

$\mathsf{Tamb}(\mathcal{C})$ inherits a tensor given by a quotient of the tensor $\otimes_H$ on $\mathsf{Prof}(\mathcal{C})$ [83]. More precisely, suppose P and Q are now Tambara modules, then we take the coequaliser of the following pair of arrows.



(7.10)

Note that for presentational simplicity we have freely neglected the associators α. As a result we get a tensor $\otimes^{\sim}_H$ (which for notational simplicity we now will also refer to as $\otimes_H$) on $\mathsf{Tamb}(\mathcal{C})$. This tensor is also closed [83], with the internal-hom (for the left closure) constructed by an equaliser, similar to that presented in [15].

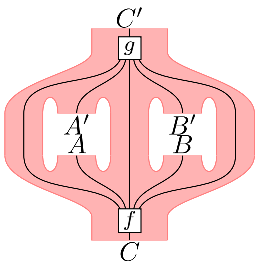


The internal-hom constructed above is precisely the object of natural transformations that equalise the left strengths of Q and R. The internal-hom for the right closure is similar.

Since $\otimes_H$ is closed on $\mathsf{Tamb}(\mathcal{C})$ we yield another promonoidal structure on $\mathsf{Optic}(\mathcal{C})$. We can find the action of this promonoidal structure by calculating the coequaliser (7.10) on representable presheaves.

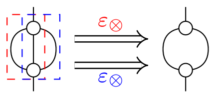
Proposition 52. *The tensor $\otimes_H$ acts on representable presheaves as:*

$$(よ_{\mathbf{A}} \otimes_H よ_{\mathbf{B}})(-,=) = \int^{XYZ} \mathcal{C}(-, X \otimes A \otimes Y \otimes B \otimes Z) \boxtimes \mathcal{C}(X \otimes A' \otimes Y \otimes B' \otimes Z, =)$$



(7.11)

Proof. The proof essentially comes down to finding the coequaliser of the following parallel pair (freely neglecting associators).



(7.12)

This coequaliser is the value of $1_\mathcal{C} \otimes_H 1_\mathcal{C}$ where $1_\mathcal{C} = \mathcal{C}(-,=)$. It is demonstrated in [83] that $1_\mathcal{C}$ is the unit of the monoidal structure $\otimes_H$ so that $1_\mathcal{C} \otimes_H 1_\mathcal{C} \cong 1_\mathcal{C}$. □

So we have seen that $\mathsf{Tamb}(\mathcal{C})$ has two closed monoidal structures $\otimes_V$ and $\otimes_H$ inherited from the analogous ones on $\mathsf{Prof}(\mathcal{C})$. It is demonstrated in [83] that the duoidal structure of $\mathsf{Prof}(\mathcal{C})$ also transfers to $\mathsf{Tamb}(\mathcal{C})$, which now has the same unit for both tensors and is therefore a normal duoidal category. We can also discuss these tensors at the level of $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ and $\mathsf{Optic}(\mathcal{C})$ but we require the notion of a *pro*duoidal category.

Definition 67 (Produoidal Category [33])**.** A category $\mathcal{C}$ is produoidal when it is equipped with two promonoidal structures and natural transformations analogous to those of Definition 66 (where we now interpret the string diagrams in Prof), satisfying analogous coherence conditions. A produoidal category is normal when the unit presheaves of the two promonoidal structures are equal.

The relationship between duoidal and produoidal categories is akin to that between monoidal and promonoidal ones. Day's theorem can be extended to this setting to show that there is an equivalence between produoidal structures on $\mathcal{C}$ and closed duoidal structures on $\widehat{\mathcal{C}}$.

The end result of this discussion is that, given a monoidal category $\mathcal{C}$, we now have a normal produoidal category $\mathsf{Optic}(\mathcal{C})$ which allows us to discuss the horizontal and vertical composition of holes in $\mathcal{C}$. That is, it is an ideal replacement for the polycategorical structure of Section 7.3 allowing us to study these structures without having to leave the world of monoidal categories.

Whilst the tensor $\otimes_V$ has been known since the original work on the category of optics [127], its use in applications has been rather neglected. Perhaps this can be attributed to the complexity and initial conceptual overhead in working with promonoidal categories. Nevertheless, given the natural interpretation of $\otimes_V$ as the vertical composition of holes, it seems an ideal setting for the study of n-combs.

On the other hand, the tensor $\otimes_H$ has only been discussed on the category of Tambara modules and its interpretation as a promonoidal structure on optics does not seem to have been noticed until very recently [72]. One interesting aspect of this tensor is that is does *not* require symmetry of $\mathcal{C}$ in order to define horizontal composition of holes. Typically, $\mathcal{C}$ is assumed to be symmetric from the outset, in which case $\mathsf{LOptic}(\mathcal{C})$ permits a tensor product given by horizontal composition of combs [134]. Let us see how this arises from the promonoidal structure $\otimes_H$.

Recall that $P \otimes_H Q$ coequalises the *right* action on P and the *left* action on Q. This was possible in $\mathsf{Tamb}(\mathcal{C})$ because all objects are bimodules and thus have both left and right actions. Given two left modules or two right modules there is no clear way of defining an analogous tensor, but in the presence of symmetry on $\mathcal{C}$, all left Tambara modules are canonically right Tambara modules, and vice versa. This allows us to extend the tensor $\otimes_H$ to the categories $\mathsf{LTamb}(\mathcal{C})$ and $\mathsf{RTamb}(\mathcal{C})$ - to define $P \otimes_H Q$ we simply regard P as a right module and Q as a left module and take the coequaliser as before.

Remark. It is worth noting that $\otimes_H$ is conceptually no different than the tensor product of modules over a ring, and on $\mathsf{LTamb}(\mathcal{C})$ and $\mathsf{RTamb}(\mathcal{C})$, $\otimes_H$ is analogous to the tensor product of modules over a commutative ring.

The following result recovers the tensor on the category of optics in the presence of symmetry.

Proposition 53. *Let $\mathcal{C}$ be symmetric. Then* $\mathsf{LTamb}(\mathcal{C})$ *has a tensor product* $\otimes_H$ *given by the coequaliser* (7.10)*. On representable presheaves of* $\mathsf{LOptic}(\mathcal{C})$ *this acts as follows.*

$$(よ_{\mathbf{A}} \otimes_H よ_{\mathbf{B}})(-,=) = \int^{X} \mathcal{C}(-, X \otimes A \otimes B) \boxtimes \mathcal{C}(X \otimes A' \otimes B', =)$$

As a result, $\otimes_H$ *is representable and descends to a tensor on* $\mathsf{LOptic}(\mathcal{C})$ *which acts on objects as* $(A, A') \otimes_H (B, B') = (A \otimes B, A' \otimes B')$*.*

Proof. Given left Tambara modules P and Q define their tensor by the coequaliser (7.10) by considering P with its canonical right action given by applying the symmetry to the left action. Taking representable presheaves and calculating the coequaliser proceeds much like Proposition 52 and yields the claimed expression. One then notes that this is precisely the representable Tambara module at $(A \otimes B, A' \otimes B')$. $\square$

Consequently, $\mathsf{LTamb}(\mathcal{C})$ has both a tensor product $\otimes_H$ and a promonoidal tensor $\otimes_V$. These are precisely the structures required to capture the polycategorical structure described in Section 7.3. For instance, a polymorphism (7.8) in $\mathsf{OPTIC}(\mathcal{C})$ is given by a morphism in $\mathsf{LTamb}(\mathcal{C})$ between the the representable presheaves at $\otimes_H$ tensor products of objects of $\mathsf{Optic}(\mathcal{C})$ and presheaves in the image of $\otimes_V$. For instance, consider an arrow in $\mathsf{LTamb}(\mathcal{C})$ of the type:

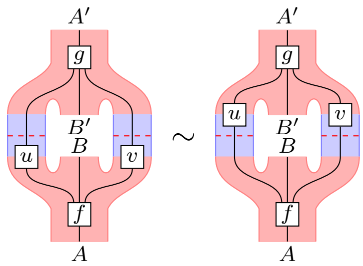
$$(よ_{\mathbf{B_1} \otimes_H \cdots \otimes_H \mathbf{B_m}})(-,=) \to (\mathbf{A_1} \otimes_V \mathbf{A_2} \otimes_V \cdots \otimes_V \mathbf{A_n})(-,=)$$

By the Yoneda lemma there is a natural isomorphism between such arrows and the object of arrows of $\mathsf{OPTIC}(\mathcal{C})$.

$$(\mathbf{A_1} \otimes_V \mathbf{A_2} \otimes_V \cdots \otimes_V \mathbf{A_n})(\mathbf{B_1} \otimes_H \cdots \otimes_H \mathbf{B_m}) =$$
$$\int^{X_1 \ldots X_n} \mathcal{C}\left(\bigotimes_{i=i}^{m} B_i, X_1 \otimes A_1\right) \boxtimes \mathcal{C}(X_1 \otimes A_1', X \otimes A_2) \boxtimes \cdots \boxtimes \mathcal{C}\left(X_n \otimes A_n', \bigotimes_{i=1}^{m} B_i'\right)$$

7.4 Premonoidal Optics

In this section we will generalise optics over a monoidal base $\mathcal{C}$ to allow for a premonoidal base $\mathcal{C}$. Since not all morphisms in a premonoidal category interchange, there is now additional subtlety to formalising optics. Firstly, the coends that are usually used to quotient to allow for the sliding of morphisms can only be taken over the centre $Z\mathcal{C}$ of the premonoidal category $\mathcal{C}$. This leads us to consider the category $\mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})$ with objects given by pairs (A, A') of those of $\mathcal{C}$ and homs given by the coend $\int^{XY \in Z\mathcal{C}} \mathcal{C}(A, X \otimes B \otimes Y) \boxtimes \mathcal{C}(X \otimes B' \otimes Y, A')$, allowing us to slide only central morphisms between the top and bottom of the optic.



We might then hope to equip $\mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})$ with two tensors analogous to those in equations (7.9) and (7.11). While the vertical tensor $\otimes_V$ poses no immediate difficulties, the horizontal tensor $\otimes_H$ does: we cannot expect this to be promonoidal because $\mathcal{C}$ does not satisfy interchange. This leads us to our first bona-fide example of a pro-effectful category.

Secondly, there is an additional challenge with premonoidal optics: there is the category $\mathsf{Optic}(Z\mathcal{C})$ of optics over the monoidal centre and an embedding, $\mathsf{Optic}(Z\mathcal{C}) \to \mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})$, of these central optics into the optics over the entire premonoidal category. $\mathsf{Optic}(Z\mathcal{C})$ is equipped with the two promonoidal structures, $\otimes_H$ and $\otimes_V$, and we would like to understand how these behave in relation to any tensors we can define on $\mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})$. This requires us to keep track of the centre and understand fully how it behaves in relation to the rest of the premonoidal category. Thus we must take seriously the $\mathcal{V}^2$-enrichment of the category of optics over a premonoidal base.

Whilst optics over a premonoidal base may at first seem niche there are several plausible applications we foresee. In a seminal work on optics, Riley [134] introduced the notion of "effectful optics": optics over the Kleisli category of a strong monad. These optics allow the emergence of side-effects, and extend the optics of pure functional programming to other programming languages with effects; with a similar purpose, Abou-Saleh et al. [1] have introduced "monadic lenses". More recently, much applied category theory has been written about optics that create effects in different categories [31, 36, 47, 150]. The novel definition of optic over an effectful category introduced here serves to justify this previous terminology: optics over the Kleisli category of a strong monad are particular cases of our effectful optics. Furthermore, our approach leads to a pro-effectful algebra over them that had been previously neglected.

We also envisage applications in physics. If premonoidal categories are deemed to be a useful tool for modelling spacetime [28, 60, 97] or causal structure more generally, we might expect such structure to arise in models of supermaps. Similarly to how we have argued for the usage of optics to model combs over monoidal categories, the premonoidal optics developed here could be a natural category to model combs over premonoidal spacetime categories.

Let us now present the category of optics over a premonoidal category and outline its two tensor-like structures. Suppose we fix a premonoidal category $\mathcal{C}$ and write $J : Z\mathcal{C} \to \mathcal{C}$ for the inclusion of the centre. There is a $\mathcal{V}^2$-category $\mathsf{Optic}(J)$ with objects given by pairs $\mathbf{A} := (A, A')$ of those of J, i.e. pairs of those of the underlying premonoidal category $\mathcal{C}$. The homs are given by

$$\int^{XY} Z\mathcal{C}(A, X \otimes B \otimes Y) \boxtimes Z\mathcal{C}(X \otimes B' \otimes Y, A') \to \int^{XY \in Z\mathcal{C}} \mathcal{C}(A, X \otimes B \otimes Y) \boxtimes \mathcal{C}(X \otimes B' \otimes Y, A')$$

as in Figure 7.1. Thus $\mathsf{Optic}(J)_0 = \mathsf{Optic}(Z\mathcal{C})$ is the usual category of optics over the centre and $\mathsf{Optic}(J)_1 = \mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})$ is the category of optics given by the action of the centre $Z\mathcal{C}$ on the whole premonoidal category $\mathcal{C}$. The identity on objects functor $\mathsf{Optic}(Z\mathcal{C}) \to \mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})$ is the one induced by J.

Theorem 23. $\mathsf{Optic}(J)$ *is a promonoidal* $\mathcal{V}^2$*-category. The* $\mathcal{V}^2$*-profunctors forming the tensor product* $P : \mathsf{Optic}(J) \boxtimes \mathsf{Optic}(J) \nrightarrow \mathsf{Optic}(J)$ *and unit* $I : 1 \nrightarrow \mathsf{Optic}(J)$ *have components given in Figures 7.2 and 7.3. These are explicitly,*

$$P_0(\mathbf{C}, \mathbf{A}, \mathbf{B}) = \int^{XX'YY'} Z\mathcal{C}(C, X \otimes A \otimes X') \boxtimes Z\mathcal{C}(X \otimes A' \otimes X', Y \otimes B \otimes Y') \boxtimes Z\mathcal{C}(Y \otimes B' \otimes Y', C'),$$

$$P_1(\mathbf{C}, \mathbf{A}, \mathbf{B}) = \int^{XX'YY' \in Z\mathcal{C}} \mathcal{C}(C, X \otimes A \otimes X') \boxtimes \mathcal{C}(X \otimes A' \otimes X', Y \otimes B \otimes Y') \boxtimes \mathcal{C}(Y \otimes B' \otimes Y', C'),$$

$$I_0(\mathbf{A}) = Z\mathcal{C}(A, A'), \qquad I_1(\mathbf{A}) = \mathcal{C}(A, A').$$

Proof. J has commutative left and right actions by the monoidal $\mathcal{V}^2$-category $1_{Z\mathcal{C}} : Z\mathcal{C} \to Z\mathcal{C}$. Consider the $\mathcal{V}^2$-category $\mathsf{Tamb}(J)$ of Tambara modules on J [127, 47], whose objects are the $\mathcal{V}^2$-endoprofunctors $P : J \nrightarrow J$ equipped with left and right strengths over the action by $1_{Z\mathcal{C}}$. The morphisms are the bistrong $\mathcal{V}^2$-natural transformations. We can use Proposition 39 to unpack $\mathsf{Tamb}(J)$ into two $\mathcal{V}$-categories and an identity on objects functor, $\mathsf{Tamb}(J)_0 \to \mathsf{Tamb}(J)_1$. The objects of $\mathsf{Tamb}(J)_0$ and $\mathsf{Tamb}(J)_1$ are the bistrong endoprofunctors $P : J \nrightarrow J$ which are equivalently triples $(P_0 : Z\mathcal{C} \nrightarrow Z\mathcal{C}, P_1 : \mathcal{C} \nrightarrow \mathcal{C}, \eta : P_0 \Rightarrow P_1(J^{\text{op}} \boxtimes J))$. $\mathsf{Tamb}(J)_0$ has arrows $\phi : P \Rightarrow Q$ given by pairs $(\phi_0 : P_0 \Rightarrow Q_0, \phi_1 : P_1 \Rightarrow Q_1)$ while $\mathsf{Tamb}(J)_1$ has only the ϕ_1 as arrows.

It is known that the category of Tambara modules is equivalent to the presheaf category of the category of optics [127, 47], which in this particular case implies

$[\mathsf{Optic}(J)^{\mathrm{op}}, \mathcal{V}^2] \cong \mathsf{Tamb}(J)$. The $\mathcal{V}^2$-category $\mathsf{Optic}(J)$ has objects given by pairs $\mathbf{A} = (A, A')$ of $\mathsf{Optic}(J)$ and homs given by

$$\mathsf{Optic}(J)(\mathbf{A}, \mathbf{B}) = \int^{XY \in 1_{Z\mathcal{C}}} J(A, X \otimes B \otimes Y) \boxtimes J(X \otimes B' \otimes Y, A')$$

where $J(-,-) := Z\mathcal{C}(-,-) \to \mathcal{C}(-,-)$ is the hom of J as a $\mathcal{V}^2$-category and the coend is taken in this fully enriched setting. By Proposition 40 this coend is given by the following arrow.

$$\int^{XY} Z\mathcal{C}(A, X \otimes B \otimes Y) \boxtimes Z\mathcal{C}(X \otimes B' \otimes Y, A') \to \int^{XY \in Z\mathcal{C}} \mathcal{C}(A, X \otimes B \otimes Y) \boxtimes \mathcal{C}(X \otimes B' \otimes Y, A')$$

This recovers the expected the identity on objects functor $\mathsf{Optic}(Z\mathcal{C}) \to \mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})$ equivalent to $\mathsf{Optic}(J)$.

Now, since $\mathsf{Tamb}(J)$ has a closed monoidal structure given by composition of the profunctors, there is an induced promonoidal structure on $\mathsf{Optic}(J)$. To arrive at the explicit expressions claimed in the Theorem, take objects $\mathbf{A}$ and $\mathbf{B}$ of $\mathsf{Optic}(J)$ and consider the tensor (i.e. composition as profunctors) of the associated representable presheaves.

$$(y_{\mathbf{A}} \boxtimes y_{\mathbf{B}})(-) \cong \int^{WXYZ \in 1_{Z\mathcal{C}}} J(-, W \otimes A \otimes X) \boxtimes J(W \otimes A' \otimes X, Y \otimes B \otimes Z) \boxtimes J(Y \otimes B' \otimes Z, -)$$

This can be unpacked by Proposition 40 to give the result.

Finally note that the unit of the monoidal structure on $\mathsf{Tamb}(J)$ is $1_J : J \nrightarrow J$, which is $(1_{Z\mathcal{C}}, 1_{\mathcal{C}}, \eta : 1_{Z\mathcal{C}} \Rightarrow y^J y_J)$. $\square$

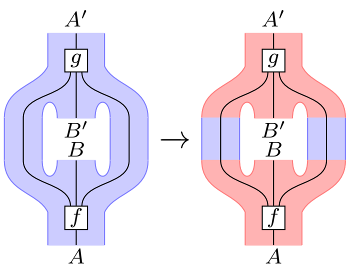


Figure 7.1: Homs.

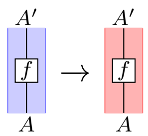


Figure 7.2: Promonoidal unit I.

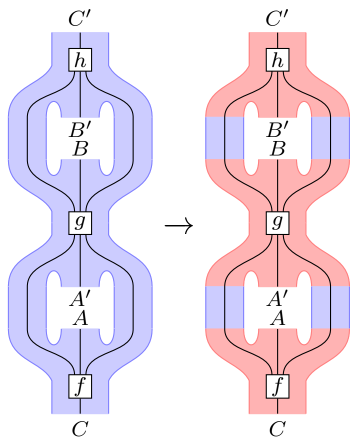


Figure 7.3: Promonoidal tensor P.

Now let us turn our attention to another tensor-like structure on $\mathsf{Optic}_{1_{Z\mathcal{C}}}(J)$, this one induced by the premonoidal structure on $\mathcal{C}$.

Theorem 24. $\mathsf{Optic}(J)$ *is a pro-effectful category. The tight $\mathcal{V}^2$-profunctors forming the tensor product $P : \mathsf{Optic}(J) \boxtimes \mathsf{Optic}(J) \to \overline{\mathsf{Optic}(J)}$ and unit $I : 1 \to \overline{\mathsf{Optic}(J)}$ have components which act on objects as,*

$$\begin{aligned} &P_0(\mathbf{C}, \mathbf{A}, \mathbf{B}) = P_1(\mathbf{C}, \mathbf{A}, \mathbf{B}) \\ &= \int^{XYZ} Z\mathcal{C}(C, X \otimes A \otimes Y \otimes B \otimes Z) \boxtimes Z\mathcal{C}(X \otimes A' \otimes Y \otimes B' \otimes Z, C'), \qquad (7.13) \\ &I_0(\mathbf{A}) = I_1(\mathbf{A}) = Z\mathcal{C}(A, A'). \end{aligned}$$

Proof. The free tight cocompletion of $\mathsf{Optic}(J)$ is given by $[\mathsf{Optic}(Z\mathcal{C})^{\mathrm{op}}, \mathcal{V}] \to \overline{\mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})}$. We will show that this is a closed effectful category and then by Theorem 21 we will be done.

Start by considering the effectful category $J^{\mathrm{op}} \boxtimes J : Z\mathcal{C}^{\mathrm{op}} \boxtimes Z\mathcal{C} \to \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C}$. The free tight cocompletion of this category is $\mathrm{Lan}^L_{J^{\mathrm{op}} \boxtimes J} : \mathsf{Prof}(Z\mathcal{C}) \to \overline{\mathsf{Prof}(\mathcal{C})}$ which is closed effectful. The domain is the duoidal category $\mathsf{Prof}(Z\mathcal{C})$ of endoprofunctors on $Z\mathcal{C}$ and it has a closed monoidal structure given by Day convolution over the monoidal structure of $Z\mathcal{C}$:

$$P * Q := \int^{AA'BB'} Z\mathcal{C}(-, A \otimes A') \boxtimes P(A, B) \boxtimes Q(A', B') \boxtimes Z\mathcal{C}(B \otimes B', -) \qquad (7.14)$$

The premonoidal structure on $\overline{\mathsf{Prof}(\mathcal{C})}$ is given on objects by (7.14). On homs, given a $\eta : P \Rightarrow P'$ in $\overline{\mathsf{Prof}(\mathcal{C})}$ (that is, a $\eta : \mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J} P \Rightarrow \mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J} P'$) the left side of the premonoidal structure is given by:

$$\begin{aligned} \mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J}(P * Q) &\cong \int^{AA'BB'} \mathcal{C}(-, J(A \otimes A')) \boxtimes P(A, B) \boxtimes Q(A', B') \boxtimes \mathcal{C}(J(B \otimes B'), -) \\ &\cong \int^{A'B' \in Z\mathcal{C}, CD \in \mathcal{C}} \mathcal{C}(-, C \rtimes A')) \boxtimes (\mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J} P)(C, D) \boxtimes Q(A', B') \boxtimes \mathcal{C}(D \rtimes B', -) \\ &\xRightarrow{\int \eta} \int^{A'B' \in Z\mathcal{C}, CD \in \mathcal{C}} \mathcal{C}(-, C \rtimes A')) \boxtimes (\mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J} P')(C, D) \boxtimes Q(A', B') \boxtimes \mathcal{C}(D \rtimes B', -) \\ &\cong \mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J}(P' * Q) \end{aligned}$$

Since $\mathrm{Lan}^L_{J^{\mathrm{op}} \boxtimes J}$ is a left adjoint, it follows that it is a closed effectful category.

There is a $\mathcal{V}^2$-category $\mathsf{Tamb}(Z\mathcal{C}) \to \overline{\mathsf{Tamb}(\mathcal{C})}$ with objects given by the Tambara modules on $Z\mathcal{C}$. The homs of $\mathsf{Tamb}(Z\mathcal{C})$ are the bistrong natural transformations while the homs of $\overline{\mathsf{Tamb}(\mathcal{C})}$ are the bistrong natural transformations between the left Kan extensions along $J^{\mathrm{op}} \boxtimes J$ of the Tambara modules. This $\mathcal{V}^2$-category inherits a closed effectful structure from $\mathrm{Lan}^L_{J^{\mathrm{op}} \boxtimes J}$ given by a certain quotient of (7.14) which acts to normalise the duoidal structure on $\mathsf{Prof}(Z\mathcal{C})$ [83, 72].

Finally note that the presheaf category of optics is equivalent to the category of Tambara modules, $\widehat{\mathsf{Optic}(Z\mathcal{C})^{\mathrm{op}}} \cong \mathsf{Tamb}(Z\mathcal{C})$ [47], and we can finally check that we also have $\overline{\mathsf{Optic}_{Z\mathcal{C}}(\mathcal{C})} \cong \overline{\mathsf{Tamb}(\mathcal{C})}$. $\square$

On the homs of $Z\mathcal{C}$, P_0 and I_0 act in the expected way, essentially by nesting of optics. On the homs of $\mathcal{C}$, P_1 and I_1 act somewhat unusually. Formally non-central optics are sent to natural transformations between left Kan extensions of the expressions in (7.13), that is between presheaves of the form:

$$
\begin{aligned}
&(\mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J} P_0)(\mathbf{C}, \mathbf{A}, \mathbf{B}) \\
&\cong \int^{WVXYZ} \mathcal{C}(C, JW) \boxtimes Z\mathcal{C}(W, X \otimes A \otimes Y \otimes B \otimes Z) \boxtimes Z\mathcal{C}(X \otimes A' \otimes Y \otimes B' \otimes Z, V) \boxtimes \mathcal{C}(JV, C') \\
&\cong \int^{XYZ \in Z\mathcal{C}} \mathcal{C}(C, X \otimes A \otimes Y \otimes B \otimes Z) \boxtimes \mathcal{C}(X \otimes A' \otimes Y \otimes B' \otimes Z, C')
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{Lan}_{J^{\mathrm{op}} \boxtimes J} I_0)(\mathbf{A}) &\cong \int^{XY} \mathcal{C}(A, JX) \boxtimes Z\mathcal{C}(X, Y) \boxtimes \mathcal{C}(JY, A') \\
&\cong \int^{X \in Z\mathcal{C}} \mathcal{C}(A, X) \boxtimes \mathcal{C}(X, A')
\end{aligned}
$$

This justifies thinking of the pro-effectful structure as having the components described in Figures 7.4 and 7.5.

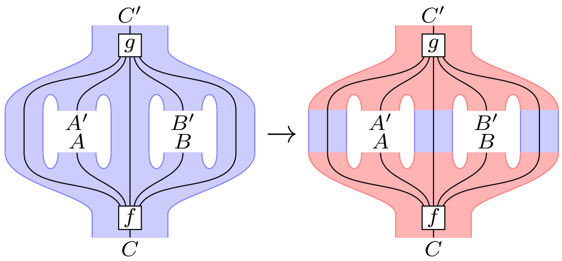


Figure 7.4: Pro-effectful tensor.

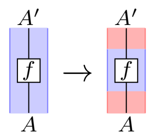
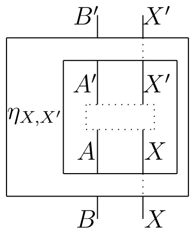


Figure 7.5: Pro-effectful unit.

7.5 Supermaps as Tambara Module Homomorphisms

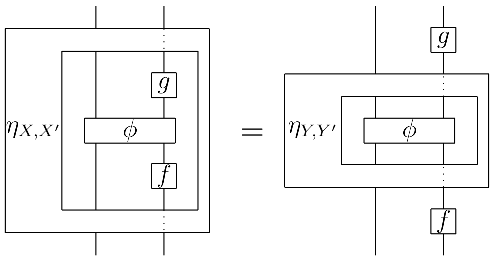
In this final section we will discuss some connections between our suggestion of using optics to model combs and the approach of [161, 160]. There, an abstract framework is developed for supermaps over any symmetric monoidal category using a notion they introduce called a *locally-applicable transformation*. When the base category is taken to be the category of quantum channels they demonstrate that their definition recovers the more traditional definition of deterministic quantum supermap [45]. This is pleasing since most other approaches require additional structure such as compact closure [110], on the category of first-order processes, and therefore their framework offers a vast generalisation of the notion of supermap applicable to many other settings.

Given a symmetric monoidal category, [161] defines a locally-applicable transformation $\eta : (A, A') \to (B, B')$ to consist of a family of maps for each pair of systems X, X' of the form:



These maps must satisfy the following two laws, capturing the idea that the supermap only acts locally on the systems A and A'.

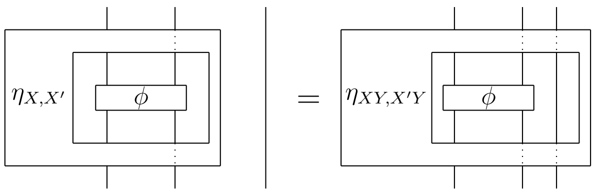
- Naturality



(7.15)

  states that the supermap commutes with the actions of agents on other systems.

- Strength (known originally as *dragging*):



(7.16)

  states that $\otimes$-separable parties are not affected by the supermap.

Wilson et al. describe how the naturality law (7.15) can be captured by saying that η is a natural transformation of the following type.

$$\eta : \mathcal{C}(A \otimes -, A' \otimes =) \Rightarrow \mathcal{C}(B \otimes -, B' \otimes =). \tag{7.17}$$

This makes their locally-applicable transformations arrows of the category $\mathsf{Prof}(\mathcal{C})$.

What they do not discuss is how to capture the strength law (7.16). The first thing to note is that for any A and A', $\mathcal{C}(A \otimes -, A' \otimes =)$ is a Tambara module with the strength maps given as follows.

$$\mathcal{C}(A \otimes X, A' \otimes X') \to \mathcal{C}(A \otimes X \otimes Y, A' \otimes X' \otimes Y) :: \phi \mapsto \phi \otimes 1_Y$$

It is straightforward to check that to ask for a strong natural transformation of the type (7.17) is exactly to ask for the strength law (7.16). As a result locally-applicable transformations are precisely arrows of the category $\mathsf{Tamb}(\mathcal{C})$. Since Tambara modules are the presheaf category of optics, this unites general supermaps with the optics approach to combs suggested here. Moreover it suggests that combs inhabit a very special place in the study of these general supermaps.

Additionally the supermaps are immediately endowed with the tensor products $\otimes_H$ and $\otimes_V$. The suitability of these for modelling physically relevant compositions of supermaps is worthy of further investigation, particularly comparing this structure with the polycategorical structure developed in [160].

Afterword, Conclusion and Future Work

In Part II we have considered profunctorial approaches to modelling supermaps and quantum supermaps. There are several lines of future work we feel are worthy of active investigation.

- Chapter 6 introduced the theory of pro-effectful categories as a way of combining premonoidal and promonoidal categories. Sadly, there is a missing part of this theory as we were not able to define "pre-promonoidal" categories - that is pro-effectful categories *without* specified centre. It is straightforward to define pro-binoidal categories by upgrading the definition of a binoidal category to Prof. The problem with this endeavour is adequately "restricting" the profunctors to give a good notion of centre. Perhaps the notion of centre pieces can be further generalised to solve this?

- It is not clear whether the category Slice is a pro-effectful category and it would be interesting to know whether this is true. One issue is that it is not obvious what the centre of the category should be. Which sets of causal paths satisfy the interchange law on the presheaves $(X \oslash Y)(-)$?

- It would be interesting to compare the profunctorial methods suggested here for spacetime to the approaches of [99, 78, 77]. Can anything be learnt by generalising e.g. the definition of idempotent subunits to the promonoidal setting?

- Comparing the spacetime categories Slice with the approach of [144] would also be interesting. Here and in [72] we discussed the weaker notion of a normal produoidal category and showed the category of optics to have this structure. This connects duoidal categories for compositional dependency with the promonoidal methods used here.

- It would be clarifying to pin down precisely when $\mathsf{Optic}(\mathcal{C}) \cong \mathsf{Comb}(\mathcal{C})$, or at least know whether this holds in cases beyond the few investigated here. Particularly for quantum theory we would like to know what happens in the case of $\mathsf{Isometry}$ and CPTP. The cases of $*$-autonomous categories and monoidally closed categories would also be interesting so we could better understand any connections with the Caus-construction [110].

- In Section 7.5 we argued the category of Tambara modules is a good setting for modelling general quantum supermaps in terms of the locally-applicable transformations of [161]. It would be good to understand how maps like the quantum `switch` can be studied alongside combs in this category. It would also be vital to understand the behaviour of the tensors $\otimes_H$ and $\otimes_V$ on general supermaps and compare this with the polycategorical semantics of [160]. Do any of the structures there become representable when thought of through the lens of Tambara modules? Are there good physical interpretations of the tensors? How does the produoidal structure of $\mathsf{Optic}(\mathcal{C})$ compare with the isomix structure of the Caus-construction [110, 147]? We note that any normal duoidal category is immediately isomix (see e.g. [83]), so that a normal produoidal category is "pro-isomix".

- It may be possible to use profunctors to capture the causal structure of maps. Informally, one can replace causal graphs with profunctor tubes whose topology acts to restrict the families of maps that are compatible with the causal structure, for instance by enforcing one-way signalling constraints. By taking these profunctors to be the domains and codomains of arrows in $\mathsf{Tamb}(\mathcal{C})$ one would be able to study the supermaps that only act on certain classes of processes.

- The methods for studying n-combs suggested here could be compared to the double categorical framework of [29].

Bibliography

[1] Faris Abou-Saleh, James Cheney, Jeremy Gibbons, James McKinna, and Perdita Stevens. Reflections on Monadic Lenses. In *A List of Successes That Can Change the World - Essays Dedicated to Philip Wadler on the Occasion of His 60th Birthday*, volume 9600 of *Lecture Notes in Computer Science*, pages 1–31. Springer, 2016. DOI: `10.1007/978-3-319-30936-1\_1` (cited on page 171).

[2] Samson Abramsky and Adam Brandenburger. An Operational Interpretation of Negative Probabilities and No-Signalling Models. In *Horizons of the mind. A tribute to Prakash Panangaden*, pages 59–75. Springer, 2014. DOI: `10.1007/978-3-319-06880-0_3` (cited on page 10).

[3] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004*, pages 415–425, 2004. DOI: `10.1109/LICS.2004.1319636` (cited on pages 1, 8, 12, 18, 108, 150).

[4] Marcelo Aguiar and Swapneel Mahajan. *Monoidal Functors, Species and Hopf Algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, 2010. DOI: `10.1090/crmm/029` (cited on page 167).

[5] Kazuyuki Asada. Arrows are strong monads. In *Proceedings of the Third ACM SIGPLAN Workshop on Mathematically Structured Functional Programming*, MSFP '10, pages 33–42, Baltimore, Maryland, USA. Association for Computing Machinery, 2010. DOI: `10.1145/1863597.1863607` (cited on pages 2, 126, 131).

[6] Daniela Ashoush and Bob Coecke. Dual Density Operators and Natural Language Meaning. *EPTCS*, 221:1–10, 2016. DOI: `10.4204/EPTCS.221.1` (cited on page 34).

[7] Michael F. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988 (cited on pages 1, 108).

[8] Léon Autonne. Sur les matrices hypohermitiennes et sur les matrices unitaires. *Annales de l'Université de Lyon*, 38:1–77, 1915 (cited on page 37).

[9] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1995. DOI: `10.1063/1.531236` (cited on pages 1, 108).

[10] Samuel Balco. *Display calculi and nominal string diagrams*. PhD thesis, University of Leicester, 2019. DOI: 10.25392/leicester.data.13315373.v1 (cited on page 94).

[11] Samuel Balco and Alexander Kurz. Nominal String Diagrams. In *8th Conference on Algebra and Coalgebra in Computer Science (CALCO 2019)*, volume 139 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 18:1–18:20, Dagstuhl, Germany. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019. DOI: 10.4230/LIPIcs.CALCO.2019.18 (cited on page 94).

[12] Samuel Balco and Alexander Kurz. Partially monoidal categories and the algebra of simultaneous substitutions, 2018. URL: https://gdlyrttnap.pl/resources/papers/syco1.pdf (cited on page 94).

[13] Cornel Balteanu, Zbigniew Fiedorowicz, Roland Schwänzl, and Rainer Vogt. Iterated monoidal categories. *Advances in Mathematics*, 176(2):277–349, 2003. DOI: 10.1016/S0001-8708(03)00065-3 (cited on page 167).

[14] Howard Barnum, Markus Müller, and Cozmin Ududec. Higher-order interference and single-system postulates characterizing quantum theory. *New Journal of Physics*, 16, 2014. DOI: 10.1088/1367-2630/16/12/123029 (cited on page 76).

[15] Michael Barr. The Chu construction. *Theory and Applications of Categories*, 2(2):17–35, 1996 (cited on page 167).

[16] Jonathan Barrett. Information processing in generalized probabilistic theories. *Phys. Rev. A*, 75:032304, 3, 2007. DOI: 10.1103/PhysRevA.75.032304 (cited on page 150).

[17] Bruce Bartlett. Quasistrict symmetric monoidal 2-categories via wire diagrams, 2014. DOI: 10.48550/ARXIV.1409.2148. arXiv: 1409.2148 (cited on page 87).

[18] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Extended 3-dimensional bordism as the theory of modular objects, 2014. DOI: 10.48550/arXiv.1411.0945. arXiv: 1411.0945 (cited on pages 2, 92).

[19] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Modular categories as representations of the 3-dimensional bordism 2-category, 2015. DOI: 10.48550/ARXIV.1509.06811. arXiv: 1509.06811 (cited on pages 2, 88–90, 92).

[20] Michael Batanin and Martin Markl. Centers and homotopy centers in enriched monoidal categories. *Advances in Mathematics*, 230(4):1811–1858, 2012. DOI: 10.1016/j.aim.2012.04.011 (cited on page 167).

[21] Ämin Baumeler, Adrien Feix, and Stefan Wolf. Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios. *Phys. Rev. A*, 90:042106, 4, 2014. DOI: 10.1103/PhysRevA.90.042106 (cited on pages 3, 150).

[22] Ämin Baumeler and Stefan Wolf. The space of logically consistent classical processes without causal order. *New Journal of Physics*, 18(1):013036, 2016. DOI: `10.1088/1367-2630/18/1/013036` (cited on pages 3, 150).

[23] Thorsten Beckmann and Georg Oberdieck. Notes on equivariant categories, 2020. DOI: `10.48550/arXiv.2006.13626`. arXiv: `2006.13626v1` (cited on page 25).

[24] Jean Bénabou. Catégories Avec Multiplication. *Comptes Rendue Acad. Sci. Paris*, 256:1887–1890, 1963 (cited on page 8).

[25] Jean Bénabou. Distributors at Work. Notes for a course at TU Darmstadt, 2000 (cited on pages vi, 2, 98, 100).

[26] Marco Benini, Marco Perin, Alexander Schenkel, and Lukas Woike. Categorification of algebraic quantum field theories. *Letters in Mathematical Physics*, 111(2):1–49, 2021. DOI: `10.1007/s11005-021-01371-8` (cited on page 109).

[27] Marco Benini, Alexander Schenkel, and Lukas Woike. Operads for algebraic quantum field theory. *Communications in Contemporary Mathematics*, 23(02), 2050007, 2020. DOI: `10.1142/s0219199720500078` (cited on page 109).

[28] Richard Blute and Marc Comeau. Von Neumann Categories. *Appl Categor Struct*, 23:725–740, 2015. DOI: `10.1007/s10485-014-9375-6` (cited on pages 111, 171).

[29] Guillaume Boisseau, Chad Nester, and Mario Román. Cornering Optics. *EPTCS*, 380:97–110, 2023. DOI: `10.4204/EPTCS.380.6`. In Proceedings ACT 2022 (cited on page 179).

[30] Guillaume Boisseau and Paweł Sobociński. String Diagrammatic Electrical Circuit Theory. *EPTCS*, 372:178–191, 2022. DOI: `10.4204/EPTCS.372.13`. In Proceedings ACT 2021 (cited on page 150).

[31] Joe Bolt, Jules Hedges, and Philipp Zahn. Bayesian open games, 2019. DOI: `10.48550/arXiv.1910.03656`. arXiv: `1910.03656` (cited on page 171).

[32] Filippo Bonchi, Robin Piedeleu, Pawel Sobociński, and Fabio Zanasi. Graphical Affine Algebra. In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12. IEEE, 2019. DOI: `10.1109/LICS.2019.8785877` (cited on page 150).

[33] Thomas Booker and Street. Ross. Tannaka duality and convolution for duoidal categories. *Theory and Applications of Categories*, 28(6):166–205, 2013 (cited on page 168).

[34] Francis Borceux and Dominique Dejean. Cauchy completion in category theory. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 27(2):133–146, 1986 (cited on page 92).

[35] Nicolas Bourbaki. *Algebra II. Chapters 4-7.* Springer-Verlag Berlin Heidelberg, 2003. DOI: `10.1007/978-3-642-61698-3` (cited on page 54).

[36] Dylan Braithwaite and Jules Hedges. Dependent Bayesian Lenses: Categories of Bidirectional Markov Kernels with Canonical Bayesian Inversion, 2022. DOI: 10.48550/arXiv.2209.14728. arXiv: 2209.14728 (cited on page 171).

[37] Philippe Brax and Carsten van de Bruck. Cosmology and brane worlds: a review. *Classical and Quantum Gravity*, 20(9):R201–R232, 2003. DOI: 10.1088/0264-9381/20/9/202 (cited on page 43).

[38] Romeo Brunetti and Klaus Fredenhagen. Quantum Field Theory on Curved Backgrounds. In *Quantum field theory on curved spacetimes*, pages 129–155. Springer, 2009 (cited on page 108).

[39] Alexander Campbell. Skew-Enriched Categories. *Applied Categorical Structures*, 26:597–615, 2018. DOI: 10.1007/s10485-017-9504-0 (cited on pages 143, 146).

[40] Giulio Chiribella. Perfect discrimination of no-signalling channels via quantum superposition of causal structures. *Phys. Rev. A*, 86:040301, 4, 2012. DOI: 10.1103/PhysRevA.86.040301 (cited on page 151).

[41] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Informational derivation of quantum theory. *Phys. Rev. A*, 84:012311, 1, 2011. DOI: 10.1103/PhysRevA.84.012311 (cited on page 75).

[42] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Probabilistic theories with purification. *Phys. Rev. A*, 81:062348, 6, 2010. DOI: 10.1103/PhysRevA.81.062348 (cited on pages 1, 8, 12, 36, 75, 150).

[43] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Quantum Circuit Architecture. *Physical Review Letters*, 101(6), 2008. DOI: 10.1103/physrevlett.101.060401 (cited on pages 3, 151–153).

[44] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. *Phys. Rev. A*, 80:022339, 2, 2009. DOI: 10.1103/PhysRevA.80.022339 (cited on pages 3, 151, 152).

[45] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. Transforming quantum operations: Quantum supermaps. *EPL (Europhysics Letters)*, 83(3):30004, 2008. DOI: 10.1209/0295-5075/83/30004 (cited on pages 3, 150, 151, 175).

[46] Giulio Chiribella, Giacomo Mauro D'Ariano, Paolo Perinotti, and Benoit Valiron. Quantum computations without definite causal structure. *Phys. Rev. A*, 88:022318, 2, 2013. DOI: 10.1103/PhysRevA.88.022318 (cited on page 151).

[47] Bryce Clarke, Derek Elkins, Jeremy Gibbons, Fosco Loregian, Bartosz Milewski, Emily Pillmore, and Mario Román. Profunctor optics, a categorical update, 2020. DOI: 10.48550/arXiv.2001.07488. arXiv: 2001.07488 (cited on pages 2, 154, 157, 160, 164, 165, 171, 172, 175).

[48] Bob Coecke. Axiomatic Description of Mixed States From Selinger's CPM-construction. *Electronic Notes in Theoretical Computer Science*, 210:3–13, 2008. ISSN: 1571-0661. DOI: `10.1016/j.entcs.2008.04.014`. Proceedings of the 4th International Workshop on Quantum Programming Languages (QPL 2006) (cited on page 11).

[49] Bob Coecke. Quantum picturalism. *Contemporary Physics*, 51(1):59–83, 2010. DOI: `10.1080/00107510903257624` (cited on page 150).

[50] Bob Coecke. Terminality Implies No-signalling ...and Much More Than That. *New Generation Computing*, 34(1-2):69–85, 2016. DOI: `10.1007/s00354-016-0201-6` (cited on pages 11, 36).

[51] Bob Coecke and Ross Duncan. Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics*, 13(043016), 2011. DOI: `10.1088/1367-2630/13/4/043016` (cited on pages 27, 44, 108, 150).

[52] Bob Coecke, Tobias Fritz, and Robert W. Spekkens. A mathematical theory of resources. *Information and Computation*, 250:59–86, 2016. DOI: `10.1016/j.ic.2016.02.008` (cited on pages 154, 161).

[53] Bob Coecke, Chris Heunen, and Aleks Kissinger. Categories of quantum and classical channels. *Quantum Inf. Process.*, 15:5179–5209, 2016. DOI: `10.1007/s11128-014-0837-4` (cited on page 19).

[54] Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press, 2017. DOI: `10.1017/9781316219317` (cited on pages 1, 30).

[55] Bob Coecke and Raymond Lal. Causal Categories: Relativistically Interacting Processes. *Found Phys*, 43:458–501, 2013. DOI: `10.1007/s10701-012-9646-8` (cited on pages 2, 36, 93, 98, 109, 110).

[56] Bob Coecke and Konstantinos Meichanetzidis. Meaning updating of density matrices, 2020. DOI: `10.48550/arXiv.2001.00862`. arXiv: `2001.00862v1` (cited on pages 32, 34).

[57] Bob Coecke, Dusko Pavlovic, and Jamie Vicary. A new description of orthogonal bases. *Mathematical Structures in Computer Science*, 23(3):555–567, 2013. DOI: `10.1017/S0960129512000047` (cited on pages 6, 27).

[58] Bob Coecke and Simon Perdrix. Environment and classical channels in categorical quantum mechanics. *Logical Methods in Computer Science*, 8, 4, 2012. DOI: `10.2168/LMCS-8(4:14)2012` (cited on pages 6, 11).

[59] Bob Coecke, John Selby, and Sean Tull. Two Roads to Classicality. *EPTCS*, 266:104–118, 2018. DOI: `10.4204/eptcs.266.7`. In Proceedings QPL 2017 (cited on pages 12, 19).

[60] Marc Comeau. *Premonoidal *-Categories and Algebraic Quantum Field Theory*. PhD thesis, University of Ottawa, 2012. DOI: `10.20381/ruor-5524` (cited on pages 111, 171).

[61] Cole Comfort and Aleks Kissinger. A Graphical Calculus for Lagrangian Relations. *EPTCS*, 372:338–351, 2022. DOI: 10.4204/EPTCS.372.24. In Proceedings ACT 2021 (cited on page 150).

[62] Oscar Cunningham and Chris Heunen. Axiomatizing complete positivity. *EPTCS*, 195:148–157, 2015. DOI: 10.4204/eptcs.195.11. In Proceedings QPL 2015 (cited on page 19).

[63] Borivoje Dakić and Časlav Brukner. *Quantum Theory and Beyond: Is Entanglement Special?* In *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*. Cambridge University Press, 2011, pages 365–392. DOI: 10.1017/CBO9780511976971.011 (cited on page 7).

[64] Borivoje Dakić, Tomasz Paterek, and Časlav Brukner. Density cubes and higher-order interference theories. *New Journal of Physics*, 16(023028), 2014. DOI: 10.1088/1367-2630/16/2/023028 (cited on pages 7, 15, 16).

[65] Brian Day. *Construction of Biclosed Categories*. PhD thesis, University of New South Wales, 1970. DOI: 10.26190/unsworks/8048 (cited on pages 3, 96, 139, 146).

[66] Brian Day. Monoidal functor categories and graphic fourier transforms. *Theory and Applications of Categories*, 25(5):118–141, 2011 (cited on page 124).

[67] Brian Day. Note on monoidal monads. *Journal of the Australian Mathematical Society*, 23(3):292–311, 1977. DOI: 10.1017/S1446788700018929 (cited on page 145).

[68] Brian Day. On closed categories of functors. In *Reports of the Midwest Category Seminar IV*, volume 137, pages 1–38, Berlin, Heidelberg. Springer Berlin Heidelberg, 1970. DOI: 10.1007/BFb0060438 (cited on pages 3, 11, 96, 110, 135, 139, 146).

[69] Brian Day and Ross Street. Monoidal bicategories and hopf algebroids. *Advances in Mathematics*, 129(1):99–157, 1997. DOI: 10.1006/aima.1997.1649 (cited on page 90).

[70] Christopher L. Douglas and David J. Reutter. Fusion 2-categories and a state-sum invariant for 4-manifolds, 2018. DOI: 10.48550/arXiv.1812.11933. arXiv: 1812.11933 (cited on page 2).

[71] Lawrence Dunn and Jamie Vicary. Coherence for Frobenius pseudomonoids and the geometry of linear proofs. *Logical Methods in Computer Science*, Volume 15, Issue 3, 2019. DOI: 10.23638/LMCS-15(3:5)2019 (cited on page 90).

[72] Matt Earnshaw, James Hefford, and Mario Román. The Produoidal Algebra of Process Decomposition, 2023. DOI: 10.48550/arXiv.2301.11867. arXiv: 2301.11867 (cited on pages v, vii, 169, 174, 178).

[73] Tilo Eggeling, Dirk Schlingemann, and Reinhard F Werner. Semicausal operations are semilocalizable. *Europhysics Letters (EPL)*, 57(6):782–788, 2002. DOI: 10.1209/epl/i2002-00579-4 (cited on page 151).

[74] Alexey Elagin. On equivariant triangulated categories, 2015. DOI: 10.48550/arXiv.1403.7027. arXiv: 1403.7027 (cited on pages 20, 21).

[75] William Ellison. Waring's Problem for Fields. *Séminaire de théorie des nombres de Bordeaux*:1–8, 1970-1971. URL: www.numdam.org/item/STNB_1970-1971___A8_0/ (cited on pages 62, 63).

[76] William Ellison. Waring's problem for fields, 2013. DOI: 10.48550/arXiv.1303.4818. arXiv: 1303.4818 (cited on pages 62, 63).

[77] Pau Enrique Moliner. *Space and time in monoidal categories.* PhD thesis, The University of Edinburgh, 2021. DOI: 10.7488/era/1259 (cited on page 178).

[78] Pau Enrique Moliner, Chris Heunen, and Sean Tull. Space in Monoidal Categories. *EPTCS*, 266:399–410, 2018. DOI: 10.4204/eptcs.266.25. In Proceedings QPL 2017 (cited on page 178).

[79] Marcelo Fiore, Nicola Gambino, Martin Hyland, and Glynn Winskel. Relative pseudomonads, kleisli bicategories, and substitution monoidal structures. *Selecta Mathematica*, 24:2791–2830, 2018. DOI: 10.1007/s00029-017-0361-3 (cited on page 141).

[80] François Foltz, Christian Lair, and Gregory M. Kelly. Algebraic categories with few monoidal biclosed structures or none. *Journal of Pure and Applied Algebra*, 17:171–177, 2, 1980. DOI: 10.1016/0022-4049(80)90082-1 (cited on page 106).

[81] Tobias Fritz. A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics. *Advances in Mathematics*, 370:107239, 2020. DOI: 10.1016/j.aim.2020.107239 (cited on page 150).

[82] Nora Ganter and Mikhail Kapranov. Symmetric and exterior powers of categories. *Transformation Groups*, 19:57–103, 2014. DOI: 10.1007/s00031-014-9255-z (cited on pages 20, 21).

[83] Richard Garner and Ignacio López Franco. Commutativity. *Journal of Pure and Applied Algebra*, 220(5):1707–1751, 2016. DOI: 10.1016/j.jpaa.2015.09.003 (cited on pages vii, 131, 167, 168, 174, 179).

[84] Robert Geroch. Domain of Dependence. *Journal of Mathematical Physics*, 11(2):437–449, 1970. DOI: 10.1063/1.1665157 (cited on page 112).

[85] Stefano Gogioso. A Process-Theoretic Church of the Larger Hilbert Space, 2019. DOI: 10.48550/arXiv.1905.13117. arXiv: 1905.13117 (cited on pages 2, 93, 98, 110).

[86] Stefano Gogioso. Fantastic Quantum Theories and Where to Find Them, 2017. DOI: 10.48550/arXiv.1703.10576. arXiv: 1703.10576 (cited on pages 10, 70).

[87] Stefano Gogioso. Higher-order CPM Constructions. *EPTCS*, 287:145–162, 2019. DOI: 10.4204/EPTCS.287.8. In Proceedings QPL 2018 (cited on pages v, 2, 18, 19, 22, 23, 26, 27).

[88] Stefano Gogioso and Aleks Kissinger. Fully graphical treatment of the quantum algorithm for the Hidden Subgroup Problem, 2017. DOI: 10.48550/arXiv.1701.08669. arXiv: 1701.08669 (cited on page 70).

[89] Stefano Gogioso and Carlo Maria Scandolo. Categorical Probabilistic Theories. *EPTCS*, 266:367–385, 2018. DOI: 10.4204/EPTCS.266.23. In Proceedings QPL 2017 (cited on pages v, 8, 10, 12, 13, 75, 150).

[90] Stefano Gogioso and Carlo Maria Scandolo. Density Hypercubes, Higher Order Interference and Hyper-decoherence: A Categorical Approach. In *Quantum Interaction*, pages 141–160. Springer International Publishing, 2019. DOI: 10.1007/978-3-030-35895-2_10 (cited on pages 2, 7, 15, 32, 33, 35, 36, 38, 39, 41, 49, 76).

[91] Stefano Gogioso, Maria E. Stasinou, and Bob Coecke. Functorial Evolution of Quantum Fields. *Frontiers in Physics*, 9, 2021. ISSN: 2296-424X. DOI: 10.3389/fphy.2021.534265 (cited on pages 2, 98, 110, 112).

[92] Stefano Gogioso and William Zeng. Fourier transforms from strongly complementary observables, 2015. DOI: 10.48550/arXiv.1501.04995. arXiv: 1501.04995 (cited on page 27).

[93] Rudolf Haag and Daniel Kastler. An Algebraic Approach to Quantum Field Theory. *Journal of Mathematical Physics*, 5(7):848–861, 1964 (cited on pages 1, 108).

[94] James Hefford and Cole Comfort. Coend Optics for Quantum Combs. *EPTCS*, 380:63–76, 2023. DOI: 10.4204/EPTCS.380.4. In Proceedings ACT 2022 (cited on pages v–vii).

[95] James Hefford and Stefano Gogioso. CPM Categories for Galois Extensions. *EPTCS*, 343:165–192, 2021. DOI: 10.4204/eptcs.343.9. In Proceedings QPL 2021 (cited on page v).

[96] James Hefford and Stefano Gogioso. Hyper-decoherence in Density Hypercubes. *EPTCS*, 340:141–159, 2021. DOI: 10.4204/eptcs.340.7. In Proceedings QPL 2020 (cited on pages v, 2, 7, 15, 32, 36, 75).

[97] James Hefford and Aleks Kissinger. On the Pre- and Promonoidal Structure of Spacetime. *EPTCS*, 380:284–306, 2023. DOI: 10.4204/EPTCS.380.17. In Proceedings ACT 2022 (cited on pages v, vi, 98, 171).

[98] James Hefford and Mario Román. Optics for Premonoidal Categories, 2023. DOI: 10.48550/arXiv.2305.02906. arXiv: 2305.02906. To appear in Proceedings ACT 2023 (cited on pages v–vii).

[99] Chris Heunen and Nesta van der Schaaf. Ordered Locales, 2023. DOI: 10.48550/arXiv.2303.03813. arXiv: 2303.03813 (cited on page 178).

[100] Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: An Introduction*. Oxford University Press, 2019. DOI: 10.1093/oso/9780198739623.001.0001 (cited on pages 10, 30).

[101] Nick Hu and Jamie Vicary. Traced Monoidal Categories as Algebraic Structures in Prof. *EPTCS*, 351:84–97, 2021. DOI: 10.4204/eptcs.351.6. In Proceedings MFPS 2021 (cited on pages 88, 90).

[102] Bart Jacobs, Chris Heunen, and Ichiro Hasuo. Categorical semantics for arrows. *Journal of Functional Programming*, 19(3-4):403–438, 2009. DOI: 10.1017/S0956796809007308 (cited on pages 2, 126, 128, 131, 145).

[103] Nathan Jacobson. *Basic Algebra II: Second Edition*. W. H. Freeman and Company, 1989 (cited on page 62).

[104] George Janelidze and Gregory M. Kelly. A Note On Actions of a Monoidal Category. *Theory and Applications of Categories*, 9(4):61–91, 2001 (cited on page 146).

[105] André Joyal and Ross Street. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78, 1993. DOI: 10.1006/aima.1993.1055 (cited on page 90).

[106] André Joyal and Ross Street. The geometry of tensor calculus, I. *Advances in Mathematics*, 88(1):55–112, 1991. DOI: 10.1016/0001-8708(91)90003-P (cited on page 8).

[107] André Joyal and Ross Street. The geometry of tensor calculus, II. 1991 (cited on page 8).

[108] Gregory Kelly. *Basic Concepts of Enriched Category Theory*. Cambridge University Press, 1982 (cited on pages 81, 82, 84, 140).

[109] Gregory Kelly. On MacLane's conditions for coherence of natural associativities, commutativities, etc. *Journal of Algebra*, 1(4):397–402, 1964. DOI: 10.1016/0021-8693(64)90018-3 (cited on page 8).

[110] Aleks Kissinger and Sander Uijlen. A categorical semantics for causal structure. *Logical Methods in Computer Science*, 2019. DOI: 10.23638/LMCS-15(3:15)2019 (cited on pages 3, 150, 151, 153, 176, 179).

[111] Aleks Kissinger and John van de Wetering. Reducing T-count with the ZX-calculus, 2019. DOI: 10.48550/arXiv.1903.10477. arXiv: 1903.10477 (cited on page 46).

[112] E. H. Kronheimer and Roger Penrose. On the structure of causal spaces. *Mathematical Proceedings of the Cambridge Philosophical Society*, 63(2):481–501, 1967. DOI: 10.1017/S030500410004144X (cited on page 112).

[113] Stephen Lack. Composing PROPs. *Theory and Applications of Categories*, 13(9):147–163, 2004 (cited on page 30).

[114] Stephen Lack and Michael Shulman. Enhanced 2-categories and limits for lax morphisms. *Advances in Mathematics*, 229(1):294–356, 2012. DOI: 10.1016/j.aim.2011.08.014 (cited on page 136).

[115] Raymond Lal. *Causal Structure in Categorical Quantum Mechanics*. PhD thesis, Oxford University, UK, 2012 (cited on page 93).

[116] David Langlois. Brane Cosmology. *Progress of Theoretical Physics Supplement*, 148:181–212, 2002. DOI: 10.1143/PTPS.148.181 (cited on page 43).

[117] Ciarán M. Lee and John H. Selby. A no-go theorem for theories that decohere to quantum mechanics. *Proc. R. Soc. A*, 474(20170732), 2018. DOI: 10.1098/rspa.2017.0732 (cited on pages v, 2, 7, 14, 43, 52, 76).

[118] Ciarán M. Lee and John H. Selby. Higher-Order Interference in Extensions of Quantum Theory. *Found Phys*, 47:89–112, 2017. DOI: 10.1007/s10701-016-0045-4 (cited on pages 7, 16, 17, 76).

[119] Tom Leinster. *Higher Operads, Higher Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2004. DOI: 10.1017/CBO9780511525896 (cited on page 97).

[120] Paul Levy. *Call-By-Push-Value*. Springer Dordrecht, 2003. DOI: 10.1007/978-94-007-0954-6 (cited on pages 127, 128).

[121] Fosco Loregian. *(Co)end Calculus*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2021. DOI: 10.1017/9781108778657 (cited on pages 2, 82, 84).

[122] Fosco Loregian and Emily Riehl. Categorical notions of fibration. *Expositiones Mathematicae*, 38(4):496–514, 2020. DOI: 10.1016/j.exmath.2019.02.004 (cited on page 100).

[123] Jacob Lurie. On the classification of topological field theories. *Current Developments in Mathematics*, 2008(1):129–280, 2008 (cited on page 108).

[124] Saunders Mac Lane. Natural Associativity and Commutativity. *Rice University Studies*, 49:28–46, 1963 (cited on pages 8, 9).

[125] David B. Malament. The class of continuous timelike curves determines the topology of spacetime. *Journal of Mathematical Physics*, 18(7):1399–1404, 1977. DOI: 10.1063/1.523436 (cited on page 112).

[126] Ognyan Oreshkov, Fabio Costa, and Časlav Brukner. Quantum correlations with no causal order. *Nature Communications*, 3(1092), 2012. DOI: 10.1038/ncomms2076 (cited on pages 3, 150).

[127] Craig Pastro and Ross Street. Doubles for Monoidal Categories. *Theory and Applications of Categories*, 21(4):61–75, 2008 (cited on pages 3, 154, 157, 163–165, 169, 172).

[128] Roger Penrose. *Applications of negative dimensional tensors*. In Academic Press, 1971 (cited on page 8).

[129] Robin Piedeleu, Dimitri Kartsaklis, Bob Coecke, and Mehrnoosh Sadrzadeh. Open System Categorical Quantum Semantics in Natural Language Processing, 2015. DOI: 10.48550/arXiv.1502.00831. arXiv: 1502.00831 (cited on pages 32, 34).

[130] John Power. Generic models for computational effects. *Theoretical Computer Science*, 364(2):254–269, 2006. DOI: 10.1016/j.tcs.2006.08.006 (cited on pages 3, 126, 127, 132, 133, 136, 138, 146).

[131] John Power. Premonoidal categories as categories with algebraic structure. *Theoretical Computer Science*, 278(1):303–321, 2002. DOI: 10.1016/S0304-3975(00)00340-6 (cited on pages 3, 126, 127, 132, 135, 146).

[132] John Power and Edmund Robinson. Premonoidal categories and notions of computation. *Mathematical Structures in Computer Science*, 7(5):453–468, 1997. DOI: 10.1017/S0960129597002375 (cited on pages 3, 101, 102, 110, 137).

[133] David Reutter. *Higher linear algebra in topology and quantum information theory*. PhD thesis, University of Oxford, UK, 2019 (cited on page 2).

[134] Mitchell Riley. Categories of Optics, 2018. DOI: 10.48550/arXiv.1809.00738. eprint: 1809.00738 (cited on pages 2, 3, 154, 157, 169, 171).

[135] Mario Román. Comb Diagrams for Discrete-Time Feedback, 2020. DOI: 10.48550/arXiv.2003.06214. arXiv: 2003.06214 (cited on pages 153, 154, 158, 161).

[136] Mario Román. Open Diagrams via Coend Calculus. *EPTCS*, 333:65–78, 2021. DOI: 10.4204/eptcs.333.5. In Proceedings ACT 2020 (cited on pages 88, 89, 153, 154).

[137] Mario Román. Profunctor optics and traversals, 2020. DOI: 10.48550/arXiv.2001.08045. arXiv: 2001.08045 (cited on pages 2, 154).

[138] Mario Román. Promonads and String Diagrams for Effectful Categories. *EPTCS*, 380:344–361, 2023. DOI: 10.4204/EPTCS.380.20. In Proceedings ACT 2022 (cited on pages vi, 126–128, 135).

[139] Joseph J. Rotman. *Advanced Modern Algebra*. Prentice Hall, 2002 (cited on pages 54, 57).

[140] Christopher J. Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. PhD thesis, University of California, Berkeley, 2009. DOI: 10.48550/ARXIV.1112.1000 (cited on pages 2, 89, 108).

[141] John H. Selby. *A process theoretic triptych*. PhD thesis, Imperial College London, 2017 (cited on page 7).

[142] Peter Selinger. Dagger Compact Closed Categories and Completely Positive Maps. *Electronic Notes in Theoretical Computer Science*, 170:139–163, 2007. DOI: 10.1016/j.entcs.2006.12.018 (cited on pages 2, 18, 20).

[143] Peter Selinger. Idempotents in Dagger Categories: (Extended Abstract). *Electronic Notes in Theoretical Computer Science*, 210:107–122, 2008. DOI: 10.1016/j.entcs.2008.04.021 (cited on pages 12, 18, 20).

[144] Brandon T. Shapiro and David I. Spivak. Duoidal Structures for Compositional Dependence, 2022. DOI: 10.48550/ARXIV.2210.01962. arXiv: 2210.01962 (cited on pages 167, 178).

[145] Evgeny Shinder. Group actions on categories and Elagin's theorem revisited. *European Journal of Mathematics*, 4:413–422, 2018. DOI: 10.1007/s40879-017-0150-8 (cited on page 20).

[146] Carl Siegel. Darstellung total positiver Zahlen durch Quadrate. *Math Z*, 11:246–275, 1921. DOI: `10.1007/BF01203627` (cited on page 63).

[147] Will Simmons and Aleks Kissinger. Higher-order causal theories are models of BV-logic, 2022. DOI: `10.48550/arXiv.2205.11219`. arXiv: `2205.11219` (cited on page 179).

[148] Rafael D. Sorkin. *Quantum Classical Correspondence: Proceedings of the 4th Drexel Symposium on Quantum Nonintegrability*. In International Press, Cambridge Mass., 1997. Chapter Quantum Measure Theory and its Interpretation, pages 229–251 (cited on page 76).

[149] Rafael D. Sorkin. Quantum Mechanics as Quantum Measure Theory. *Mod. Phys. Lett. A*, 9(33):3119–3127, 1994. DOI: `10.1142/S021773239400294X` (cited on page 76).

[150] David I Spivak. Generalized Lens Categories via functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$, 2019. DOI: `10.48550/arXiv.1908.02202`. arXiv: `1908.02202` (cited on page 171).

[151] Sam Staton and Paul Blain Levy. Universal Properties of Impure Programming Languages. *SIGPLAN Not.*, 48(1):179–192, 2013. DOI: `10.1145/2480359.2429091` (cited on page 148).

[152] Ross Street. Frobenius monads and pseudomonoids. *Journal of Mathematical Physics*, 45(10):3930–3948, 2004. DOI: `10.1063/1.1788852` (cited on page 90).

[153] Ross Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, 1972. DOI: `10.1016/0022-4049(72)90019-9` (cited on page 129).

[154] Teiji Takagi. On an Algebraic Problem Reluted to an Analytic Theorem of Carathéodory and Fejér and on an Allied Theorem of Landau. *Japanese journal of mathematics :transactions and abstracts*, 1:83–93, 1924. DOI: `10.4099/jjm1924.1.0_83` (cited on page 37).

[155] Daisuke Tambara. Distributors on a tensor category. *Hokkaido Mathematical Journal*, 35(2):379–425, 2006. DOI: `10.14492/hokmj/1285766362` (cited on pages 4, 163, 164).

[156] Jamie Vicary. Topological Structure of Quantum Algorithms. In *2013 28th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 93–102, 2013. DOI: `10.1109/LICS.2013.14` (cited on page 70).

[157] Mark Weber. Free Products of Higher Operad Algebras. *Theory and Applications of Categories*, 28(2), 2013 (cited on page 106).

[158] Julian Wechs, Hippolyte Dourdent, Alastair A. Abbott, and Cyril Branciard. Quantum Circuits with Classical Versus Quantum Control of Causal Order. *PRX Quantum*, 2:030335, 3, 2021. DOI: `10.1103/PRXQuantum.2.030335` (cited on page 150).

[159] John van de Wetering. ZX-calculus for the working quantum computer scientist, 2020. DOI: `10.48550/arXiv.2012.13966`. arXiv: `2012.13966` (cited on page 150).

[160] Matt Wilson and Giulio Chiribella. Free Polycategories for Unitary Supermaps of Arbitrary Dimension, 2022. DOI: 10.48550/ARXIV.2207.09180. arXiv: 2207.09180 (cited on pages vii, 152, 175, 177, 179).

[161] Matt Wilson, Giulio Chiribella, and Aleks Kissinger. Quantum Supermaps are Characterized by Locality, 2022. DOI: 10.48550/ARXIV.2205.09844. arXiv: 2205.09844 (cited on pages vii, 4, 152, 175, 176, 179).

[162] Harvey Wolff. V-cat and V-graph. *Journal of Pure and Applied Algebra*, 4(2):123–135, 1974. DOI: 10.1016/0022-4049(74)90018-8 (cited on page 133).

[163] Maaike Zwart and Bob Coecke. Double Dilation $\neq$ Double Mixing. *EPTCS*, 266:133–146, 2018. DOI: 10.4204/EPTCS.266.9 (cited on pages 32–34, 36).

[164] Karol Życzkowski. Quartic quantum theory: an extension of the standard quantum mechanics. *J. Phys. A: Math. Theor.*, 41(355302), 2008. DOI: 10.1088/1751-8113/41/35/355302 (cited on pages 7, 15, 17).