Compositional Frameworks for Supermaps and Causality

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Contents

Abstract

Quantum supermaps are transformations of quantum processes, and have found many applications in quantum foundations and quantum information theory in the past two decades, particularly in the study of causality. Whilst the concept of a supermap is a simple and intuitive one, the current state-of-the-art formalisations of supermaps cannot be applied to arbitrary Hilbert spaces or Operational Probabilistic Theories (OPTs). We review the standard approaches to defining supermaps in quantum theory, wherever possible highlighting the background compositional principles at play using diagrammatic languages and referring to their algebraic formalisation in the field of category theory. The core argument of this thesis is that a more principled and general approach to defining quantum supermaps exists, using a definition of *locally-applicable transformation*, which can be applied to any symmetric monoidal category. As a consequence this approach can be applied to all quantum processes on general quantum degrees of freedom and to all transformations in OPTs. We identify key compositional features for entire theories of supermaps and show that the supermaps of those theories are always operationally described by locally-applicable transformations. Two tests for a good construction of supermaps on symmetric monoidal categories are identified, recovery of standard physicists definitions for quantum supermaps when applied to categories of standard quantum processes, and existence of key compositional features. By the end of the thesis we find a way to strengthen locally-applicable transformations to construct the theory of *polyslots*, which passes both tests. Applications of this new general framework for the study of quantum causality and quantum information theory are identified as future potential research directions.

Acknowledgements

To begin, I would like to thank my supervisor Giulio Chiribella, you were a consistent source of enthusiasm, creativity, and insight throughout my time at Oxford. I am particularly grateful for the encouragement and self-belief which you gave me, and I hope that it will stay with me as I move on from here.

Thanks to Bob Coecke and Jonathan Barrett for your supervision, and to Bob Coecke and Aleks Kissinger for your communication of categorical quantum mechanics to people like me who grew up without a background in abstract algebra, and so might never have bumped into it otherwise. I hope that having seen the presentation of this thesis almost entirely in terms of diagrams you will be proud.

Although not directly involved in the supervision or work that went into this thesis, a special thanks goes to Pablo Arrighi for being the friendly and supportive face who introduced me to quantum foundations research, with whom I am glad to have been able to collaborate throughout this period.

Thanks to Aleks Kissinger, John Selby, Mio Murao, Tim Forrer, Oscar Higgot, Dan Browne, Simon Burton, James Hefford, Vincent Wang, Guillaume Boisseau, Amélia Durbec, Augustin Vanrietvelde, Martti Karvonen, Sina Salek, Daniel Ebler, Hoi Fung Chau, Farhan Hanif, James Dborin, Kathleen Barse, Bob Coecke, Maria Stasinou, and Hlér Kristjánsson for collaborations past, present, and with any luck future too. It has been an honour to have learned from and worked with you all. I have also had the pleasure of having a variety of fascinating conversations with Carlo Maria Scandalo, Luca Apadula, Timothée Hoffreumon, Stefano Gogioso, Benoît Valiron, Ognyan Oreshkov, Philip Taranto, Nicola Pinzani, Mario Román, Peter Selinger, Vilasini Venkatesh, Cole Comfort, and Alastair Abbott. In particular I would like to thank Mio Murao for an invitation to visit the University of Tokyo, and John Selby for an invitation to visit the University of Gdansk. Thanks to the organisers of QPL, ACT, Causal Worlds, and QISS, for giving me a chance to meet many of the above people, and to see many new places.

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Preface

The content of this thesis is mostly based on four publications produced under the supervision of Giulio Chiribella, along with a manuscript in preparation based on section 4.5.

- M Wilson, G Chiribella: *Causality in higher order process theories* [1].
- M Wilson, G Chiribella: *A Mathematical Framework for Transformations of Physical Processes* [2].
- M Wilson, G Chiribella, A Kissinger: *Quantum Supermaps are Characterized by Locality* [3].
- M Wilson, G Chiribella: *Free Polycategories for Unitary Supermaps of Arbitrary Dimension* [4].

All major results presented in this thesis were produced by me, with the exception of Lemma 8 of Chapter 5 which was proven by Aleks Kissinger, and Lemma 9 of Chapter 5 which was proven collaboratively by Giulio Chiribella and myself. Other publications produced during the period of the PhD which did not make it to here, due to space-time constraints, are the following:

- G Chiribella, M Wilson, HF Chau: *Quantum and classical data transmission through completely depolarizing channels in a superposition of cyclic orders* [5].
- M Wilson, G Chiribella: *A Diagrammatic Approach to Information Transmission in Generalised Switches* [6].
- M Wilson, A Vanrietvelde, M Karvonnen: *Composable constraints* [7].
- P Arrighi, A Durbec, M Wilson: *Quantum networks theory* [8]
- O Higgott, M Wilson, J Hefford, J Dborin, F Hanif, S Burton, DE Browne: *Optimal local unitary encoding circuits for the surface code* [9].
- H Kristjánsson, G Chiribella, S Salek, D Ebler, M Wilson: *Resource theories of communication* [10].

- J Hefford, V Wang, M Wilson: *Categories of Semantic Concepts* [11].

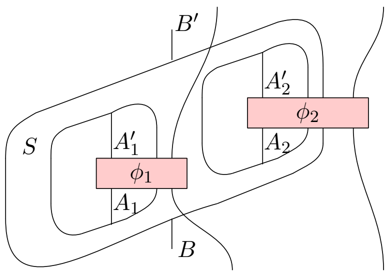
With publications [6] and [9] partially completed beforehand.

To Stuart Knox, for getting me hooked on the conceptual stuff.

Chapter 1

Introduction

This thesis is an attempt to give an abstract model for a simple picture of a box with some holes, the kind of box drawn here in white



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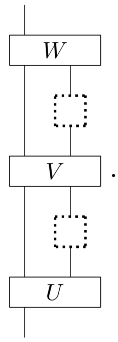
into which more normal looking boxes drawn in pink can be inserted. Variants of this picture have been formalised in quantum foundations and quantum information theory many times in the last two decades [12–19]. This is likely because at its heart it represents a simple concept, a higher-order transformation which can be applied to standard transformations. We will adopt the naming convention of the first formalisation of higher-order transformations in quantum theory, calling them supermaps [12].

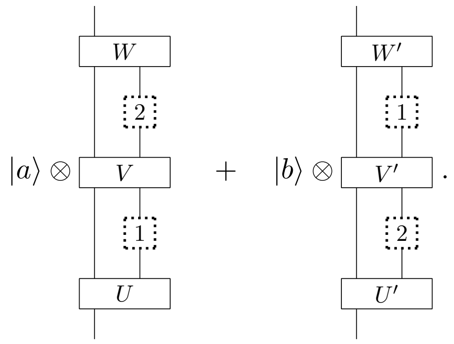
The representation of quantum transformations between physical degrees of freedom, as boxes between wires, has its roots in two fields of research. The first is the study of quantum computation, where quantum circuit diagrams [20] are regularly used to represent quantum transformations and their composition rules in time-like and space-like directions. The second is the study of category theory [21], and specifically monoidal category theory; a meta-theory consisting of the entire class of theories which can be represented in terms of boxes, wires, and their composition to form flowchart-like diagrams [22–24].

So, why does the notion of a transformation of a transformation keep on appearing in foundational physics? Broadly, in quantum information theory interventions by agents are regularly represented by transformations [18, 19, 25–27]. Contexts within which interventions occur, such as circuit boards or spacetimes, can then be considered as things which can be applied to interventions to return either new transformations [12] or probability distributions [18].

The least abstract examples of supermaps are those constructed by puncturing holes into quantum circuit diagrams [13, 15, 28, 29] as in the following picture:



There are, however, more general supermaps than circuits-with-holes, those which model indefinite causal structures [17, 18, 30–43]. Examples of such supermaps which have received a great deal of attention are switches [14, 17], which are controlled combinations of open circuit diagrams



The development of the theory of *quantum switches* and more general supermaps with indefinite causal structures has motivated a variety of research projects in which quantum causal structures are used as a resource for transformations between devices [10, 44–48]. As a consequence of the introduction of this new perspective on information processing, new protocols have been been formulated and discovered in quantum information theory [17, 49–72], quantum communication [5, 6, 68, 73–92], quantum metrology [93–95], and quantum thermodynamics [96–100].

In terms of the interventional perspective, supermaps have been re-axiomatised twice as *process tensors* [19, 101, 102] and *process matrices* [18, 103, 104]. In the process tensor framework, the notion of transformation of transformation is used to model non-Markovian (I.E non-memoryless) processes in quantum theory. Treating the fundamental object of study as the process, rather than the state, allows the bypass of the tension between the non-locality of pure states and the notion of Markovianity [19]. In the process matrix framework the notion of transformation of transformation is used to define the general possible ways to send events, modelled by non-deterministic interventions, to probabilities

[18]. A key application of this viewpoint has been the development of theory-independent measures of indefinite causality in terms of ideas such as causal inequality violation or use of causal witnesses [35, 105–117], a generalisation of tests for non-locality in terms of Bell-inequality violation.

Another independent area of research in which the simplest forms of supermaps, circuits with open holes, have been appearing is in the emerging field of applied category theory [118]. In this context open diagrams have been formalised for all monoidal categories, using the theory of profunctor-optics [28, 29, 119–121]. This approach has been used to model aspects of bidirectional data accessors in functional programming languages [122], compositional game theory [123], financial trading protocols [124], and Markov processes [125].

In all of the above formalisations of supermaps, at least one of two core issues is present:

- In the field of quantum information theory: *the definitions are black-box but cannot be formulated on general symmetric monoidal categories* [12–17, 19, 126–130].
- In the field of applied category theory: *the definitions can be applied to all symmetric monoidal categories but are not black-box, instead only giving models for open circuit diagrams* [28, 29, 119–121, 131].

The aim of this thesis is to resolve these issues and at the same time organise the study of supermaps, by establishing some general principles for theories of supermaps and establishing a unifying general construction for black-box supermaps on arbitrary symmetric monoidal categories. This is a concrete goal, which we are motivated to work towards for a variety of independent reasons:

- *Aesthetics*: To understand the motivating picture of a supermap seems only to require understanding of the notion of a multi-partite process. However, our current mathematical formalisation of supermaps seems to require us to know other mathematical features specific to finite-dimensional quantum theory, such as compact closure of categories [126, 132, 133], or existence of a Choi-Jamiolkowski isomorphism [12–18, 127, 134, 135]. Furthermore, in each of these cases linearity is assumed. None of those mathematical features seem to be needed to understand the intuitive concept of a supermap, so the fact that these features seem to be needed is at the very least a surprise. One motivation of the work of this thesis is to establish more well-motivated principles for supermaps which imply linearity and representation in terms of compact closure.
- *General Quantum Degrees of Freedom:* As noted in the previous bullet point, the current definition methods for quantum supermaps typically require specific aspects

of finite-dimensional quantum theory. Extensions to definitions of supermaps on separable Hilbert spaces exist [16, 136], however the generality of these definition methods and their composability is left unclear. For Hilbert spaces which are non-seperable there has yet to be even a proposal for a suitable definition of supermap. By defining a framework for supermaps on general monoidal categories, we will have solved this problem by constructing a framework for supermaps on arbitrary Hilbert spaces as a special case. A key application in this direction will be the generalisation of the study of indefinite causal structures on arbitrary Hilbert spaces, and the study of devices on arbitrary quantum degrees of freedom as resources.

- *Operational Probabilistic Theories:* Another key aspect of the past two decades of quantum foundations has been the study of post-quantum correlations in operational probabilistic theories (OPTs) [137–141] and a variety of related frameworks for studying broad classes of physical theories such as categorical probabilistic theories [142, 143] and effectus theories [144]. The study of such a broader class of theories has provided a wealth of new insights into the structure of fundemental physics, giving a way to study the consequences of stronger-than-quantum correlations [140, 141, 145–176], and to reconstruct quantum theory and classical theory in a more principled way [137, 177–184]. Without a framework for supermaps on OPTs and their variants we are without an organised way to study post-quantum correlations between causal structures, or a way to reconstruct higher-order quantum theory in an analogous way.

- *Flowcharts Outside of Physics:* As we have discussed, in recent years the study of the special class of supermaps given by circuits-with-holes has received attention within applied category theory. It is as-yet unclear whether black-box pictures such as the motivating picture of supermaps will become of interest in this area, however, it is our expectation that the concept of supermap once formalised could allow new protocols to be imagined and phrased in applied category theory, as was the case in the fields of quantum information theory and quantum foundations. Indeed, for any monoidal category enriched in convex spaces [185], one can easily imagine the notion of a convex combination of open circuits, and so a convex combination of compositional structures.

Encouraged by these motivations, we set out to develop a compositional framework for supermaps by finding suitable axioms for the behaviour of individual supermaps, and for entire theories of supermaps. We will use both of these perspectives to reason about supermaps on theories of processes and recover the physicists' definitions of supermaps when applied to the categories of interest in quantum foundations and quantum information theory.

The attitude of this thesis is that of the process-theoretic approach [186–189] and operational probabilistic approach [137] to quantum theory. The language of category theory [21] is used here, as a map or a reference - a way to communicate the kinds of compositional structures being used. However, some of the subtle concerns of category theory are barely grappled with. More concretely, we do not in general concern ourselves with issues of non-strictness and *coherence* in monoidal categories [21]. For the non-categorically initiated, this means that for instance we choose to not worry about the difference between the following two sets built from Cartesian products

$$(A \times B) \times C \neq A \times (B \times C).$$

We choose to treat them the same and represent them both diagrammatically using string-diagrams [190]. This is a choice made without hesitation within the quantum information community [20, 22], but at odds with some of the primary concerns of the category theory community [21]. There are many aspects of this thesis which would benefit in the future from being studied without dropping concerns of coherence and strictness, this is outside of the scope of the thesis and would have so far probably hindered a more rapid development of a framework for supermaps in process-theoretic and operational probabilistic contexts.

1.1 Content of the Thesis

Here we give an outline of the content of the thesis and the main original contributions of the author. We begin with a brief summary and then give a more detailed outline. First, relevant concepts and applications in the theory of quantum supermaps [12–17, 126] are introduced, compositional features of supermaps are highlighted throughout this introduction, using the language of category theory [21] where appropriate. Next, we follow the first contributions [1, 2] in which we identify some additional as-yet unobserved compositional features of supermaps. The existence of certain structural supermaps is identified with property of enrichment [191], and some consequences for the notion of causality in higher-order quantum theories are studied. We then follow the contribution of [3], our main conceptual proposal, by defining and developing the theory of *locally-applicable transformations* on general symmetric monoidal categories. We motivate such transformations from direct intuition, and show that they indeed always enrich the category from which they were constructed. We then see that locally-applicable transformations can be formally justified from a top-down approach, by building on enrichment to define some general axioms for theories of supermaps and then showing that morphisms in such a theory can always be used to construct locally-applicable transformations. In the main technical contribution of the thesis we show that locally-applicable transformations on quantum channels are exactly quantum superchannels, proving that black-box supermaps can be generalised to arbitrary symmetric monoidal categories. We then take a step back, and follow [4] noting that local-applicable transformations do not recover the standard notion of supermap on the unitaries, and cannot be given a polycategorical composition rule. We find that a stronger notion of local-applicability can be defined which solves both of these problems. The associated theory of *polyslots* we construct is black-box, recovers standard definitions of supermaps on both the quantum channels and the unitaries, has a polycategorical composition rule, and has sequential and parallel composition supermaps. Having established a working theory of supermaps for all symmetric monoidal categories, we then conclude by discussing the potential future research directions which could be developed by building from this point.

Here is a breakdown of the chapters covering the above in more detail.

Chapter 2 In this chapter, enough quantum information theory is introduced to present an adapted version of the original formalisations of quantum supermaps [12–17, 126]. Each compositional feature of quantum transformations and quantum supermaps that we would later like to reference is introduced along the way using the language of category-theory. In particular, we introduce:

- Categories: As theories of processes with sequential composition [21].
- Monoidal Categories: As theories of multi-input multi-output processes with sequential and parallel composition [21].
- Multicategories: As theories of multi-input processes with composition along one-wire-at-a-time [192, 193].
- Polycategories: As theories of multi-input multi-output processes with composition along one-wire-at-a-time [194–197].

In passing we define the polycategory of superunitaries, which is not to the author's knowledge explicitly written down anywhere in the literature. A conceptual contribution made here is to explain the relevance of polycategorical composition rules for superchannels and superunitaries in terms of the forbidding of the creation of time-loops. We also review iterated theories in which transformations of supermaps are studied.

Chapter 3 Here we present some core features of [2] and [1], making our first claim on the expected properties of theories of supermaps - the existence of sequential and parallel composition supermaps. We note that this requirement can be technically formalised using enriched category theory [191] and then examine the philosophical and technical consequences of this observation, focusing on the case in which the enriching multicategory of supermaps is representable by a monoidal product. We give a proof that faithful linked self-enriched enriched monoidal categories are closed monoidal categories, which in intuitive terms shows that currying can be treated as a derived concept for more basic principles for supermaps [2]. We then study causality in such theories of supermaps, discovering that no-signalling conditions in closed monoidal categories can be derived from basic restrictions on the strength of correlations on bipartite states of a theory [1].

Chapter 4 In this chapter we outline the core proposal of [3], to model supermaps using locally-applicable transformations. We then establish some basic features of locally-applicable transformations:

- Locally-applicable transformations on a monoidal category define a multicategory which enriches the symmetric monoidal category they act on.
- Locally-applicable transformations are broad enough to model indefinite causal structures on arbitrary quantum systems.
- Locally-applicable transformations can be defined on constrained spaces of transformations.

We then build on the top-down approach of chapter 3, and combine the expected properties of enrichment and polycategorical structure with some additional reasonable laws for *theories of supermaps*. We discover that any such theory can be mapped into the theory of locally-applicable transformations.

Chapter 5 Here we present the main technical result of [3], by showing that quantum superchannels are exactly the locally-applicable transformations on the symmetric monoidal category of quantum channels.

- We show an equivalence between locally-applicable transformations on compact closed categories and morphisms of a compact closed category.
- We use this to motivate a more complex proof that locally-applicable transformations on quantum channels are exactly the quantum superchannels. We show this holds for supermaps on a large class of constrained spaces, including those defined by the satisfaction of signalling constraints.
- We show that this characterisation can be repackaged as the claim that the space of quantum superchannels is simply the space of natural transformations between some well-chosen functors.

Consequently, we show that superchannels are reconstructable from a simple, reasonable, theory-independent categorical principle.

Chapter 6 Here we present the key results of [4], showing how to freely construct a theory of supermaps as defined in chapter 4 over any monoidal category.

- We give a construction **pslot**[−] which sends each symmetric monoidal category **C** to a polycategory **pslot**[**C**] of supermaps on **C**.
- We show that this construction recovers general superchannels and superunitaries. Concretely, taking **U** and **QC** to be the symmetric monoidal categories of unitaries and quantum channels, we prove that **pslot**[**U**] is the polycategory of superunitaries and that **pslot**[**QC**] is the polycategory of quantum superchannels.

These results can be summarised by saying that **pslot**[−] is a construction for theories of supermaps on symmetric monoidal categories, which can be used to reconstruct the standard physicists definitions of supermaps when applied to the relevant symmetric monoidal categories.

Chapter 7 In the final chapter we summarise the results of the thesis and then discuss the future work that could stem from these results. In particular we discuss what has yet to be developed - a formalisation of super-supermaps, tensor products, compositional types, and concrete applications to quantum information processing. We suggest some future research directions in the study of infinite-dimensional quantum causal structures and infinite-dimensional quantum information processing, and discuss a broader perspective on the potential applications of phrasing local-applicability in terms of families of functions. Finally, we discuss a need for a meta-theory of theories of supermaps based on adaptation of the axioms of theories of supermaps, in particular the need for a theory of structure preserving maps between theories of supermaps, so that particular theories can be characterised by universal properties in the spirit of [198].

Chapter 2

Quantum Information Theory and Quantum Supermaps

In this chapter we are going to take a whistle-stop tour through some aspects of quantum theory. In particular, we will discuss those aspects which are relevant to the study of supermaps and causal structures in quantum information theory. Throughout the tour, we take care to stop and identify compositional features which are present, and the names which category theorists use to describe them. To remain as readable as possible to quantum information theorists, we do not prioritise formality of our introduction of categorical structures, and we use diagrammatic presentations at every possible turn.

2.1 Pure Quantum Theory

Let us begin at the start, with the most easy to introduce form of quantum theory - finite-dimensional pure quantum theory [199,200]. We from now on assume finite-dimensionality throughout the thesis unless we explicitly state otherwise. In finite-dimensional quantum theory, physical degrees of freedom are modelled as having values given by elements of Hilbert spaces. The possible values of degrees of freedom are typically referred to as states of the system. We will often choose to use bra-ket notation for elements of Hilbert spaces, representing elements of a Hilbert space H by $|\phi\rangle$ and representing the inner product (ϕ, ψ) of the Hilbert space by $\langle\psi| |\phi\rangle$. We will consider the dynamics of pure-quantum theory to be described by unitary linear maps.

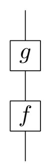
The most fine-grained form of measurement outcome of a degree of freedom is given by choosing some element of an orthonormal basis $\{|a\rangle\}$ for its associated Hilbert space. The probability $P(a : \psi)$ to measure state $|\psi\rangle$ to be in state $|a\rangle$ is then given by $P(a : \psi) = |\langle a| |\psi\rangle|^2$. In this thesis we will focus less on the probabilistic structure of quantum theory and more on its compositional structure.

2.1.1 Sequential Composition Rules

Given a linear map $L : H_1 \to H_2$ and another linear map $L' : H_2 \to H_3$ then those linear maps can be composed, as functions, to produce a new linear map $L' \circ L : H_1 \to H_3$. The same is true of unitary linear maps since both linearity and unitarity are preserved under function composition. This, and the fact that the identity function is a linear map, can be packaged into the statements that linear maps and unitary linear maps between Hilbert spaces form categories [21]. We will introduce the language of category theory throughout this chapter, however, we will not be concerned with some subtleties in category theory concerning coherences, or the differences between classes and sets.

Ignoring issues of size, a category is a set of objects $A, B, \ldots$ with for each pair A, B a set $\mathbf{C}(A, B)$ of morphisms. When considering an element $f \in \mathbf{C}(A, B)$ we will refer to it as a morphism of type $A \to B$ when the background category within which the morphism lives is clear from context. A category is furthermore equipped with, for each triple (A, B, C) of objects a composition function $\circ_{ABC} : \mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, B)$ satisfying two key properties. Firstly it is associative, meaning that $f \circ (g \circ h) = (f \circ g) \circ h$. Note that we have dropped the object indeces of $\circ_{ABC}$, and we will do so whenver the labels are either irrelevant or clear from context. Second, there is a unit morphism $i_A : A \to A$ for each object A such that for all $f : A \to B$ then $f \circ i = f = i \circ f$, where we have again immediately adopted the convention of dropping object labels for i.

Diagrammatically, a morphism $f : A \to B$ may be written as a box labelled f with input wire A and output wire B [201]. In terms of this notation, the sequential composition operation can be represented by plugging the output wire of one box into the input wire of the other box as in the following diagram:



This notation has the convenient property that it absorbs the structural equations of a category. There is no graphical difference for instance between the diagram representing $(f \circ g) \circ h$ and the diagram representing $f \circ (g \circ h)$. Representing the identity by a bare wire, there is also no graphical difference between the diagrams representing $f \circ i$, f, and $i \circ f$.

A morphism $f : A \to B$ is called an *isomorphism* if there exists some $f^{-1} : B \to A$ such that $f \circ f^{-1} = i_B$ and $f^{-1} \circ f = i_A$. If every morphism of a category is an isomorphism then it is called a *groupoid.* We say that $\mathbf{C}$ is a *subcategory* of $\mathbf{D}$ denoted $\mathbf{C} \subseteq \mathbf{D}$ if every object of $\mathbf{C}$ is an object of $\mathbf{D}$, every morphism $A \to B$ in $\mathbf{C}$ is a morphism $A \to B$ in $\mathbf{D}$, and finally identities and composition of $\mathbf{C}$ are the same as those in $\mathbf{D}$. Let us now see our first example of a category, one on top of which many other examples are constructed.

Example 1. *The prototypical category is* **Set** *[21]. The objects of* **Set** *are given by sets and the morphisms are given by functions. Function composition is associative and there is an identity function.*

Many categories are constructed by imposing structure on objects and structure preservation on functions - this is the case for the category of linear maps between Hilbert spaces.

Example 2. *The linear maps between finite dimensional Hilbert spaces form a category* **FHilb** *[202], with objects given by Hilbert spaces, and morphisms $A \to B$ given by linear maps of the same type. Sequential composition is given by function composition, and the identity is given by the identity function, which is indeed always linear. All unitary linear maps give examples of isomorphisms, where in particular $U^{-1} = U^{\dagger}$ with $U^{\dagger}$ the Hermitian adjoint. The unitaries in-fact form a subcategory $\mathbf{U} \subseteq \mathbf{FHilb}$ of the category of linear maps, and since every unitary is an isomorphism with unitary-inverse this subcategory is a groupoid.*

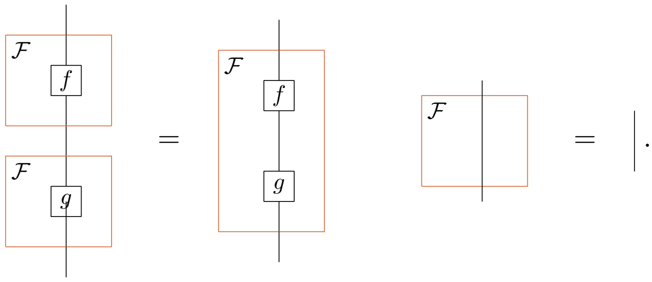
There are some general constructions which build new categories from old ones, one of the most useful is the construction of the reversed or *opposite* version of any category [21].

Example 3 (Opposite Category)**.** *For any category $\mathbf{C}$ one can construct the opposite category $\mathbf{C}^{op}$ which has the same objects as $\mathbf{C}$ and has for each morphism $f \in \mathbf{C}(A, B)$ a corresponding morphism $f^{op} \in \mathbf{C}^{op}(B, A)$. Composition $f^{op} \circ g^{op}$ is defined as $(g \circ f)^{op}$ and identity is given by i^{op}.*

Some of those general constructions use many input categories to construct new categories. For instance, the notion of cartesian product of sets can be generalised to a notion of cartesian product of categories [21].

Example 4 (Product Category)**.** *Given any pair $\mathbf{C}, \mathbf{D}$ of categories one can construct the product category $\mathbf{C} \times \mathbf{D}$ with objects given by pairs (c, d) of objects and morphisms $(c, d) \to (c', d')$ given by pairs (f, g) with $f \in \mathbf{C}(c, c')$ and $g \in \mathbf{D}(d, d')$. Composition is inherited from $\mathbf{C}$ and $\mathbf{D}$ by defining $(f, g) \circ (f', g') := (f \circ f', g \circ g')$. Identities are given by (i, i).*

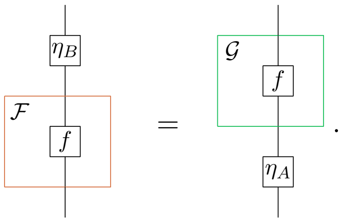
Finally, all of the categories can together be used to construct a more abstract category, where the objects are categories and the morphisms between categories are a new kind of structure-preserving map referred to as a functor. A functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ from category $\mathbf{C}$ to category $\mathbf{D}$ is an assignment of an object $\mathcal{F}A$ to each object A along with for each pair A, B a function $\mathcal{F}_{AB} : \mathbf{C}(A, B) \to \mathbf{D}(\mathcal{F}A, \mathcal{F}B)$ which preserves composition in the sense that $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ and $\mathcal{F}(i) = i$. Note that again, as soon as the context is clear we dropped object indices for $\mathcal{F}$, we will from now on choose to drop these kinds of labels wherever they appear without comment. Diagrammatically the action of a functor on a morphism can be represented by surrounding the morphism with a larger box [203]. The defining equations for functors are then representable diagrammatically by box-merging and deletion from wires in the following way:



It turns out that functors can be composed and so define an even more abstract kind of category, the category of categories!

Example 5. *The category* **Cat** *has as objects the categories and as morphisms from* **C** *to* **D** *the functors of the same type. It is easy enough to check that the composition rule* $\mathcal{F} \circ_{\mathbf{Cat}} \mathcal{G}(f) = \mathcal{F}(\mathcal{G}(f))$ *returns a new functor, and that in terms of this composition rule there is an identity functor* $\mathcal{I}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ *given by the identity function on objects and the identity function on morphisms.*

As a final step into the many layers of the categorical onion, even transformations between functors can be defined, to make a category in which the functors are the objects. Letting $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{D}$ be functors, a natural transformation $\eta : F \Rightarrow G$ is for every object A a morphism $\eta_A : \mathcal{F}A \to \mathcal{G}A$ such that for every $f : A \to B$ then $\eta_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_A$. Diagrammatically natural transformations are the kinds of things which can be pulled through functor boxes in the following way:



Natural transformations can also be composed, leading to the construction of a category of natural transformations between functors.

Example 6. *The category* $\mathbf{Cat}(\mathbf{C}, \mathbf{D})$ *has as objects the functors from* **C** *to* **D** *and as morphisms the natural transformations between functors. Composition of natural transformations* $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ *and* $\mu : \mathcal{G} \Rightarrow \mathcal{H}$ *is given by* $(\mu \circ \eta)_A = \mu_{\mathcal{F}A} \circ \mathcal{F}(\eta_A)$. *That this composition is associative and returns a natural transformation is easily verified using functor box notation.*

Note that these transformations of functors can be understood as transformations of transformations, these are words not dissimilar to those used to describe supermaps in the introductory chapter. Indeed, we will discover in this thesis that the most commonly studied class of quantum supermaps is no more and no less than the set of natural transformations between some carefully chosen functors. We will occasionally refer to some categories as being equivalent, what is encoded by equivalence of two categories $\mathbf{C} \cong \mathbf{D}$ is a pair of functors $\mathbf{C} \leftrightarrow \mathbf{D}$ which are inverse to eachother[1].

2.1.2 Parallel Composition Rules

There are two key ways of joining Hilbert spaces together to form larger composite Hilbert spaces, the tensor product $\otimes$ and the direct sum $\oplus$ [202]. Given a pair of Hilbert spaces H, K their tensor product $H \otimes K$ is given by the set of pairs of elements of H and K separately, up to equivalence by shuffling around scalars and multilinearity. For instance an element $(h, \lambda k)$ is taken to be equal to the element $(\lambda h, k)$. In either case the element $(h, \lambda k) = (\lambda h, k)$ is typically notated by $\lambda h \otimes k$. Furthermore, the elements $h \otimes k + h' \otimes k$ and $(h + h') \otimes k$ are identified. Given a pair of Hilbert spaces H, K their direct sum $H \oplus K$ is given again by the set of pairs (h, k), this time denoted as $h \oplus k$. However, equivalence by shuffling of scalars is not enforced, but a stronger notion of linearity is instead imposed. Precisely, $h \oplus k + h' \oplus k'$ is defined to be equal to $(h + h') \oplus (k + k')$. Tensor products and direct sums come with quite distinct physical interpretations, a decomposition $H = H_1 \otimes H_2$ represents recognition of two independent degrees of freedom of a system, whereas a decomposition of the form $H = H_1 \oplus H_2$ represents a partition of the possible values a degree of freedom may take into two distinct classes. To see the stark distinction between the two consider the difference between $\mathbb{C} \otimes \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C}$. The former represents a pair of systems with essentially one potential value for their degrees of freedom, consequently their composite system is a new degree of freedom with only one value, given by the unique available value for each subsystem. On the other hand the latter expression $\mathbb{C} \oplus \mathbb{C}$ represents two possible values assigned to a single degree of freedom. In short $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ whereas $\mathbb{C} \oplus \mathbb{C} \ncong \mathbb{C}$.

Each of these ways of joining together Hilbert spaces gives a way to join together linear maps on those Hilbert spaces. First, given linear maps $L_i : H_i \to K_i$ one can define $L_1 \otimes L_2 : H_1 \otimes H_2 \to K_1 \otimes K_2$ by $L_1 \otimes L_2(v_1 \otimes v_2) = L_1(v_1) \otimes L_2(v_2)$. For the direct sum one can define $L_1 \oplus L_2(v_1 \oplus v_2) = L_1(v_1) \oplus L_2(v_2)$. One can check both that

$$(L_1' \otimes L_2') \circ (L_1 \otimes L_2) = (L_1' \circ L_1) \otimes (L_2' \circ L_2),$$

[1] By inverse here we really mean inverse up-to natural isomorphism [21].

and furthermore that

$$(L_1' \oplus L_2') \circ (L_1 \oplus L_2) = (L_1' \circ L_1) \oplus (L_2' \circ L_2).$$

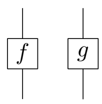
Each of these laws is respectively referred to as an *interchange law*, a key feature of *monoidal categories* [21] used to model the abstract notion of parallel composition.

Somewhat formally, a monoidal category is a category equipped with a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and object I along with some natural isomorphisms $I \otimes (-) \cong \mathcal{I}(-) \cong (-) \otimes I$ and $((- \otimes -) \otimes -) \cong (- \otimes (- \otimes -))$ which satisfy additional laws called coherences which can be found in [21, 202]. Functorality of $(- \otimes -)$ encodes the interchange law since functorality precisely says that

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g').$$

In other words given (f, g) and (f', g') one can either find their tensor products and then compose, or first compose in $\mathbf{C} \times \mathbf{C}$ and afterwards compute the tensor product. Second, a notion of empty space is given by the unit object I, the existence of a natural isomorphism $I \otimes A \cong A$ says that whenever I is placed next-to another object, it may as well be ignored.

Throughout this thesis we are going to represent monoidal categories with string diagrams [24, 201], representing parallel composition by placing boxes next-to-each-other in the following way



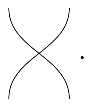
.

Note that the picture used to represent $(f \otimes g) \otimes h$ will be exactly the same as that which is used to represent $f \otimes (g \otimes h)$. In contrast to the case for sequential composition, this diagrammatic representation therefore appears to encode slightly more than the equivalence up-to isomorphism of $(f \otimes g) \otimes h$ and $f \otimes (g \otimes h)$. In general, this is justified since every monoidal category is equivalent in a formal sense to a *strict* monoidal category [204, 205], one in which the $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ and $I \otimes A = A = A \otimes I$. To simplify a variety of definitions, proofs, and discussions throughout the thesis, we choose to work entirely in terms of strict monoidal categories and their associated diagrammatic language. When it is convenient for us, usually when we need to save space, we will take advantage of strictness by notating $A \otimes B$ as AB and $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ as ABC.

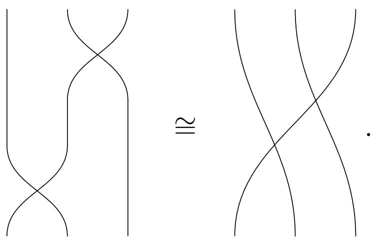
Example 7. *The linear maps define a monoidal category with monoidal product given by the tensor product of vector spaces and tensor product of linear maps. The unit is given by the Hilbert space $\mathbb{C}$ of complex scalars. Indeed for any other Hilbert space H it is true that there exists isomorphisms $\mathbb{C} \otimes H \cong H \cong H \otimes \mathbb{C}$. Associativity isomorphisms can also be defined and shown to satisfy the required coherence conditions. For a fuller discussion of these subtleties see [202].*

Example 8. *The linear maps define a monoidal category with monoidal product given by the direct sum of vector spaces and direct sum of linear maps. The unit is given by the* $\{0\}$ *Hilbert space. Indeed for any other Hilbert space* $\mathbb{H}$ *it is true that there exists isomorphisms* $\{0\} \oplus H \cong H \cong H \oplus \{0\}$*. Associativity isomorphisms can also be defined and shown to satisfy the required coherence conditions. For a fuller discussion of these subtleties see [202].*

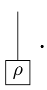
Linear maps also have an additional feature, the existence of a linear map $\beta_{A,B} : A \otimes B \to B \otimes A$ which is an isomorphism which swaps states, meaning that $\beta_{A,B}(a, b) = (b, a)$. This behaviour can be modelled in categorical language as naturality of $\beta_{A,B}$, which formally entails that $\beta_{A',B'} \circ (f \otimes g) = g \otimes f \circ \beta_{A,B}$. Monoidal categories equipped with such natural transformations also satisfying braiding laws such as $(i_A \otimes \beta_{B,C}) \circ (\beta_{A,B} \otimes i_C) = \beta_{A,B\otimes C}$, and invertibility laws $\beta_{A,B} \circ \beta_{B,A} = i_{A\otimes B}$, are referred to as *symmetric monoidal categories* [206]. Diagrammatically, a symmetric monoidal category is one in which wires can be passed across each-other in the following way [201]:



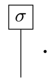
The braiding laws for symmetric monoidal categories are then absorbed diagrammatically by



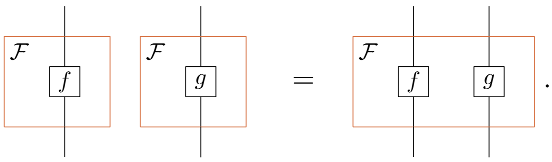
In the process-theoretic approach to physics, in which the transformations are treated as the fundamental objects of study, one might worry that there is no way to speak about states, however, in monoidal categories the empty space unit object I comes to the rescue. States $a \in H$ of Hilbert spaces are in one-to-one correspondence with the linear maps of type $|a\rangle : \mathbb{C} \to H$, this is in-fact the core principle of bra-ket notation, which allows us to think of the inner product as a composition of linear maps. In monoidal category theory, this is a very general feature, it is common to see such a correspondence whenever categories are defined as having sets with structure and morphisms as functions which preserve that structure. In string diagrams the unit I can be safely omitted from all drawings, so that if we define states on A of a monoidal category $\mathbf{C}$ as the elements of $\mathbf{C}(I, A)$ then we can draw states as boxes which only have outputs as follows



We will sometimes refer to the opposite notion of effect too, an effect on A is a morphism of type $\sigma : A \to I$ and can be drawn as a morphism which only has an input



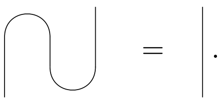
Finally, the notions of functor and monoidal category can be combined to define monoidal functors [21]. A (strong) monoidal functor between monoidal categories is a functor equipped with some well behaved natural isomorphisms between the functors $\mathcal{F}(- \otimes =)$ and $\mathcal{F}(-) \otimes \mathcal{F}(=)$ which allow to phrase up-to-isomorphism the requirement of preservation of parallel composition by $\mathcal{F}(f \otimes g) = \mathcal{F}(f) \otimes \mathcal{F}(g)$. For the details the reader is directed to [21], for now let us just introduce the graphical language of functor boxes between monoidal categories [203]. The key point modulo details concerning isomorphisms and equalities [203] is that one can merge boxes vertically as before, but now also horizontally, to represent $\mathcal{F}(f) \otimes \mathcal{F}(g) = \mathcal{F}(f \otimes g)$ in the following way:



Functors between symmetric monoidal categories can also be defined, being those which preserve swap morphisms, meaning $\mathcal{F}(\beta_{A,B}) = \beta'_{A,B}$ up-to isomorphism [21]. Sub-monoidal categories, monoidal natural transformations, and equivalences of monoidal categories, along with their symmetric variants, are all then defined analogously to their counterparts for standard categories [21].

2.1.3 Perfectly Correlated States

A key conceptual feature of pure quantum theory is the existence of perfectly correlated states, in this context referred to as entangled states [199]. The existence of a perfectly correlated state can be modelled by the notion of compact closure [207, 208]. A compact closed category is a monoidal category equipped with for each object A a *dual* object A^* and a pair $(\cup_A : I \to A \otimes A^*, \cap_A : A^* \otimes A \to I)$ of morphisms such that $(i_A \otimes \cap_A) \circ (\cup_A \otimes i_A) = i_A$. For subtleties and details see [202]. Diagrammatically, we represent these states and effects by bent wires and write the defining equation for compact closure as wire bending [201]



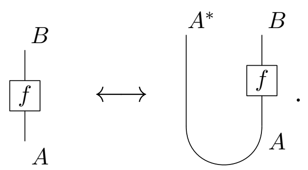
Note that in this equation we have neglected to write associators and unitors, imagining our monoidal category to be strict. The diagrammatic language for compact closed categories models the perfect correlations between spatial locations using a bent identity wire between those locations.

Example 9. *The linear maps form a compact closed category with A^* given by the dual Hilbert space to A and with $\cup_A : \sum_i |i\rangle \otimes |i\rangle^*$ and $\cap_A := \cup_A^\dagger$ (up to a swap). In other words, it is perfect entanglement in the category of linear maps with makes it compact closed.*

Another compact closed category of interest in the study of classical information processing, is the category of positive valued matrices.

Example 10. *The category $\mathbf{Mat}_{\mathbb{R}+}$ [126] has objects given by natural numbers and morphisms given by matrices with elements given by positive real numbers. Composition is given by standard matrix composition, and the identity matrix is a morphism since all of its entries are positive. Parallel composition is given by the standard tensor product for matrices, the unit object is given by the number 1. From this we can deduce that states are given by column vectors and effects are given by row vectors, with compact closure given by $(1, 0, 0, 1)$ in its column and row forms.*

The existence of compact closure also gives an equivalence between the states of a category and the morphisms of a category, there is a natural isomorphism $\mathbf{C}(A, B) \cong \mathbf{C}(I, A^* \otimes B)$ given diagrammatically by



As a result, when compact closure is present, it gives an easy shortcut to defining supermaps, by finding a place where processes appear as states.

2.2 Mixed Quantum Theory

To incorporate mixtures into quantum theory, quantum information theory instead models states of a Hilbert space A using the set of *density matrices* on A, that is, particular positive linear operators [199]. Let us see how such a representation arises. First, consider some ensemble $\{(p_i, \psi_i)\}$ of states, each occuring ψ_i with some probability p_i. The probability of measurement outcome a for such an ensemble $P(a, \{(p_i, \psi_i)\})$ ought to be given by

$\sum_i p_i P(a, \psi_i)$. Consequently, we can see that for any ensemble we require

$$P(a, \{(p_i, \psi_i)\}) = \sum_i p_i P(a, \psi_i) \tag{2.1}$$
$$= \sum_i p_i | \langle a| |\psi\rangle_i |^2 \tag{2.2}$$
$$= \sum_i p_i \langle a| |\psi_i\rangle \langle \psi_i| |a\rangle \tag{2.3}$$
$$= \langle a| \sum_i p_i |\psi_i\rangle \langle \psi_i| |a\rangle \tag{2.4}$$
$$= \langle a| \rho |a\rangle . \tag{2.5}$$

Here, the linear map $\rho : H \to H$ is given by $\rho(\phi) = \sum_i p_i |\psi_i\rangle \langle \psi_i| |\phi\rangle$ and as a result of being of this form is a density matrix, that is, a positive semi-definite Hermitian operator of trace 1.

Let us now consider the definition of transformations in mixed quantum theory. We will adopt the notation $\mathbf{L}(A, A')$ for the Hilbert space of linear operators between Hilbert spaces A and A', and further adopt the compressed notation $\mathbf{L}(A) := \mathbf{L}(A, A)$. The set $st(A)$ of density matrices on a Hilbert space A is a subset $st(A) \subseteq \mathbf{L}(A)$ of the set of linear operators on A, this property can be used to define transformations of quantum states.

The transformations of quantum information theory are typically taken to be those linear maps which, when *applied locally* to states, preserve the space of states. This local-applicability was first phrased as complete-positivity [199], and requires the assumption of linearity before it can be formulated. To formally express the meaning of local application of a linear map, we will review some useful properties of finite-dimensional Hilbert spaces. First, a key isomorphism is the following[2]:

$$u_{AA'BB'} : \mathbf{L}(A, A') \otimes \mathbf{L}(B, B') \cong \mathbf{L}(A \otimes B, A' \otimes B').$$

For the compacted notation this entails that there exists an isomorphism $u_{AB} : \mathbf{L}(A) \otimes \mathbf{L}(B) \cong \mathbf{L}(A \otimes B)$. Using this isomorphism and assuming that transformations of quantum theory should be linear maps, local-application of some linear map $\mathcal{E} : \mathbf{L}(A) \to \mathbf{L}(A')$ to some $\rho \in st(A \otimes X)$ is given by the linear map $(\mathcal{E} \otimes_u I)(\rho)$ with $(\mathcal{E} \otimes_u I)$ defined by the following composition

$$\begin{array}{ccc} \mathbf{L}(A) \otimes \mathbf{L}(X) & \xrightarrow{\mathcal{E}\otimes i} & \mathbf{L}(B) \otimes \mathbf{L}(X) \\ {\scriptstyle u_{ax}^{-1}}\uparrow & & \downarrow{\scriptstyle u_{bx}} \\ \mathbf{L}(A \otimes X) & \xrightarrow[(\mathcal{E}\otimes_u i)]{} & \mathbf{L}(B \otimes X) \end{array} . \tag{2.6}$$

[2]This isomorphism works explicitly by taking $(u_{AA'BB'}(\sum_{ij} a_i \otimes b_j))(\phi) := \sum_{ij}((a_i \otimes b_j)(\phi))$ and taking $u^{-1}_{AA'BB'}(m) = \sum_{ijkl}((\langle i| \otimes \langle j|)m(|k\rangle \otimes |l\rangle)(|i\rangle \langle k|) \otimes (|j\rangle \langle l|)$.

The standard definition of transformation of type $\mathcal{E} : A \to A'$ in quantum information theory is taken to that of a linear map $\mathcal{E} : \mathbf{L}(A) \to \mathbf{L}(A')$ which *completely preserves* the quantum states [199]. Formally, complete-preservation is the requirement that whenever $\rho \in st(A \otimes X)$ then $(\mathcal{E} \otimes_u I)(\rho) \in st(B \otimes X)$.

If we were not concerned with normalisation of probabilities, we could have defined $st(A)$ to be the space of positive operators, in which case transformations as defined above would be the completely positive maps. The completely positive maps form a compact closed category, again compact closure is given by the existence of the Bell-state or more generally perfect entanglement. When $st(A)$ is defined to be the space of normalised density matrices, what is returned is the space of completely positive trace preserving maps, also referred to in the literature as quantum channels.

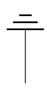
2.2.1 Composition of Quantum Transformations

The transformations of quantum theory can be composed in sequence or in parallel [22, 199]. Given any quantum transformations $\mathcal{E} : A \to A'$ and $\mathcal{N} : B \to B'$ one can define $(\mathcal{E} \otimes_u \mathcal{N}) : A \otimes B \to A' \otimes B'$ through the following composition of linear maps[3]:

$$\begin{array}{ccc} \mathbf{L}(A) \otimes \mathbf{L}(B) & \xrightarrow{\mathcal{E} \otimes \mathcal{N}} & \mathbf{L}(A') \otimes \mathbf{L}(B') \\ {\scriptstyle u_{ab}^{-1}} \uparrow & & \downarrow {\scriptstyle u_{a'b'}} \\ \mathbf{L}(A \otimes B) & \xrightarrow[(\mathcal{E} \otimes_u \mathcal{N})]{} & \mathbf{L}(A' \otimes B') \end{array} \quad . \tag{2.7}$$

The sequential composition $\mathcal{E} \circ \mathcal{N}$ is given simply by sequential composition of $\mathcal{E}$ and $\mathcal{N}$ as functions. All together this is the majority of the data required to state that quantum transformations define monoidal categories, we will refer to the monoidal category of completely positive maps as **CP** and the monoidal category of quantum channels as **QC**. In this thesis we interpret the quantum channels as the deterministic transformations of quantum theory [137], and we treat the completely positive maps as a reasonably well-behaved background theory which we will use to construct a variety of other kinds of transformations.

The quantum channels have an additional property, for each object A there is one and only one morphism of type $A \to I$ called the trace [137,199,209]. A monoidal category with a unique morphism of type $A \to I$ for each object A is typically referred to in the quantum foundations literature as a causal category [209]. In the wider literature of category theory, such categories are referred to as affine monoidal, or semi-cartesian categories [198, 210]. We will commonly refer to this unique effect using the ground symbol

 .

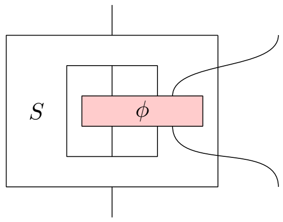
[3]This composition rule works whenever the state sets are closed under swaps in the sense that $(\beta_{AB} \otimes i) \circ u_{ABC}^{-1} st(A \otimes B \otimes C) \subseteq st(B \otimes A \otimes C)$

Example 11. *The category* **Stoch** $\subseteq$ $\mathbf{Mat}_{\mathbb{R}+}$ *of matrices in which every column sums to* 1 *is called the category of stochastic matrices [126]. The stochastic matrices represent finite-dimensional classical channels, and inherit their monoidal structure from* $\mathbf{Mat}_{\mathbb{R}+}$*. There is one and only one effect* $(1, \ldots, 1)$ *on each object with each column summing to one, since each effect is a row vector.*

This completes our review of the standard transformations of quantum theory, we will now move on to more recent and elaborate notions in recent developments in quantum information theory and quantum foundations.

2.3 Supermaps

We will now return to the motivating picture for this thesis, the picture of a supermap [12], a box with holes. For now to simplify the presentation we will begin with single-input supermaps, and in section 2.4 return to multi-input supermaps. The intuitive picture of the action of such a supermap is the following



and the most commonly studied setting is that in which the pink box represents a quantum channel. When applied to quantum channels, supermaps will be referred to as superchannels.

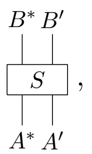
The key piece of mathematical technology needed to formally define superchannels is similar to that which is used to define transformations of states in standard quantum information theory. Usually, the definition of supermap is phrased using the Choi-Jamiolkowski isomorphism [134] for completely positive maps, a special case of compact closure. Rather than introducing the Choi-Jamiolkowski isomorphism, we will find it more convenient to phrase the definition in terms of the explicit isomorphisms for linear maps between finite-dimensional Hilbert spaces that we have already used to define standard transformations. First, note that the space $\mathbf{QC}(A, A')$ of quantum channels is a subset $\mathbf{QC}(A, A') \subseteq \mathbf{L}(\mathbf{L}(A), \mathbf{L}(A'))$ of the set of linear maps. Second, given any linear map $S : \mathbf{L}(\mathbf{L}(A), \mathbf{L}(A')) \to \mathbf{L}(\mathbf{L}(B), \mathbf{L}(B'))$ one can define its local-application $S \otimes_u I(\mathcal{E})$ on an element $\mathcal{E} \in \mathbf{QC}(A \otimes X, A' \otimes X') \subseteq \mathbf{L}(\mathbf{L}(A \otimes X), \mathbf{L}(A' \otimes X'))$ by the following composition of isomorphisms:

$$\begin{array}{ccc}
\mathbf{L}(\mathbf{L}(A),\mathbf{L}(A'))\otimes\mathbf{L}(\mathbf{L}(X),\mathbf{L}(X')) & \xrightarrow{\mathcal{S}\otimes i} & \mathbf{L}(\mathbf{L}(B),\mathbf{L}(B'))\otimes\mathbf{L}(\mathbf{L}(X),\mathbf{L}(X')) \\
\uparrow u^{-1}_{\mathbf{L}A\mathbf{L}A'\mathbf{L}X\mathbf{L}X'} & & \downarrow u_{\mathbf{L}B\mathbf{L}B'\mathbf{L}X\mathbf{L}X'} \\
\mathbf{L}(\mathbf{L}(A)\otimes\mathbf{L}(X),\mathbf{L}(A')\otimes\mathbf{L}(X')) & & \mathbf{L}(\mathbf{L}(B)\otimes\mathbf{L}(X),\mathbf{L}(B')\otimes\mathbf{L}(X')) \\
\uparrow \mathbf{L}(u_{AX},u^{-1}_{A'X'}) & & \downarrow \mathbf{L}(u^{-1}_{BX},u_{B'X'}) \\
\mathbf{L}(\mathbf{L}(A\otimes X),\mathbf{L}(A'\otimes X')) & \xrightarrow[(\mathcal{S}\otimes_u i)]{} & \mathbf{L}(\mathbf{L}(B\otimes X),\mathbf{L}(B'\otimes X'))
\end{array} \quad . \tag{2.8}$$

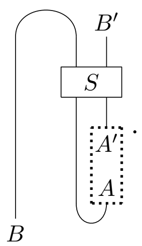
Where in writing $\mathbf{L}(f,g)$ for linear maps f,g we mean the linear map $\mathbf{L}(M,N)\to\mathbf{L}(M',N')$ given by $\mathbf{L}(f,g)(h)=g\circ h\circ f$. We will call S a superchannel if for every $\mathcal{E}\in\mathbf{QC}(A\otimes X,A'\otimes X')$ then $(S\otimes_u i)(\mathcal{E})\in\mathbf{QC}(B\otimes X,B'\otimes X')$. One can take analogous routes to defining the supermaps on general completely positive maps, or to defining supermaps on unitary channels, which we will call superunitaries. In the literature, supermaps on some set of transformations such as the unitaries are typically equivalently defined to be those superchannels which furthermore preserve those transformations [26, 211]. Rather than taking the care to introduce each definition of supermap algebraically, in all its detail and original subtlety, we now move on to a more structural and diagrammatic approach.

2.3.1 Diagrammatic Representation and Minor Generalisation of Supermaps

The definition of a supermap can be written in diagrammatic form, using the compact closure of the categories of completely positive maps and linear maps. In doing so, supermaps can be written in a way which generalises them to all monoidal categories which are subcategories of compact closed categories. Let us now outline this approach, a morphism

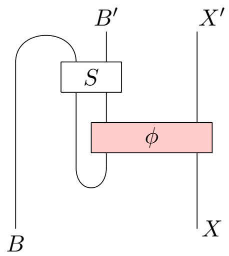


of a compact closed category can informally interpreted as a diagram with a hole by bending wires [126]



This observation, leads to an adaptation of the approach of [126, 132] as written down in [3] which does not reference some additional notions used there such as causality and closure. These are instead replaced by an abstraction of complete-preservation [12].

Definition 1. *Let* $\mathbf{C} \subseteq \mathbf{D}$ *be an inclusion of a symmetric monoidal category* $\mathbf{C}$ *into a compact closed category* $\mathbf{D}$*. A* $\mathbf{D}$*-supermap on* $\mathbf{C}$ *of type* $S : [A, A'] \to [B, B']$ *is a morphism in* $\mathbf{D}$ *of type* $S : A^* \otimes A' \to B^* \otimes B'$ *such that for every* $\phi \in \mathbf{C}(A \otimes X, A' \otimes X')$ *then*



$$\in \quad \mathbf{C}(B \otimes X, B' \otimes X').$$

We will sometimes as a shorthand represent this expression simply as $(S \otimes i)(\phi_1, \ldots, \phi_n)$, recalling the interpretation of supermaps as transformations which can be applied to $\phi_1 \ldots \phi_n$. Throughout this thesis, whenever it seems convenient to distinguish between boxes being used to represent standard transformations and boxes being used to represent supermaps, we will shade the boxes to be interpreted as standard transformations in pink. Briefly, this definition approach for supermaps can be phrased as saying that $\mathbf{D}$ is raw-material category from which we can make holes using compact closure. Of those holes in $\mathbf{D}$, the good ones for $\mathbf{C} \subseteq \mathbf{D}$ are the ones which preserve morphisms of $\mathbf{C}$ when applied locally. It is easy enough to check that $\mathbf{D}$-supermaps on $\mathbf{C}$ form a category, given $S : [A, A'] \to [B, B']$ and $T : [B, B'] \to [C, C']$ then the composition $S \circ_{\mathbf{D}} T$ returns a supermap of type $[A, A'] \to [C, C']$. This category can furthermore be made monoidal and symmetric using the symmetric monoidal structure of $\mathbf{D}$. We can now write down a variety of theories of supermaps using this approach.

Definition 2. *We define the following theories of supermaps:*

- *The quantum superchannels [12] of type* $[A, A'] \to [B, B']$ *are the* $\mathbf{CP}$*-supermaps on* $\mathbf{QC}$ *of the same type* [4].
- *The superunitaries [211] of type* $[A, A'] \to [B, B']$ *are the* $\mathbf{FHilb}$*-supermaps on* $\mathbf{U}$ *of the same type* [5].
- *The classical superchannels [16, 126, 213] are the* $\mathbf{Mat}_{[\mathbb{R}+]}$*-supermaps on* $\mathbf{Stoch}$ *of the same type* [6].

Each of these classes of supermaps satisfies a key structural theorem.

[4] Note that we have made our life easier by assuming complete positivity, this complete positivity is derivable for supermaps by noting that when applied to the swap, every superchannel returns a quantum channel and so a completely positive map.

[5] One can see that any superunitary defined in this way gives a superchannel by stinespring dilation, and that any standard definition superchannel which preserves unitaries is the double [22, 212] of some linear map by [53].

[6] What we refer to here as a classical superchannel is what is referred to in [126] as a second-order causal process in the category $\mathbf{Mat}_{\mathbb{R}+}$.

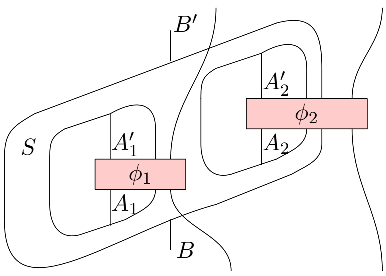
Theorem 1. *Every quantum superchannel, superunitary, and classical superchannel of type* $S : [A, A'] \to [B, B']$ *decomposes as* $S(\mathcal{E}) = S_f \circ (i \otimes \mathcal{E}) \circ S_p$ *with* S_f, S_p *quantum channels, unitaries, and stochastic matrices respectively.*

Each of these theorems is a consequence of a key theorem on decomposition of quantum transformations, that semi-causal processes decompose as semi-localisable processes [214, 215]. The first case can be seen in [12], the second is given in [53], and the third is given in [126].

Note that in the literature, some attention is also given to the unitarily-extendible superchannels [104], meaning those which are constructed up-to dilation from superunitaries.

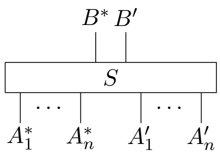
2.4 Multi-Input Supermaps:

One motivation for the study of quantum supermaps was the study of quantum and more generally indefinite causal structures [14, 18]. The supermaps used to model such structures are those which are often interpreted as having multiple independent inputs as in the following intuitive picture



We will now define multi-input supermaps, again making direct use of compact closure.

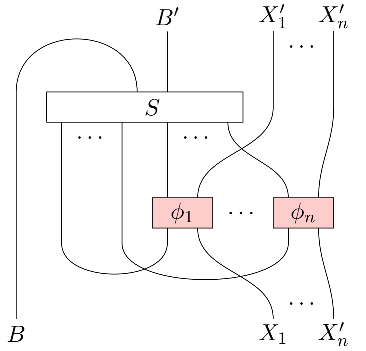
Definition 3. *Let* $\mathbf{C} \subseteq \mathbf{D}$ *be an inclusion of a symmetric monoidal category* $\mathbf{C}$ *into a compact closed category* $\mathbf{D}$*, a morphism*



in $\mathbf{D}$ *is a* $\mathbf{D}$*-supermap on* $\mathbf{C}$ *of type*

$$S : \bigtimes_{i=1}^{n} [A_i, A'_i] \to [B, B']$$

if and only if for every family $\phi_i \in \bigtimes_{i=1}^{n} \mathbf{C}(A_i X_i, A_i' X_i')$ *then*

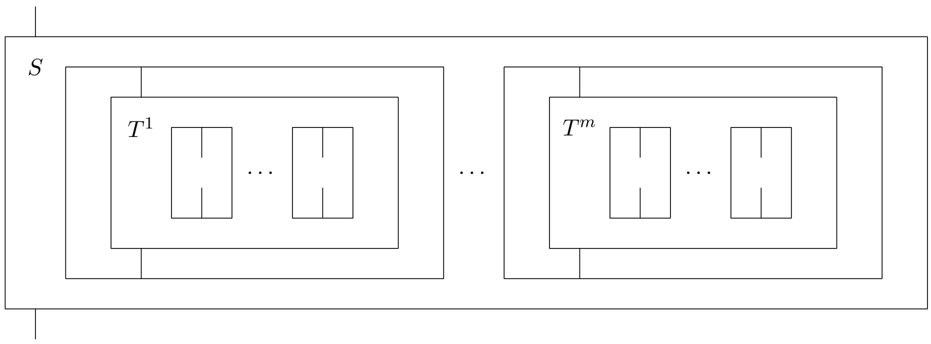


$$\in \quad \mathbf{C}(B \bigotimes_{i=1}^{n} X_i, B' \bigotimes_{j=1}^{n} X_j').$$

We define the multi-input superchannels, superunitaries, quantum channels, and classical superchannels using the same inclusions as for for the single input case. The superchannels as we define them here are equivalent to the supermaps on the product channels as defined in [14] and to the process matrices of [18] as proven in [211]. The superunitaires defined here are equivalent to the quantum superchannels which preserve unitaries [26,211].

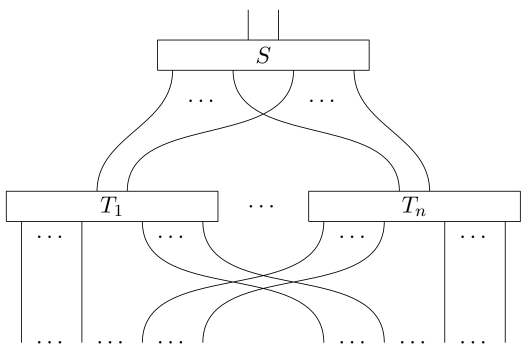
2.4.1 Many-To-One Composition Rules

The multi-input supermaps appear to exist outside of the standard categorical language introduced so far, by acting on lists of objects rather than individual objects. These supermaps can correspondingly most naturally be composed in a many-to-one way by nesting as in the following intuitive diagram



,

Formally, this nesting rule is given by



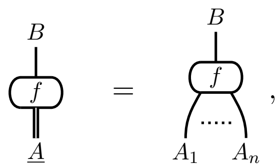
.

Such a many-to-one compositional theory is referred to in the categorical literature as a *multicategory* [192, 193], a generalisation of the notion of category to the setting in which morphisms are allowed to have multiple inputs.

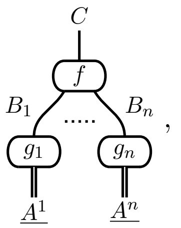
Definition 4. *A multicategory* $\mathbf{M}$ *is a specification of objects* $A, B, \dots$ *and for every (possible empty) list of objects* $\underline{A} = A_1 \dots A_n$ *and object* B *a set of morphisms* $\mathbf{M}(\underline{A}, B)$*. A multicategory comes equipped with for each non-empty* $\underline{B}$*, composition functions*[7] $\circ_{\underline{A}^{(i)}\underline{B}C}$: $\mathbf{M}(\underline{B}, C) \times_{i=1}^{|\underline{B}|} \mathbf{M}(\underline{A}^i, B_i) \to \mathbf{M}(\underline{A}^1 \dots \underline{A}^n, C)$ *written* $\circ(f, g_1, \dots, g_N) := f \circ (g_1, \dots, g_N)$[8] *and for each object* A *a specific morphism* $i \in \mathbf{M}(A, A)$*. These morphisms are required to satisfy the following laws, which model associativity, interchange, and unitality for the identity:*

- *Unit below* $f \circ (i, \dots, i) = f$.
- *Unit above* $i \circ (f) = f$.
- *Associativity* $f \circ (g_1 \circ (h_{11}, \dots, h_{1m_1}), \dots, g_n \circ (h_{n1}, \dots, h_{nm_n})) = ((f \circ (g_1, \dots, g_n)) \circ (h_{11} \dots h_{nm_n})$.

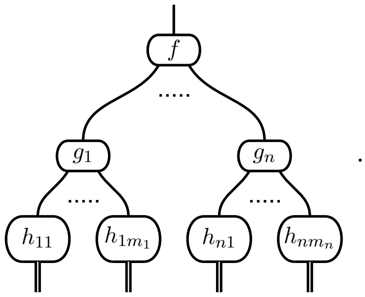
Diagrammatically, morphisms of a multicategory can be depicted as:



with composition $f \circ (g_1, \dots, g_n)$ depicted as:



and with the unitality and associativity laws meaning that unambiguous meaning can be given to diagrams such as:

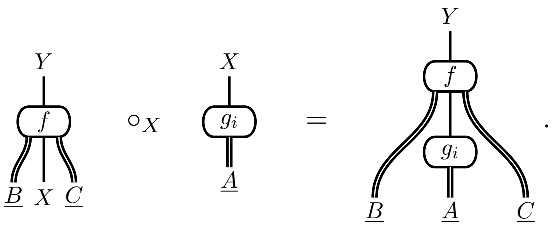


There are alternative ways to axiomatise multicategories, such as the so-called $\circ_i$ definition [193]. In this approach composition is defined along one wire at a time using functions

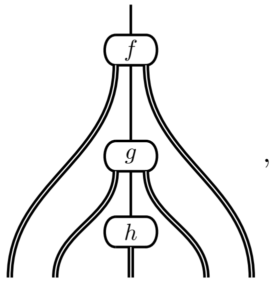
[7] We use the symbol $|\underline{B}|$ to represent the length of the list $\underline{B}$.

[8] Note that we immediately drop indeces on composition symbols whenever the meaning is clear.

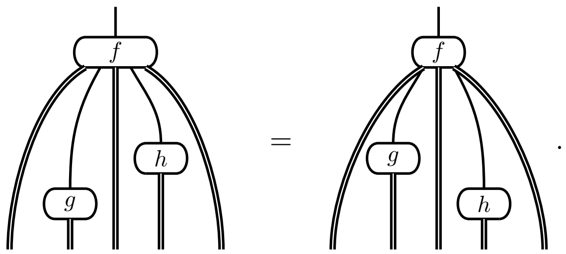
of the form $\circ_X : \mathbf{M}(\underline{B}X\underline{C}, Y) \times \mathbf{M}(\underline{A}, X) \to \mathbf{M}(\underline{BAC}, Y)$ depicted in the following way:



Associativity is then split into two laws, one which concerns sequential composition giving unambiguous meaning to diagrams such as:



and one which mimics the interchange law for monoidal categories:



One may note that multicategories look like they have some restricted subset of the properties of monoidal categories, and indeed, every monoidal category defines a multi-category.

Example 12. *Every (symmetric) monoidal category can be used to construct a (symmetric) multicategory by taking*

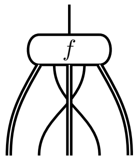
$$\mathbf{M_C}(A_1 \ldots A_n, B) := \mathbf{C}(A_1 \otimes \cdots \otimes A_n, B).$$

The morphisms $\bullet \to B$ from the empty list $\bullet$ in $\mathbf{M_C}$ are taken to be the morphisms of type $I \to B$ in $\mathbf{C}$. An explicit example of a multicategory which arises from this construction is the multicategory of multilinear maps, which is what remains when the monoidal structure of the linear maps is forgotten

Of course, our motivation for introducing multicategories is to study supermaps.

Example 13. *The $\mathbf{D}$-supermaps on $\mathbf{C}$ form a multicategory. We will not prove this here, since we will in section 2.8.1 find that the $\mathbf{D}$-supermaps form polycategories, from which the formation of a multicategorical structure would immediately follow.*

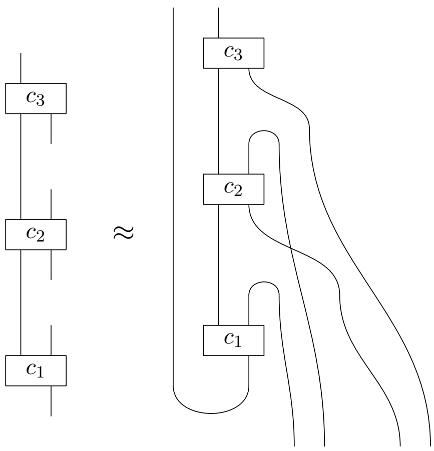
Multicategories can also be equipped with a notion of symmetry analogous to symmetry for monoidal categories [193]. This allows us safely swap wires diagrammatically as in the following picture[9]:



Such swaps can be composed just as for symmetric monoidal categories, to form arbitrary permutations. Such swaps are also required to be compatible with multicategory composition, mimicking the naturality law for swaps in symmetric monoidal categories[10]. The multicategory of **D**-supermaps on **C** is made symmetric by defining swaps of wires using pairs of associated swaps from the symmetric monoidal structure of **D**, composability and naturality are consequently directly inherited.

Multifunctors between multicategories are defined in a completely analogous way to functors between categories. A multifunctor $\mathcal{F} : \mathbf{M} \to \mathbf{N}$ assigns each object A to an object $\mathcal{F}A$ of $\mathbf{N}$ and sends morphisms of type $A_1 \dots A_n \to B$ to morphisms of type $\mathcal{F}A_1 \dots \mathcal{F}A_n \to \mathcal{F}B$ [217]. Multifunctors can then be drawn using functor boxes, with functorality again given by box-merging. Similarly to the case for functors, sub-multicategories, natural transformations between multifunctors, and equivalences of multicategories can then be defined just as for standard categories.

Examples of Supermaps: The most natural examples of **D**-supermaps are the combs, circuits into-which holes have been punctured [13]. In terms of our diagrammatic representation they can be drawn intuitively on the left and formally on the right as



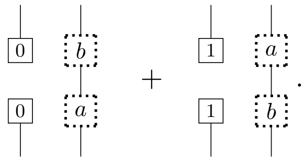
[9]The statement of symmetry is more subtle in the multicategorical setting, since each morphism of a multicategory has at most one output, the action of swapping wires is not a morphism. Instead, one can construct a category $\sigma_{\mathbf{M}}$ with objects given by lists of objects of the multicategory, and morphisms given by permutations. A notion of permutation of wires which compose analogously to the composition of swaps in a monoidal category is then modelled by giving each permutation of lists an action via equipment of $\mathbf{M}(A_1 \dots A_n, B)$ to a functor into **Set**.

[10]This compatibility of multicategory composition with an action by permutations is referred to as an equivariance law [216].

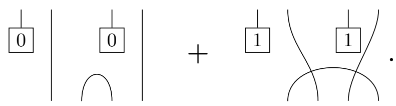
The most commonly studied example of a supermap in quantum foundations is the quantum switch, a superposition of combs [14]. Here we will introduce the switch as a superunitary, which will make it easier write down. There is an analogous superchannel which represents the quantum switch, indeed as a simple consequence of Stinespring's dilation theorem [218] every superunitary can be used to construct a superchannel using the embedding of linear maps into completely positive maps [199] known in diagrammatic quantum theory as doubling [22]. Forgetting the supermap properties for a second, the key idea of the quantum switch is that it models a two-input function on processes, which uses a control qubit to decide the order in which those processes are applied

$$qSwitch(U_1, U_2) = |0\rangle\langle 0| \otimes U_1 \circ U_2 + |1\rangle\langle 1| \otimes U_2 \circ U_1.$$

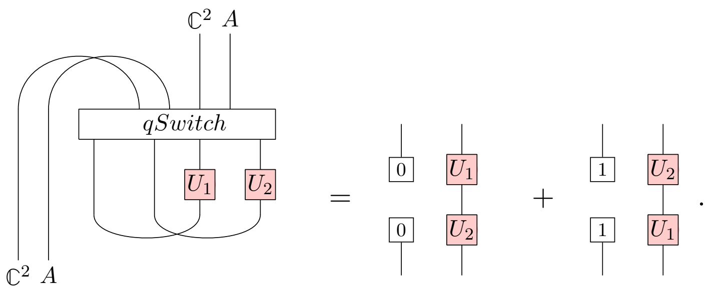
Without inserting unitaries into the entries of the function then, the switch can intuitively be thought of as representing a pair of holes $(-)_a$ and $(-)_b$ which are in an indefinite order

$qSwitch((-)_b, (-)_a) \quad \approx$ 

As an **FHilb**-supermap, the switch is modelled by the linear map $qSwitch : [A, A][A, A] \to [\mathbb{C}^2 \otimes A, \mathbb{C}^2 \otimes A]$ with[11]

$qSwitch \quad :=$ 

Indeed, then



From now on we will draw intuitive representations whenever the translation to a formal representation is clear. A main goal of this thesis, will be to be able to find formal representations of such intuitive pictures which do not rely on the existence of embedding into a compact closed category to be phrased. Now that we have discussed what supermaps are, and seen some key examples of supermaps, it is time to discuss in more detail what they are good for.

[11]The following diagrammatic expression is given formal meaning by interpretation of diagrams in the category **FHilb**, with + representing the standard addition rule for linear maps between Hilbert spaces.

2.5 Supermaps in Action

By converting quantum channels into quantum channels, supermaps allow us to study the ways in which devices can be transformed into each-other, and ultimately be treated as *resources* [10, 45, 46, 48, 66, 219]. Furthermore we will find that supermaps can be used to study indefinite causal structures [14, 18], and that together these applications can be combined to develop the study of causal structure as resource for information-theoretic protocols.

2.5.1 One-Shot Discrimination of Unitaries

We begin with a simple example of a computational protocol which can be phrased in terms of supermaps. Let us imagine that one is promised that a pair of unitaries will be provided, and that they will either perfectly commute or perfectly anti-commute meaning that either $U_1U_2 = U_2U_1$ or $U_1U_2 = -U_2U_1$. One can show that with a single use of the quantum switch, one can determine with certainty between these two cases [51]. Indeed, consider that on unitaries the action of the quantum switch with control fixed to the $|+\rangle$ state is given by

$$qSwitch(U_1, U_2)_+ := qSwitch(U_1, U_2) \circ (|+\rangle \otimes i) = \frac{1}{\sqrt{2}} |0\rangle \otimes U_1U_2 + \frac{1}{\sqrt{2}} |1\rangle \otimes U_2U_1, \quad (2.9)$$

in the commuting case then $qSwitch_+(U_1, U_2) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes U_1U_2 = |+\rangle \otimes U_1U_2$ and in the anti-commuting case then $qSwitch_+(U_1, U_2) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes U_1U_2 = |-\rangle \otimes U_1U_2$. The outcome of measurement of the auxiliary qubit in the $\{|+\rangle, |-\rangle\}$ basis is then guaranteed by the commutation or anti-commutation property of the input unitaries. It can be shown that a one-shot protocol to perfectly distinguish these cases does *not* exist for supermaps which are not quantum superpositions of causal structures, and that this result holds also for quantum channels in terms of commutation properties between their Kraus decompositions [51]. As such, this protocol involving the manipulation of devices, shows that there is a distinction between the tasks which are possible with or without the resource of quantum causal structure.

Inspired by this protocol, another is proposed in [55] and phrased in terms of communication complexity, which removes the need for promised inputs whilst demonstrating exponential advantages for quantum switches over causally ordered supermaps.

2.5.2 Capacity Activation

In quantum information theory, for the purposes of communication between parties A, B represented by Hilbert spaces, the canonical useless quantum channel is the *completely* *depolarising channel* $\mathcal{D} : st(A) \to st(B)$ defined formally by

$$\mathcal{D}(\rho) = \frac{Tr[\rho]}{dim(B)} \sum_{i=1}^{dim(B)} |i\rangle \langle i| = \texttt{The Maximally Mixed State}. \tag{2.10}$$

It is easy to intuitively see why such a channel is considered as useless for communication, since the output of the channel in-fact does not depend on the input of that channel. Formally the completely depolarising channel has a classical and quantum communication capacity of 0 [220]. However, it can be shown that coherently controlled usage of depolarising channels does allow for the transmission of information through the output channel [73]. For instance, whilst the sequential composition of two depolarising channels in either order recovers another useless depolarising channel, it can be shown that the quantum combination of those two compositional orders does in-fact produce a new channel which has a non-zero capacity for the transmission of information. Concretely

$$\mathbf{q}Switch_+(\mathcal{D}, \mathcal{D})(\rho) = \frac{1}{2} \sum_{i,j \in \{0,1\}} \left[\delta_{ij} \frac{I}{d} + (1 - \delta_{ij}) \frac{\rho}{d^2} \right] \otimes |i\rangle \langle j| , \tag{2.11}$$

Where $\mathbf{q}Switch$ represents the doubling [22] of $qSwitch$ to define its action not just on unitaries but on all quantum channels. Whilst the output of $\mathcal{D}(\rho)$ has no dependence on the input ρ the output of the quantum switch of depolarising channels on ρ *does* still depend on ρ. This phenomenon has been termed *Causal Activation* of classical communication capacity, and has been generalised to more elaborate effects involving a variety of protocols. First, whilst the above channel has classical capacity it does not have quantum capacity, there have been shown to be quantum channels with 0 classical capacity which after application of the quantum switch return a new channel with perfect quantum capacity [74]. Causal activation effects have further been studied in the N-input setting [5, 6, 76, 79, 80] where in fact it has been shown that with enough 0-capacity depolarising channels a channel with quantum capacity can be constructed [5] using superpositions of cyclic permutations of circuits with holes.

For some of the above effects, similar results can be achieved using superpositions of channels [86], or superpositions of trajectories through quantum channels [75], which can be formalised as supermaps on vacuum-extended channels [10]. In formal terms there are even plenty of supermaps which activate capacity by simply discarding their input channels and then producing cleaner ones, leading to significant subtleties in the study of resources in the capacity-activation setting, a discussion and resolution of these subtleties can be found in [10]. A comprehensive review of capacity activation phenomena can be found in [82]. The activation of capacity by quantum channels has further inspired a family of protocols concerned with the rate of cooling in thermodynamical systems, where again cyclic permutations are found to produce the greatest advantages [97].

2.5.3 Correlations Between Causal Structures

Supermaps can be used to study the quantum correlations between parties often referred to as *laboratories*, here the idea is to treat each hole of a supermap as a place where an event might occur with some probability [18]. In this context, where supermaps are instead often referred to as process matrices, the main stated goal is to define and analyse causal-inequalities [18, 106, 108, 109, 112, 115, 221, 222] for non-causal models of spacetime structures.

Let us now outline the process matrix axiomatisation for supermaps, which is more focussed on the probabilistic structure of quantum theory [18], equivalence to supermaps was folklore until a recent formal proof of [211]. First, the in the process-matrix approach a general notion of probabilistic event in quantum information theory is modelled using elements of *quantum instruments*. A quantum instrument is a twice-parameterised family $\{\mathcal{E}_i^b\}$ of completely positive trace non-increasing maps, where fixing the lower parameter as some pre-chosen experimental setting, the upper parameter represents a series of potential measurement outcomes each one corresponding to the occurrence of some completely positive trace non-increasing map $\mathcal{E}_i^a$. Given a pair of events in separate non-interacting laboratories, one can ask the question of their joint probability of having occurred. In general this probability will depend their connectivity and so on the spacetime environment which surrounds those events. It is this spacetime environment which is modelled using quantum supermaps.

In short, every supermap of type $S : [A, A'][B, B'] \to [\mathbb{C}, \mathbb{C}]$ is called a *process matrix* and represents the most general possible way to sensibly extract probabilities from events in unconnected laboratories. Probabilities are explicitly extracted from supermaps by defining $P(\mathcal{E}_i^a, \mathcal{N}_k^b) := S(\mathcal{E}_i^a, \mathcal{N}_k^b)$ where the isomorphism between linear maps of type $\mathbb{C} \to \mathbb{C}$ and elements of $\mathbb{C}$ has been used after application of the supermap to generate a number. Usually, in this context, an isomorphic object is the one referred to as the process matrix. This isomorphic object is the positive state which can be constructed from the supermap by applying the supermap to swap morphisms and then using the Choi-Jamiolkowski isomorphism [134].

We mentioned in the introduction that in this process matrix viewpoint, the key application is the generalisation from Bell-inequalities to *Causal-Inequalities*. In analogy with the field of device-independent verification of non-locality, causal inequalities define a way to study device-independent verification of the existence of indefinite causal structures [18, 106, 108, 109, 112, 115, 221, 222].

There are states which mathematically satisfy a theory-dependent/quantum-specific notion of entanglement, and yet, cannot cause probabilistic measurement outcomes outside of those which can be created by genuinely local states. It turns out the quantum switch suffers a similar fate, whilst being clearly causally indefinite, the quantum switch cannot be used to break any device-independent causal inequality. In general, there is no two-input superunitary which can be used to break causal inequalities [222][12], however, there is a two-input superchannel termed the OCB-process which *can* be used to break causal inequalities [18]. Whereas the switch has been proposed to be implementable experimentally in terms of photonic setups [14] or quantum spacetime metrics [223], the causal-inequality violating supermaps such as the OCB process have yet to be given physical interpretation. One way to rule out such processes in a top-down way is to require unitarity or purifiability of supermaps, with the OCB-process does not satisfy [104].

It turns out however, that there are three-input quantum superchannels, three-input classical superchannels, *and* three-input superunitaries which can be used to break causal inequalities [213]. The key to such processes is that rather than using any particularly quantum causal structure, they make use of cycles which are carefully constructed so that they manage to avoid creating genuine time-travel paradoxes [211, 224]. It turns out that a large potentially exhaustive class of superunitaries, including all of those currently known to break causal inequalities, can be proven to be consistent using one structural theorem [211]. This final structural theorem is built on the phrasing of sectorial constraints for linear maps [225], one of the topics of the next section in which we introduce a few key classes of constraints in quantum information theory.

2.6 Constraints

In this section we are going to take a short detour into the formalisation of constraints of quantum processes. Whilst a natural notion of a constraint on a process with respect to some decomposition, is *preservation* of decompositional structure, we will be interested in a more elaborate notion of specification of the *distribution* of structure. Two particular kinds of decompositions of objects which it is natural to consider in quantum theory are those given in terms of the two monoidal products $\otimes$ and $\oplus$. A flexible enough mathematical notion for the encoding of constraints with respect to $\otimes$ and $\oplus$ will be that of a relation, a kind of generalisation of functions which allows for one-to-many values.

Concretely, a relation $\tau : X \to Y$ between sets X and Y is a function of the form $\tau : X \times Y \to \{0, 1\}$. Typically we will say that τ relates x to y if $\tau(x, y) = 1$, and we will accordingly use the shorthand $x \sim_\tau y$. Relations can be composed to define a category, given $\tau : X \to Y$ and $\lambda : Y \to Z$ then $\lambda \circ \tau : X \to Z$ is defined by

$$\lambda \circ \tau(x, z) = 1 \iff \exists y \in Y : \quad \tau(x, y) = 1 \quad \wedge \quad \tau(y, z) = 1.$$

[12] A significant recent development is the combination of causality with locality to construct a device independent inequality which is broken by the switch [112], another significant recent development is the device independent framework for verification of indefinite causal order in analogy with the study of contextuality in quantum theory [35].

Relations can also be composed in parallel in two ways to form two monoidal categories. First, the relations form a monoidal category with monoidal product on objects $X \otimes Y$ given by the Cartesian product $X \times Y$ of sets. Given $R_1 : X_1 \to Y_1$ and $R_2 : X_2 \to Y_2$ then the monoidal product $(R_1 \otimes R_2) : X_1 \times X_2 \to Y_1 \times Y_2$ is given by

$$(R_1 \otimes R_2)((x_1, x_2), (y_1, y_2)) := 1 \iff R_1(x_1, y_1) = 1 \text{ and } R_2(x_2, y_2) = 1.$$

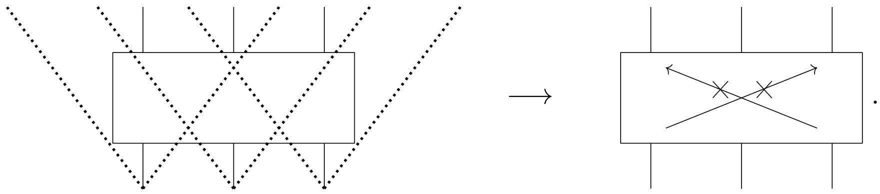
The unit object is given by the singleton set $\{\bullet\}$ since $\{\bullet\} \times X \cong X \cong X \times \{\bullet\}$ Second, the relations form another monoidal category with monoidal product $X \oplus Y$ given by the disjoint union $X \cup Y$ of sets. On morphisms the monoidal product $(R_1 \oplus R_2)$ of $R : X_1 \to Y_1$ and $R_2 : X_2 \to Y_2$ is given by

$$\begin{aligned}(R_1 \oplus R_2)(x, y) := 1 \iff & (x, y) \in X_1 \times Y_1 \text{ and } R_1(x, y) = 1 \\ & \text{or } (x, y) \in X_2 \times Y_2 \text{ and } R_2(x, y) = 1\end{aligned}$$

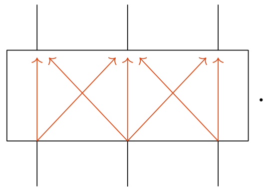
The unit object is given by the empty set $\emptyset$ since $\emptyset \cup X = X = X \cup \emptyset$. One may wonder why we choose to match the notations of monoidal products of relations and linear maps in the way we have. We do so to highlight that in each case $\oplus$ is a special kind of monoidal product given by the categorical notion of *coproduct* and that in each case $\otimes$ is the kind of monoidal product with respect to which each category is compact closed. Indeed, the relations form a compact closed category with $X^* = X$ and with $\cup_X(\bullet, (x, x')) = 1 \iff x = x'$ and $\cap_X((x, x'), \bullet) = 1 \iff x = x'$. Again it is the existence of perfectly correlated states and effects which make **Rel** compact closed. In short, relations give a general way to specify connection between various elements of sets, this generality makes them suitable for the specification of distribution of structure by quantum transformations.

2.6.1 Signalling Constraints

The constraints given by signalling relations specify which input systems of a global process are permitted to send information to each output system. These constraints are of broad interest in quantum information and quantum foundations [14,126,214,215,226,227]. The interest in signalling relations at least in part comes from the motivation of embedding global processes into spacetimes. Imagine for instance, that the inputs and outputs of a process are assigned points within a spacetime, from a physical perspective the light-cone structure of that spacetime will guarantee that the transformation cannot transmit information from some points to others. This imposed structure from a spacetime configuration of inputs and outputs can then be encoded a by no-influence relations between those input and output objects of the process:



One can instead choose to work the the complement of no-influence relations, which we will call signalling relations, such relations specify systems which *are* allowed to influence each-other:



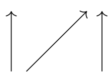
The absence of an arrow correspondingly indicates no-influence, we choose this representation because whilst it is easy to check that no-influence arrows cannot be safely composed, these orange allowed-influence arrows *can* be safely be composed [7].

We now outline how to specify the mathematical condition on the transformation which is encoded by this relation. Informally, signalling constraints in quantum theory can be encoded by choosing a relation τ on an input and output partition of spaces into tensor products, which specifies the ways in which a process is permitted to distribute information. Letting $A = \otimes_i A_i$ and $B = \otimes B_k$, a quantum channel can be said to satisfy the constraint encoded by the relation $\tau : [A_i] \to [B_k]$ if for each A_i there exists a channel $\mathcal{D}$ such that for every state ρ_A then

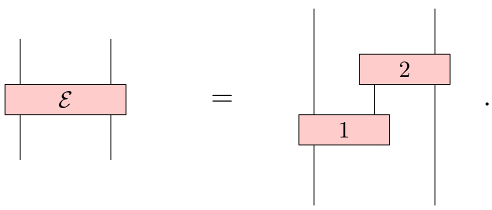
$$\mathrm{Tr}_{\tau A_i}(\mathcal{E}(\rho_A)) = \mathcal{D}(\mathrm{Tr}_{A_i})(\rho_A), \tag{2.12}$$

where $\tau(A)$ denotes the image of τ on A. A key structural theorem on signalling constraints is the following.

Theorem 2 (Causal Decomposition for Quantum Channels)**.** *Let $\mathcal{E} : A \otimes B \to A' \otimes B'$ satisfy the one-way signalling relation $(\nleftarrow)^{A'B'}_{AB} : [AA'] \to [BB']$ given by*

$(\nleftarrow)^{A'B'}_{AB} =$  ,

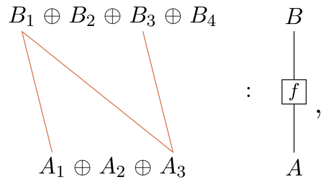
then $\mathcal{E}$ correspondingly decomposes as [214, 215, 226]:



This theorem tells us that no-faster-than light communication enforces locality of operations, and allows us to confidently study causal structures using compositional structures. Subtleties do however appear, when the tripartite and general N-partite cases are studied [227].

2.6.2 Sectorial Constraints

Some of the previously mentioned subtleties on decompositions of transformations satisfying signalling relations, can be addressed by considering sectorial constraints [225, 227]. Sectorial constraints are used to encode the fact that a given linear map is forbidden from sending some elements of orthogonal subspaces of its domain to some elements of orthogonal subspaces of its codomain. We will often refer to orthogonal subspaces of a Hilbert space as sectors of that Hilbert space. As an example, consider the following figure



(2.13)

here the intended encoded sectorial constraints correspond to the absence of links between some sectors. For instance, the absence of links from A_1 to B_2, B_3 and B_4 means that, for a f following these constraints, then $f(A_1) \subseteq B_1$. A convenient way to formalise sectorial constraints is in terms of relations and the orthogonal projectors $\pi_i : A \to A$ which project onto subspace A_i and corresponding orthogonal projectors $\sigma_k : B \to B$ which project onto subspace B_k.

Given a relation, and families of orthogonal projectors, one may say that $f : A \to B$ satisfies the sectorial constraint $\tau : [A_1 \dots A_n] \to [B_1 \dots B_m]$ if

$$\sum_{ik} \tau_i^k \sigma_k f \pi_i = f$$

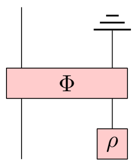
where τ_i^k is the matrix which takes values in $\{0, 1\}$ with $\tau_i^k = 1$ if and only if τ relates A_i to B_k. A key feature of sectorial constraints is that they are *composable* [225]. If $f : A \to B$ satisfies sectorial constraint $\tau : [A_i] \to [B_j]$ and $g : B \to C$ satisfies sectorial constraint $\lambda : [B_j] \to [C_k]$ then $g \circ f$ satisfies the sectorial constraint $\lambda \circ \tau$ given by composition of relations. It turns out that in fact the formalisation of sectorial and signalling constraints, and their compositionally are given by applying a single construction for relational constraints on causal categories, to the causal categories of linear maps with the direct sum and quantum channels with the tensor product respectively [7].

2.7 Constraints in Action

Here we will review a few places in the literature on quantum information theory and foundations where constrained sets of processes appear, focussing mainly on those of the above form, given by sectorial or signalling relations.

2.7.1 Supermaps on Non-Simple Types

Originally, what we call multi-input supermaps were actually formalised as acting not just on sequences of processes but instead on all global processes which satisfy non-signalling constraints [17]. Essential to defining supermaps on restricted sets, is the possibility to define the set of extended processes with respect to those restricted sets. Consider some subset $K \subseteq \mathbf{QC}(A, A')$, then the dilation extension $\texttt{dext}_{X,X'}(K)$ of K can be defined by taking $\texttt{dext}_{X,X'}(K)$ to be the set of all $\phi \in \mathbf{QC}(A \otimes X, A' \otimes X')$ such that for any ρ then



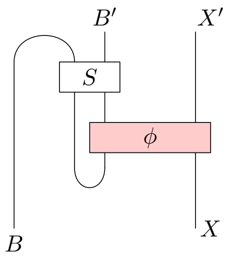
$\in \quad K.$

So, $\texttt{dext}_{X,X'}(K)$ represents all processes which are only ever dilations of processes in K. In terms of these extensions we will define the supermaps of type $K \to M$ with $K \subseteq \mathbf{QC}(A, A')$ and $M \subseteq \mathbf{QC}(B, B')$ by adapting the diagrammatic approach [13] [14].

Definition 5. *A completely positive map $S : A^* \otimes A' \to B^* \otimes B$ is a* **CP***-supermap on* **QC** *of type*

$$S : K \to M$$

if and only if for every family $\phi \in \texttt{dext}_{X,X'}(K)$ then



$\in \quad \texttt{dext}_{X,X'}(M).$

A natural example of a **CP**-supermap is the sequential composition supermap:

$$S_{\circ_{\leftarrow}} : (\not\leftarrow)^{AA}_{AA} \to \mathbf{QC}(A, A)$$

[13] Here we avoid stating the definition of supermaps on constrained sets for general symmetric monoidal subcategories of compact closed categories, we do this to avoid interrupting the flow at this stage. For a definition of supermaps on dilation extensions in general symmetric monoidal categories see section 4.4.

[14] We chose to assume complete positivity from the beginning here, this is a natural choice for normal sets as defined in Chapter 5, since the swap is in the dilation extension of any such set. For more general sets, the right definition of supermap is less clear, and suggests a general categorical framework may be needed for comparing different constructions of supermaps.

which wires together an output of a biparitite channel to an input of that bipartite channel in the following way:

$$S_{\circ_{\leftarrow}}(\mathcal{E}(-)) := \sum_{jk} i \otimes \langle j| \mathcal{E}(|j\rangle \langle k| \otimes -)i \otimes |k\rangle \,.$$

This sequential composition supermap earns its title for the following reason, when the channel $\mathcal{E}(-)$ separates as $\mathcal{E}_1(-) \otimes \mathcal{E}_2(-)$ then

$$\begin{aligned}
S_{\circ_{\leftarrow}}(\mathcal{E}(\rho)) :=& \sum_{jk} i \otimes \langle j| \mathcal{E}(|j\rangle \langle k| \otimes -)i \otimes |k\rangle \\
=& \sum_{jk} i \otimes \langle j| (\mathcal{E}_1(|j\rangle \langle k|) \otimes \mathcal{E}_1(\rho))i \otimes |k\rangle \\
=& \sum_{jk} \mathcal{E}_1(|j\rangle \langle k|) \langle j| \mathcal{E}_2(\rho) |k\rangle \\
=& \sum_{jk} \mathcal{E}_1(|j\rangle \langle j| \mathcal{E}_2(\rho) |k\rangle \langle k|) \\
=& \mathcal{E}_1(\mathcal{E}_2(\rho)).
\end{aligned}$$

Naturally, there is the alternative supermap $S_{\circ_{\rightarrow}}$ which also composes its inputs, but in the opposite direction.

2.7.2 Characterisation of Supermaps on Independent Parties

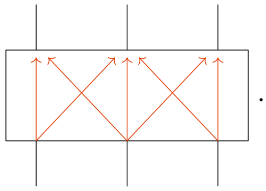
Just as one can define the one-way non-signalling channels by $(\not\leftarrow)_{AB}^{A'B'}$ one can define the neither-way signalling channels $(\not\leftrightarrow)_{AB}^{A'B'}$, more commonly referred to as the non-signalling channels. For every multi-input supermap $S : [A_1, A_1'] \dots [A_n, A_n'] \to [B, B']$ there exists a unique supermap $\hat{S} : (\not\leftrightarrow)_{A_1 \dots A_n}^{A_1' \dots A_n.'} \to [B, B']$ which does the same as S when acting on any non-signalling channels constructed from a list of independent processes [17, 126, 127]. In short, the supermaps give a way to characterise the non-signalling channels via a universal property, and the non-signalling channels correspondingly provide a notion of tensor product space for multi-input supermaps.

2.7.3 Characterisation of Circuits-With-Holes

We saw that on the space of one-way signalling processes $(\not\leftarrow)_{AB}^{A'B'}$ there is a sequential composition map, which can be interpreted as a special example of a comb. The one-way signalling channels can in fact be used to characterise the combs, a completely positive map S defines a supermap on the space of one-way signalling processes if and only if it decomposes as a circuit with holes [13, 15, 126, 127].

2.7.4 Causal Decompositions

Whilst we have seen that every one-way signalling process comes with a decomposition into local operations which guarantees the satisfaction of the one-way signalling constraint, the same cannot be said for the following more complex signalling relation:



For this relation, on unitaries, there does not always exist a decomposition into a circuit of unitaries which guarantees satisfaction of this signalling constraint *and* unitarity [227]. This issue can be resolved however, if one instead works with a generalisation of the quantum circuit formalism to include sectorial constraints, this generalisation of quantum circuits is referred to as routed quantum circuits [225]. In routed quantum circuits, relations are overlayed on top of linear maps. This kind of overlaying is possible because of the composability of sectorial constraints, the fact that if f satisfies τ and g satisfies λ then $f \circ g$ satisfies $\tau \circ \lambda$.

2.7.5 Constructing Consistent Circuits for Indefinite Causal Structure

The consistent overlaying of linear maps with sectorial constraints can be used to reconstruct all known examples of superunitaries from a single concrete method for constructing supermaps of spaces equipped with sectorial constraints [211]. In this method, by checking the fine-grained causal structure induced by the decompositions of wires into both kinds of monoidal product, the validity of cycles created by compact closure in diagrams can be used to verify whether the diagram in-fact specifies a valid supermap or not. The causal structure of such supermaps can also be studied [228], leading to a more subtle analysis of the notion of indefinite causal order in quantum switches.

2.8 The Caus[C] Construction

As we discovered, multiple-input supermaps are quite naturally modelled as morphisms within a multicategory, a kind of stripped-down monoidal category. We will now see more explicitly that in special cases relevant to classical and quantum information theory, one can do better and actually consider the kinds of sets which supermaps act on as the kinds of spaces which can be pieced together, analogously to the way in which Hilbert spaces can be pieced together using tensor products or direct sums. Using the language of [127], we will refer to theories of quantum supermaps in which spaces can be pieced together as higher-order quantum theories. We will review the most categorically phrased approach to higher-order quantum theories, in which the **Caus**[**C**] construction is defined [126, 132]. This construction represents the state of the art in the compositional modelling of supermaps, being equipped with monoidal products (plural) and even types representing spaces of N-input combs, and general super-supermaps, the kinds of transformations that can be applied to supermaps.

Whilst the **Caus**[**C**] construction represents the current state of the art, being simple, efficient, and providing a variety of ways to piece together spaces, we believe it is far from representing the complete picture for compositional modelling and construction of supermaps. This is for the simple reason that it makes use of categorical notions of causality and compact closure, which prevent us from applying the construction to unitary quantum theory, infinite-dimensional quantum theory, general operational probabilistic theories, and any other monoidal category in applied category theory which does not embed into a suitable compact closed category. Therefore, to develop a framework for supermaps on general monoidal categories, general operational physical theories, and even general aspects of quantum physics, we are going to need to move away from the **Caus**[**C**] construction. Nonetheless, the **Caus**[**C**] construction achieves a lot with the additional assumptions it relies on, and we can use it the develop a blueprint, identifying the key features it has which we would like to see from a more complete compositional framework for supermaps and causality.

We will now review the **Caus**[**C**] construction in more detail, the story begins with a raw-material *precausal* category **C**, meaning a compact closed category satisfying a few basic additional axioms. Examples of such categories are the the category **CP** of completely positive maps and the category $\mathbf{Mat}_{\mathbb{R}+}$ of positive-valued matrices. Using such a category **C** one can identify properties of *flatness* and *closure* for subsets $c \subseteq \mathbf{C}(I, A)$ of sets $\mathbf{C}(I, A)$ of states which allow those subsets to be composed in a variety of convenient ways. In the category **Caus**[**C**], objects are then taken to be these closed and flat states, and morphisms $f : c \to d$ are defined to simply be those such that for every $\rho \in c$ then $f \circ \rho \in c_B$. We will often use the language of [127] and refer to such morphisms as admissible morphisms from c to d.

A crucial feature of the **Caus**[**C**] construction which allows it to work so efficiently, is that it does not focus on complete-preservation directly. Compact closure is used in an identical way to the diagrammatic definition of supermap we provided, with one key difference, admissibility of morphisms does not reference extension by a tensor product. No condition is directly required of $S \otimes_u i$ as in the standard approach to defining supermaps, instead closure of sets when present implies an equivalence between admissibility of S and admissibility of each $S \otimes_u i$. As a consequence of this inherited complete-admissibility, the category **Caus**[**C**] can be equipped with two monoidal products $c \otimes d$ and $c \boxtimes d$. When **C** is taken to be the pre-causal category **CP** of completely positive maps, for any Hilbert space A the set of normalised density matrices c_A is flat and closed. Furthermore, for any two Hilbert spaces A, B the subset up-to natural isomorphism $\mathbf{QC}(A, B) \subseteq \mathbf{CP}(I, A^* \otimes B)$ is closed and flat. The equivalence between admissibility and complete-admissibility for closed flat sets can be captured by the following lemma.

Lemma 1. *Let K and M be closed flat subsets of the closed flat set of quantum channels, the morphisms of* $\mathbf{Caus}[\mathbf{C}](K, M)$ *are the supermaps of type* $K \to M$.

Proof. First let us show that morphisms in **Caus**[**C**] give quantum supermaps, up to natural isomorphisms given by applying cups and caps one can show that for any closed set K then $dext_{X,X'}(K) = K \boxtimes \mathbf{QC}(X, X')$. Since $\boxtimes$ is a monoidal product of **Caus**[**C**] then for every $S : K \to M$ in **Caus**[**C**] one can construct $S \boxtimes i_{\mathbf{QC}(X,X')} : K \boxtimes \mathbf{QC}(X, X') \to M \boxtimes \mathbf{QC}(X, X')$. This confirms by the admissibility requirement for morphisms of **Caus**[**C**] that for any $\phi \in dext_{X,X'}(K) = K \boxtimes \mathbf{QC}(X, X')$ then $S \boxtimes i_{\mathbf{QC}(X,X')}(\phi) \in M \boxtimes \mathbf{QC}(X, X') = dext_{X,X'}(M)$. Up to cups and caps this is exactly the defining requirement for quantum supermaps of type $K \to M$. To show that all quantum supermaps give morphisms in **Caus**[**C**] note that the admissibility requirement for morphisms of **Caus**[**C**] is implied by taking X and X' in the defining requirement for quantum supermaps to be the unit object I. □

Using the same methods, morphisms in **Caus**[**C**] of type

$$\mathbf{QC}(A_1, A_1') \otimes \cdots \otimes \mathbf{QC}(A_n, A_n') \to \mathbf{QC}(B, B'),$$

can be seen to be the superchannels of type $[A_1, A_1'] \ldots [A_n, A_n'] \to [B, B']$, meaning that the multicategorical structure of the multi-input quantum superchannels arises from forgetting the monoidal structure of part of the **Caus**[**C**] construction. However, there is more on the compositionality of multi-input supermaps which can be deduced from this observation.

2.8.1 Polycategorical Composition Rules

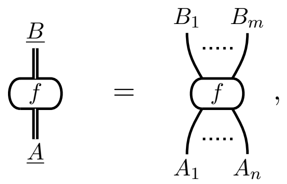
The tensor product $\boxtimes$ of **Caus**[**C**] gives complete-admissibility by the following handy result $\mathbf{QC}(A, A') \boxtimes \mathbf{QC}(B, B') \cong \mathbf{QC}(A \otimes B, A' \otimes B')$ [126]. As a consequence of this, the morphisms in **Caus**[**C**] of type

$$\mathbf{QC}(A_1, A_1') \otimes \cdots \otimes \mathbf{QC}(A_n, A_n') \to \mathbf{QC}(B_1, B_1') \boxtimes \cdots \boxtimes \mathbf{QC}(B_m, B_m'),$$

are equivalent to the superchannels of type $[A_1, A'_1] \dots [A_n, A'_n] \to [B_1 \dots B_m, B'_1 \dots B'_m]$. Together, the two monoidal products $\otimes$ and $\boxtimes$ form what is referred to in the literature as a linearly distributive structure [229]. Just as monoidal structures can be used to define multicategorical structures, linearly distributive structures can always be used to define *polycategorical* structures [194, 229]. Whereas multicategories allow only single-outputs, morphisms in polycategories are defined to have multiple inputs *and* multiple outputs. We will write $f : \underline{A} \to \underline{B}$ to denote a morphism from a list $\underline{A} = A_1 \dots A_n$ of objects to another list $\underline{B} = B_1 \dots B_m$ of objects.

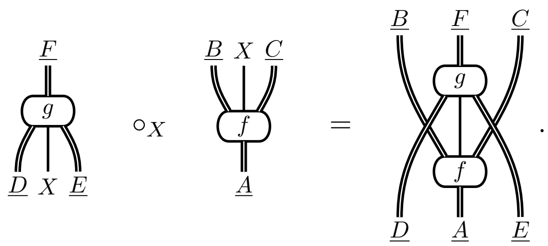
Monoidal categories also model morphisms with multiple inputs and multiple outputs, and so one may wonder, what is the difference between a monoidal category and a polycategory? The answer is simple, and of crucial relevance to the compositionality of supermaps, when working with polycategories we can only compose along *one wire at a time*. We will also require the possibility to permute or swap wires, meaning that we are really going to be working with *symmetric* polycategories [195, 197, 216]. For a presentation of polycategories without symmetries/permutations of wires the reader is pointed to the definition of *planar polycategories* of [194, 229, 230].

Definition 6. *A symmetric polycategory* $\mathbf{P}$ *is a collection of objects* $A, B, \dots,$ *for each pair of lists* $\underline{A}$ *and* $\underline{B}$ *a collection* $\mathbf{P}(\underline{A}, \underline{B})$ *of morphisms depicted:*



(2.14)

and for each pair $f : \underline{A} \to \underline{B}X\underline{C}, g : \underline{D}X\underline{E} \to \underline{F}$ *of morphisms a new composed morphism* $g \circ_X f : \underline{DAE} \to \underline{BFC}$ *depicted:*



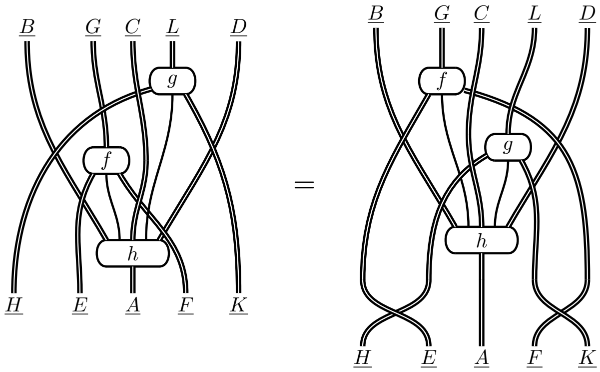
(2.15)

Symmetry (needed even to express associativity laws) is required in the form of a functorial action by permutations, meaning for each morphism $f : \underline{A} \to \underline{B}$ *and each pair of permutations* $\sigma : [n] \to [n]$ *and* $\rho : [m] \to [m]$ *a new morphism denoted* $\rho(f)\sigma$ *such that* $\rho'(\rho(f)\sigma)\sigma' = (\rho \circ \rho')(f)(\sigma' \circ \sigma)$[15]*. This gives a way to give meaning to the swapping of wires on the top or the bottom of any morphism.*

[15]Strictly, we must be more careful about permutations considering only those from the category $\sigma_{\mathbf{P}}$ of permutations of lists of objects of $\mathbf{P}$ (referred to in the literature as isomorphisms [216]). In this category objects are taken to be lists of objects of $\mathbf{P}$ and morphisms $\sigma : \underline{A} \to \underline{A}'$ are taken to be those permutations $\sigma : ||\underline{A}|| \to ||\underline{A}|| = ||\underline{A}'||$ such that for each $i \in ||\underline{A}||$ then ${A'}_i = A_{\sigma^{-1}(i)}$. Functorial action then strictly means the exstention of the specification of sets $\mathbf{P}(\underline{A}, \underline{B})$ to a functor $\mathbf{P}(-. =) : \sigma_{\mathbf{P}}^{op} \times \sigma_{\mathbf{P}} \to \mathbf{Set}$.

For each object X there is an identity morphism $i_X \in \mathbf{P}(X,X)$ and composition is subject to associativity and identity laws written $f \circ_X i_X = f$, $i_X \circ_X f = f$ and $(f \circ_X g) \circ_Y h = f \circ_X (g \circ_Y h)$ each directly absorbed in their graphical presentations, alongside:

- *Interchange 1 written $f \circ_X (g \circ_Y h) = (g \circ_Y (f \circ_X h))\rho$:*



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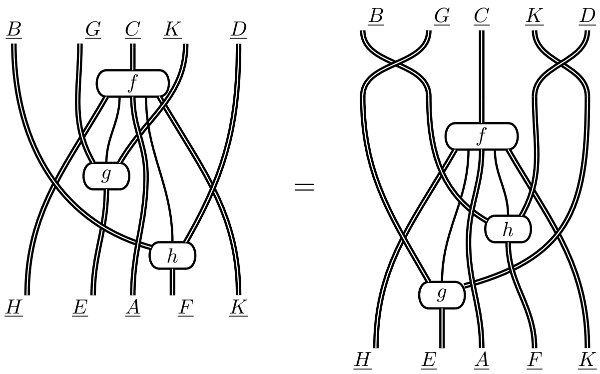
- *Interchange 2 written $(f \circ_X g) \circ_Y h = \sigma((f \circ_Y h) \circ_X g)$ and depicted graphically as:*



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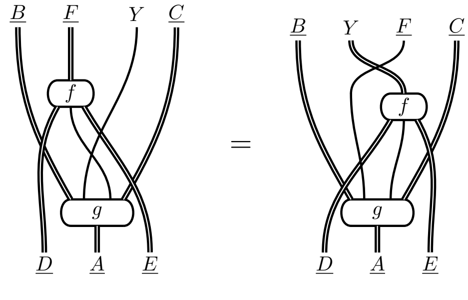
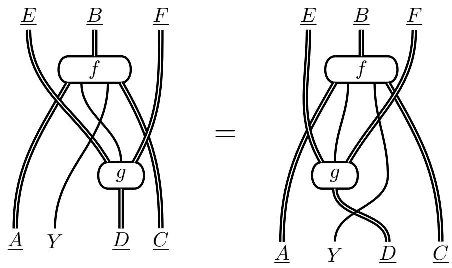
- *Equivariance with respect to permutations written $(\sigma f \rho) \circ_X (\lambda g \tau) = \alpha(f \circ_X g)\beta$ with the permutations α, β chosen such the equation is well typed. This packages a variety of graphical equivalences into one equation, most notably encoding analogues of naturality for the swaps of symmetric monoidal categories such as*



,

and further:



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As with multicategories, polycategories look like monoidal categories with a few less properties, this observation is formalised by the following examples [229].

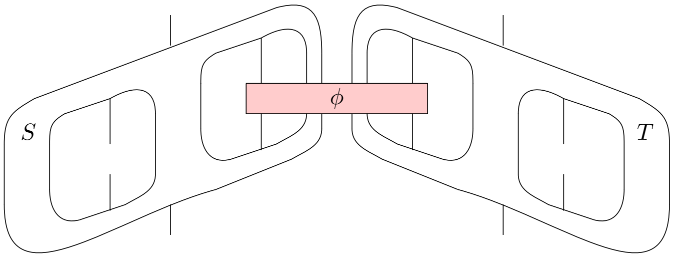
Example 14. *Every symmetric monoidal category* $\mathbf{C}$ *defines a symmetric polycategory* $\mathbf{P_C}$ *by*

$$\mathbf{P_C}(A_1 \dots A_n, B_1 \dots B_m) := \mathbf{C}(A_1 \otimes \dots \otimes A_n, B_1 \otimes \dots \otimes B_m),$$

and more generally every symmetric linearly distributive category $\mathbf{C}$ *defines a symmetric polycategory by*

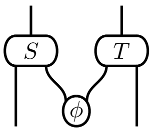
$$\mathbf{P_C}(A_1 \dots A_n, B_1 \dots B_m) := \mathbf{C}(A_1 \otimes \dots \otimes A_n, B_1 \boxtimes \dots \boxtimes B_m).$$

Using the linear-distributivity of the $\mathbf{Caus}[\mathbf{C}]$ construction, this example tells us for free that the superchannels of type $[A_1, A_1'] \dots [A_n, A_n'] \to [B_1 \dots B_m, B_1' \dots B_m']$ form a polycategory. Let us now see why it is natural to expect abstract multi-input multi-output supermaps to form polycategories. Whilst polycategorical structure allows us to give meaning to the informal picture



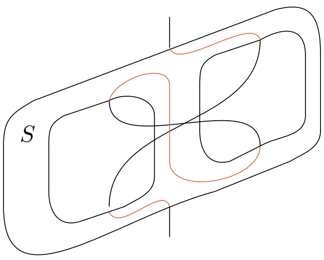
(2.16)

with the following formal diagram



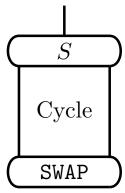
, (2.17)

there is in general no way to create the following pathological intuitive picture in which a bipartite swap morphism is inserted into two halves of the same sequential composition supermap



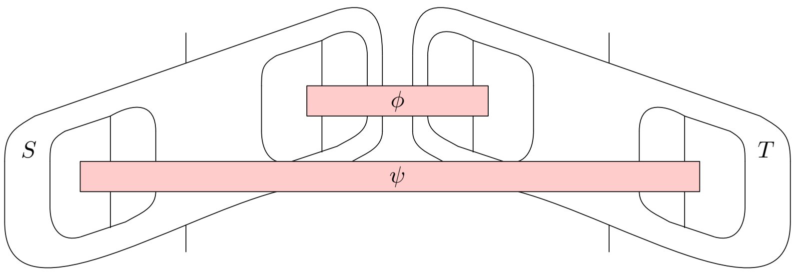
.

To model the above intuitive diagram would require us to be able to compose along more than one wire at a time



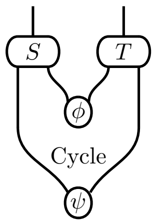
,

which is outside of the class of diagrams which can be constructed using polycategorical composition. As another more elaborate example, consider the following banned composition rule:



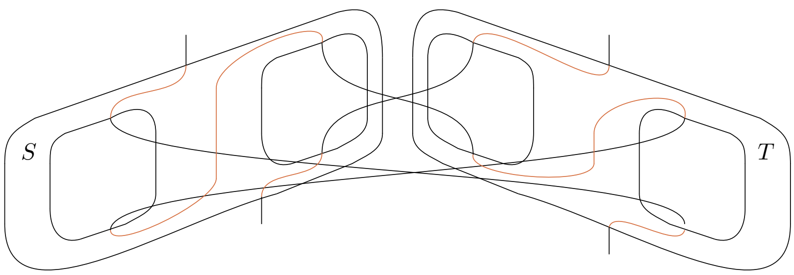
, (2.18)

in the intuitive picture this formal composition would represent the following intuitive diagram



. (2.19)

Composition along more than one wire at-a-time as above again allows for the construction of a time-loop as observed in [231], this can be seen by considering sequential composition supermaps with swap morphisms as inputs



. (2.20)

Note, that the pathology of multi-wire composition for supermaps on symmetric monoidal categories appears to be a consequence of the existence of sequential composition supermaps, this existence is a property which we will consider to be fundamental to theories of supermaps throughout this thesis.

Having seen that polycategorical composition is a rather natural consequence of the intuitive picture drawn to motivate the definition of supermaps, one would expect that good models for supermaps are polycategories. Indeed, for any monoidal subcategory $\mathbf{C} \subseteq \mathbf{D}$ of a compact closed category $\mathbf{D}$, the $\mathbf{D}$-supermaps on $\mathbf{C}$ of definition 1 form a symmetric polycategory.

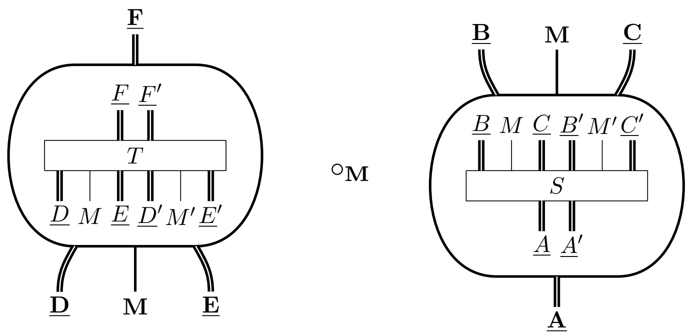
Theorem 3. *For any symmetric monoidal subcategory* $\mathbf{C} \subseteq \mathbf{D}$ *of a compact closed category* $\mathbf{D}$ *a symmetric polycategory* $\mathbf{Dsup}[\mathbf{C}]$ *can be defined with objects given by pairs* $[A, A']$ *of objects of* $\mathbf{C}$ *and morphisms of type*

$$S : \bigtimes_{i=1}^{n} [A_i, A_i'] \to \bigtimes_{j=1}^{m} [B_j, B_j']$$

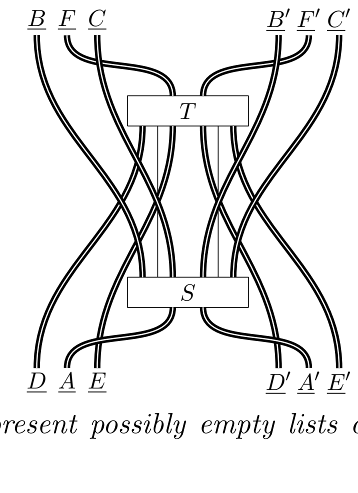
given by the $\mathbf{D}$*-supermaps of type*

$$S : \bigtimes_{i=1}^{n} [A_i, A_i'] \to [\bigotimes_{i=1}^{m} B_i, \bigotimes_{j=1}^{m} B_j].$$

The composition rule is given by taking:



to be



where we have chosen to represent possibly empty lists of wires with doubled wires and underlined objects.

Proof. Given in Appendix *B* □

As a result we can immediately see that the superunitaries define a symmetric polycategory, without taking the scenic route through quantum information theory and the theory of superchannels along the way. We will refer to this polycategory as $\mathbf{Su} := \mathbf{FHilbsup}[\mathbf{U}]$, and we will refer to the polycategory of superchannels as $\mathbf{QSc} := \mathbf{CPsup}[\mathbf{QC}]$. Since the monoidal category of sets and functions is a monoidal subcategory of the compact closed category of relations, the construction can immediately be used to return a polycategory $\mathbf{Relsup}[\mathbf{Set}]$ of super-functions. Yet another pair of polycategorical models for supermaps were recently introduced to model circuits into-which holes have been punctured in arbitrary monoidal categories [29]. These models either treat circuits with holes extensionally in-terms of the functions they can be used to perform, or intensionally in terms of explicit description of the architecture of the circuit up to equivalence by a categorical notion of sliding of boxes given by taking coends over profunctors [232].

Finally, functors between polycategories can be defined analogously to those for categories, monoidal categories and multicategories [229, 233]. A functor $\mathcal{F} : \mathbf{P} \to \mathbf{Q}$ between polycategories sends objects of $\mathbf{C}$ to objects of $\mathbf{D}$ and similarly for morphisms. Again such functors can be represented by drawing boxes around morphisms, with functorality given again by box-merging. Sub-polycategories, natural transformations of polyfunctors and equivalences between polycategories can then be defined analogously to those for categories, monoidal categories and multicategories [229].

We will now return to the $\mathbf{Caus}[\mathbf{C}]$ construction, examining the way in which it provides a model for super-supermaps and their iterations.

2.8.2 Self-Contained Higher-Order Theories

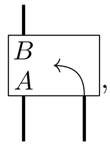
There is another method for piecing together objects of the $\mathbf{Caus}[\mathbf{C}]$ construction which builds higher-order objects $c \Rightarrow d$ from lower-order objects c and d. This method is the right one for studying supermaps, meaning that it can be used to construct $\mathbf{QC}(A, B)$ from the Hilbert spaces A and B. More precisely, letting c_A and c_B be the closed flat sets of normalised density matrices on A and B respectively, the closed flat set $c_A \Rightarrow c_B$ is isomorphic to the set $\mathbf{QC}(A, B)$ of quantum channels [126].

This outlined method for constructing higher-order objects can be used to construct super-supermaps. Indeed, first since the set of morphisms of type $\mathbf{QC}(A, A') \to \mathbf{QC}(B, B')$ in $\mathbf{Caus}[\mathbf{CP}]$ is the set of superchannels of type $[A, A'] \to [B, B']$. Using the same construction which we originally applied to sets of density matrices we can now construct the closed flat set $\mathbf{QC}(A, A') \Rightarrow \mathbf{QC}(B, B')$ of superchannels. Using sets of this kind, the $\mathbf{Caus}[\mathbf{C}]$ construction already comes with specified morphisms of type

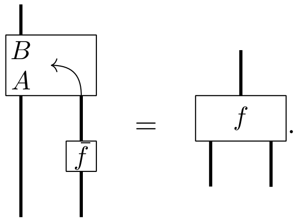
$$(\mathbf{QC}(A, A') \Rightarrow \mathbf{QC}(B, B')) \longrightarrow (\mathbf{QC}(C, C') \Rightarrow \mathbf{QC}(D, D')).$$

Transformations of this type can be understood as super-supermaps, transformations applied to the space of supermaps, and naturally this procedure can be iterated. In the background, the property being used here to iterate the notion of supermap is that of closed monoidal structure [234] of the category **Caus**[**C**].

A closed monoidal category is a category which supports *currying*, the existence of an object $[A, B]$ for each pair of objects A, B which represents the set of morphisms of type $[A, B]$ in the sense that $\mathbf{C}(C, [A, B]) \cong \mathbf{C}(A \otimes C, B)$. This currying isomorphism is an abstraction of a key property of set-like categories, which makes them suitable for functional programming. The precise definition of a closed monoidal category can be rephrased in a diagrammatically more friendly way. A closed structure on a monoidal category $\mathbf{C}$ is an assignment of an object $[A, B]$ to each pair A,B of objects in $\mathcal{C}$ along with a morphism $\epsilon_{[A,B]} : A \otimes [A, B] \to B$ denoted



such that for every morphism $f : (A \otimes C) \to B$ there exists a unique morphism $\bar{f} : C \to [A, B]$ satisfying



(2.21)

When a category is closed monoidal it indeed has a notion of transformation of transformation, reminiscent of that of the theory of supermaps. In more detail, for each A, A' and B, B' one can consider morphisms in $\mathbf{C}$ of type $[A, A'] \to [B, B']$ to be transformations of morphisms of type $A \to A'$ to morphisms of type $B \to B'$. Since there is an isomorphism $\mathbf{C}(A, A') \cong \mathbf{C}(I, [A, A'])$ then each $S : [A, A'] \to [B, B']$ can be used to define a higher-order function $S' : \mathbf{C}(A, A') \to \mathbf{C}(B, B')$ up to isomorphism by $\widehat{S'(\phi)} := S \circ \hat{\phi}$.

The higher-order functions constructed from closed monoidal categories are not necessarily analogous to supermaps. For any morphism $S : [A, A'] \to [B, B']$ of a closed monoidal category whilst one can consider the result of applying S to *part* of some bipartite state by $S \otimes I : [A, A'] \otimes [X, X'] \to [B, B'] \otimes [X, X']$. It is not true however, that in general that there is an isomorphism of type $[A, A'] \otimes [X, X'] \cong [A \otimes X, A' \otimes X']$ and so one cannot consider the local-application of S to party of *every* bipartite morphism $\phi : A \otimes X \to A' \otimes X'$. In-fact, in the **Caus**[**C**] construction it is the other tensor product $\boxtimes$ which can be used to construct the local-application of S to all bipartite processes.

The keen eyed might notice that whilst the isomorphisms $\mathbf{C}(C, [A, B]) \cong \mathbf{C}(A \otimes C, B)$ of closed monoidal categories are sufficient to give the isomorphisms of type $\mathbf{C}(I, [A, B]) \cong \mathbf{C}(A, B)$ needed to support iterated higher-order transformations, it is not clear that they are necessary. In this thesis we will show that under enough reasonable conditions on theories of supermaps, the existence of general currying isomorphisms is actually derivable from the latter restricted and more obviously well-motivated kind.

The prototypical example of a closed monoidal category is again the category of functions between sets.

Example 15. *The category of sets is closed monoidal with* $[A, B] := \mathbf{Set}(A, B)$*.*

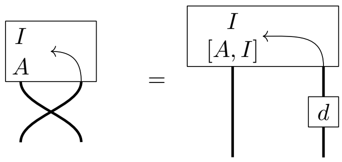
Many of the categories we have introduced are closed as a consequence of being compact closed.

Example 16. *Any compact closed category is a closed monoidal category with hom objects given by* $[A, B] := A^* \otimes B$ *and with evaluation morphisms* $e_{A,B} : A \otimes A^* \otimes B \to B$ *given by* $\cap_A \otimes i_B$*.*

Whilst any precausal category $\mathbf{C}$ is compact closed, the category $\mathbf{Caus}[\mathbf{C}]$ is not compact closed, however, it is indeed closed.

Example 17. *The category* $\mathbf{Caus}[\mathbf{C}]$ *is closed monoidal with* $[c_A, c_B] := c_A \Rightarrow c_B$*.*

The existence of the second tensor product of $\mathbf{Caus}[\mathbf{C}]$ can in-fact be deduced from observing a little extra structure on top of closed monoidal structure. The category resulting from the $\mathbf{Caus}[\mathbf{C}]$ construction is not only closed monoidal but in fact satisfies a stronger property of *-autonomy [235]. A *-autonomous category is a closed monoidal category such that the canonical map $d_A : A \to [[A, I], I]$ uniquely defined by



is an isomorphism for every object A.

When *-autonomy is present it gives a second tensor product for free, built from $\otimes$ and $[-, =]$ by $A \boxtimes B := (A^* \otimes B^*)^*$ where for any object A then $A^* := [A, I]$. Interpreting A^* as the dual of A one may consider *-autonomous categories to be those in which every object is equivalent to its double dual. Note that we chose to represent the tensor by $\boxtimes$, indeed this is how $\boxtimes$ as previously introduced is constructed in [126]. The linearly distributive, and so polycategorical, structure of $\mathbf{Caus}[\mathbf{C}]$ also follows from this *-autonomy property, suggesting to us that the suitability of polycategories for supermaps may still hold at even higher-orders.

Compact closed categories give a trivial example of *-autonomous categories.

Example 18. *Every compact closed category is a* **-autonomous category.*

A less trivial example, is the one of most interest to us, **Caus**[**C**].

Example 19. *The category* **Caus**[**C**] *is a* **-autonomous category. Indeed the fact that for each object* $d_A : A \cong [[A, I], I]$ *follows from closure of the sets which define objects of* **Caus**[**C**].

The **Caus**[**C**] construction and its OPT-based analogue "higher-order quantum theory" (HOQT) [127] have both recently been adapted to include a greater variety of type constructors [128, 132, 135], leading to the formulation of higher-order quantum transformations and their composition rules as models of various adaptations of linear logic such as BV-logic [126, 132, 135]. A key future goal outside of the scope of this thesis is to develop analogous models of higher-order quantum theories over any symmetric monoidal category.

2.9 Summary

We have reviewed the construction of supermaps and some of their compositional properties using the language of category theory where possible. The current state-of-the-art constructions of supermaps however requires some very specific features of finite-dimensional quantum theory such as compact closure [126]. In this thesis we view this as a problem, the concept of supermap as the kind of transformation that can be applied to transformations seems like one which does not require the notion of a compact closed category to be understood.

Indeed, the goal of this thesis is to develop a construction of supermaps on top of *any* symmetric monoidal category. In doing so we will have generalised supermaps to infinite-dimensional quantum theory and to arbitrary operational probabilistic theories [137]. The key tests of suitability of our constructions will be recovery of the standard physicists definitions of supermaps when applied to the quantum channels and the unitaries. We will also use some compositional features of theories of supermaps identified here and throughout the thesis as benchmarks for good constructions.

Chapter 3

Sequential and Parallel Composition Supermaps

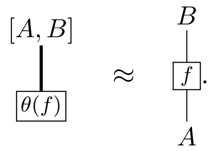
In this chapter we begin to ask the question, *what kinds of features should we expect from a theory of supermaps*? By identifying key features of features of supermaps we could hope to in the future:

- Contribute to the generalisation of resource theories to theories of supermaps, by establishing features which should be present in sub-theories [10, 44–46, 48, 219].
- Define structure-preserving maps between theories of higher order processes, and so develop a suitable mathematical language for their comparison, including possibly characterising specific theories by universal properties as has been done for fragments of standard quantum theory in [198].
- Connect the study of supermaps to other areas of computer science and applied category theory.
- Use these features as a sanity check for proposed definitions or constructions of supermaps.

The first key feature which we extract from the motivating examples of theories of supermaps, is that there are supermaps which plug processes together, in sequence or in parallel. It turns out that this requirement can be modelled in categorical language using the definition of an *enriched* symmetric monoidal category. We will give a brief definition here of an enriched symmetric monoidal category in terms of string diagrams, assuming as we do throughout that our underlying category **C** is strict monoidal to make our lives easier. Any reader interested in the concise categorical definition of enriched monoidal categories is referred to [191] or the appendix of [2]. Each of those works with monoidal categories enriched in symmetric monoidal categories, for enrichment in symmetric multicategories the reader is referred to [236][1][2]. To explain how it is that the formal string diagrams represent our informal intuitions about theories of supermaps, we will after each main axiom write formal diagrams to the left and their information interpretations on the right.

Definition 7. *A* **P***-smc* **C** *is a strict symmetric monoidal category* **C** *and a symmetric multicategory* **P** *which has*

- *For each A, B of* **C** *an object $[A, B]$ of* **P** *and a bijection $\theta : \mathbf{C}(A, B) \cong \mathbf{P}(I, [A, B])$:*

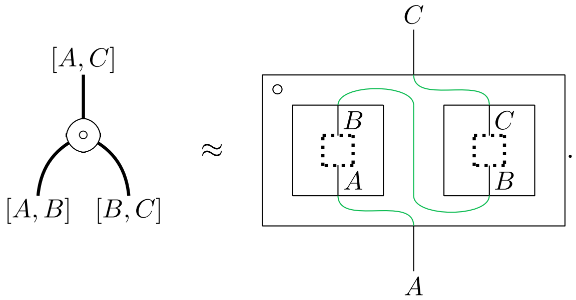


(3.1)

- *For each $[A, B]$ and $[B, C]$ a morphism in* **P** *which allows to plug their underlying processes together:*

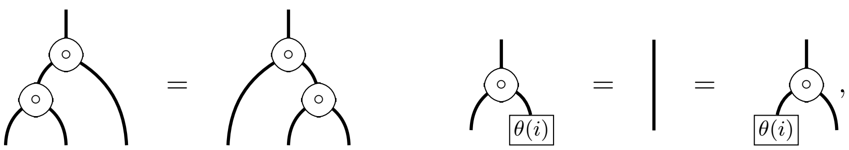
$$\bigcirc : [A, B][B, C] \to [A, C]$$

represented formally on the left as a string diagram in **P** *and informally on the right to show intuitively its action on the underlying category* **C***:*



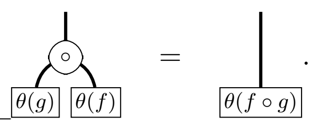
(3.2)

Associativity and unitality of the sequential composition process in **P** *is represented by:*



(3.3)

and the requirement that it actually implements sequential composition for **C***, is enforced by:*



(3.4)

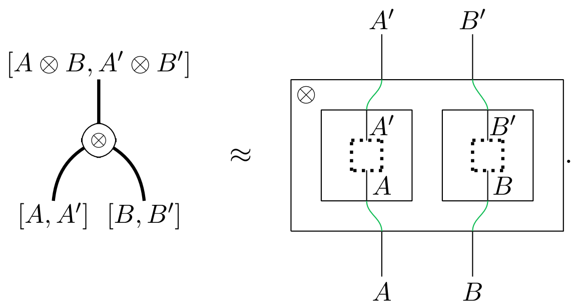
[1]Concretely we write down strict symmetric monoidal categories enriched in symmetric multicategories, which at the level of string diagrams can be seen as what remains when the monoidal structure of **P** is forgotten for any commutative monoid in the symmetric monoidal category **PCat** of **P**-categories [191].

[2]For a **P**-symmetric monoidal category without strictness, the string diagrammatic definition for **P** a symmetric multicategory can be constructed by finding the analogous string diagrammatic representation for any symmetric pseudomonoid [237] in the symmetric monoidal 2-category **PCat** [191].

- *For each A, A', B, B' a parallel composition process in $\mathbf{P}$ of type*

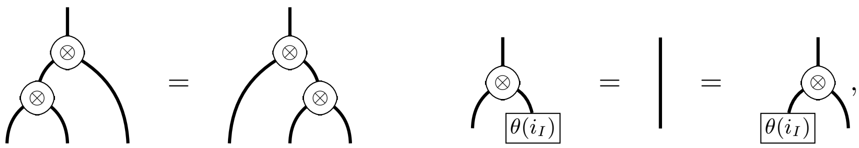
$$\otimes_{ABA'B'} : [A, A'][B, B'] \to [A \otimes B, A' \otimes B']$$

which can be represented formally and intuitively respectively by:

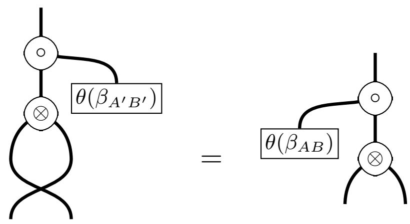


(3.5)

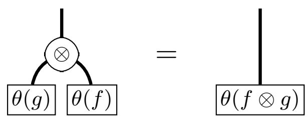
The following conditions are required, guaranteeing that $\otimes_{ABA'B'}$ really does behave like a parallel composition process:



(3.6)



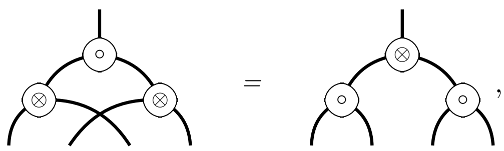
(3.7)



(3.8)

These equations represent, associativity of parallel composition, parallel composition with empty space having no effect, compatibility with symmetric structure, and finally that the morphism indeed implements the parallel composition of processes, respectively.

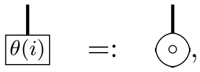
Lastly the condition:



(3.9)

is required, which represents the interchange law between sequential and parallel composition.

We will from now on adopt the notational convention of dropping θ whenever its presence is clear from context[3]. We will also adopt the notation



to emphasise the special status of $\theta(i)$ as the state which witnesses unitality.

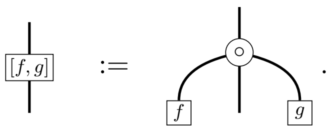
As seen by the above description, a **P**-smc **C** consists of all the data needed to say that **P** has morphisms which perform sequential and parallel composition of morphisms in **C**. We will assume that the existence of sequential and parallel composition supermaps should be seen as a structural feature of a theory **P** of supermaps. This conceptual assumption can now be encoded as a mathematical one:

Supermaps enrich the category they act on.

Such an interpretation of enrichment, has indeed been used before in [238], for the purposes of developing quantum programming languages. For the reader interested in understanding the core story of the thesis, this point is the only one of this chapter which will be referenced in future chapters. The remainder of this chapter is concerned with examining how much one can infer from the assumption of enrichment, in particular from combining it with other principles to recover key aspects of higher order quantum theory.

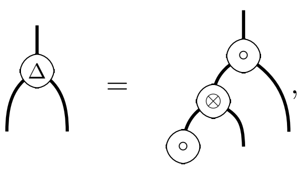
3.0.1 Review of the Properties of Monoidal Enriched Categories

In this section we outline some features of enriched monoidal categories which we will regularly reference in the remainder of this chapter. The first is the existence of *partial insertion* maps and the second is the existence of natural *usage* functions. We will also occasionally refer to the functor $[-,=] : \mathbf{C}^{op} \times \mathbf{C} \rightarrow \mathbf{P}$ defined by:



3.0.2 Partial Insertion

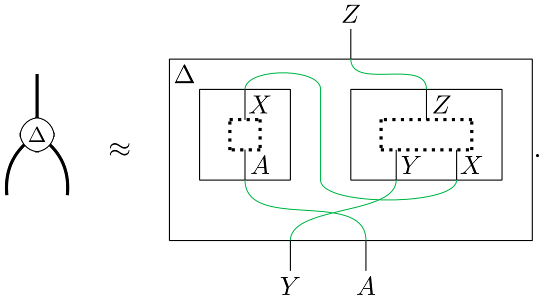
The partial insertion morphism $\Delta : [A, X][Y \otimes X, Z] \rightarrow [Y \otimes A, Z]$ takes a valid sub-input of a process and inserts a pre-processing there, leaving the rest of the inputs unchanged. Formally it is defined by:



(3.10)

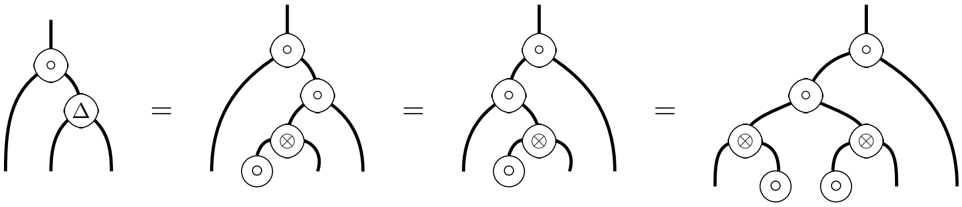
[3]It is not so standard in enriched category theory to even work with an isomorphism, this is likely a result of the fact that enriched categories are usually used to define standard categories equipped with structure. When thinking about supermaps it is more natural to think of **P** and **C** as seperate theories which are related by enrichment of one into the other.

up to unitors and associators, where $\otimes : [Y,Y][A,X] \to [YA,YX]$ and $\circ : [YA,YX][YX,Z] \to [YA,Z]$. The partial insertion can be intuitively be understood as representing the following picture:

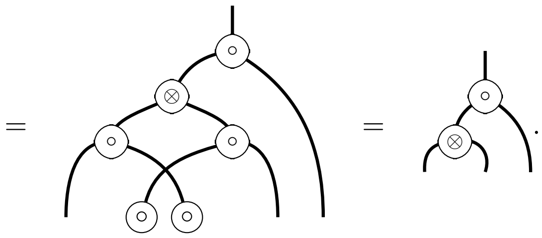


(3.11)

For $A = I$, then Δ satisfies:

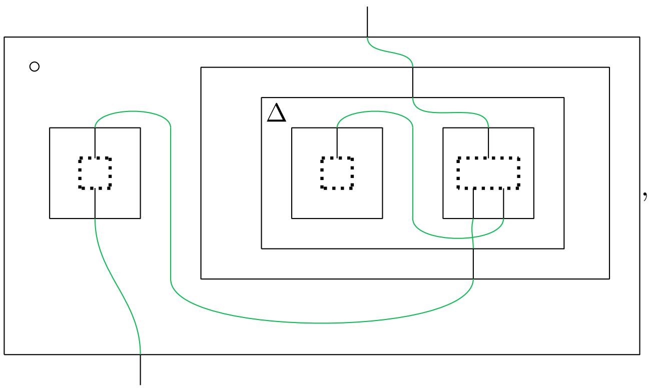


(3.12)



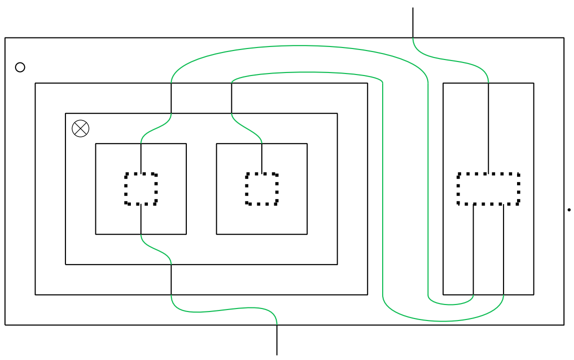
(3.13)

Intuitively the above represents the equality between



(3.14)

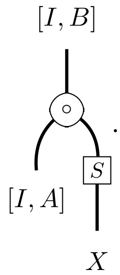
and



(3.15)

3.0.3 Usage Transformations

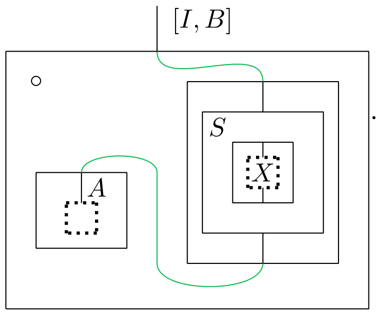
The usage transformation is a particular natural transformation $\theta : \mathbf{P}(-,[A,-]) \implies \mathbf{P}([I,A]-,[I,-])$, a family of functions θ_{BX} given by for each $S : X \to [A,B]$ taking $\theta_{BX}(S)$ to be:



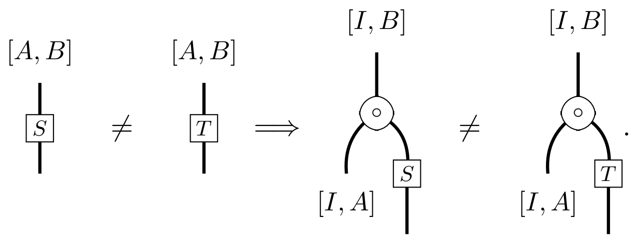
(3.16)

Intuitively, θ places S into one of the two holes of a sequential composition supermap:

$\theta(S) \quad =$



Naturally, one might expect that when two distinct processes are used, then they remain distinct; mathematically this is the requirement of injectivity of the function θ. We will use the word faithful to describe this property when present, which says that two higher order processes $S, T : X \to [A,B]$ should *only* be distinguishable if they are distinguishable when their outputs are applied to the space of states on A. Concretely, we say that a **P**-smc **C** is faithful, if for all I, A, B the composition process $\bigcirc_{IAB}$ satisfies[4]



(3.17)

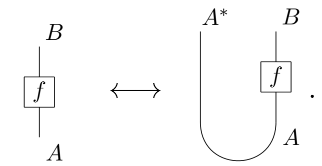
[4]This requirement can be encoded in categorical language, A **P**-smc **C** is faithful if and only if the usage transformation

$$\theta : \mathbf{P}(-,[A,-]) \implies \mathbf{P}([I,A]-,[I,-])$$

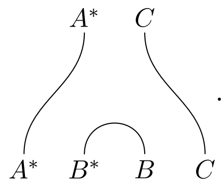
is a monomorphism in the functor category $\mathbf{Cat}(\mathbf{P}^{op} \times \mathbf{C}, \mathbf{Set})$.

3.0.4 Examples

Theories of **D**-supermaps on **C** always define a faithful enriched monoidal category. Indeed, isomorphism between morphisms of **C** and states of **Dsup**[**C**] is given by:

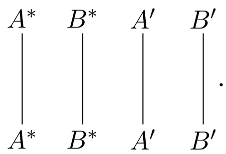


Then, for each A, B, C one can define the sequential composition supermap by:



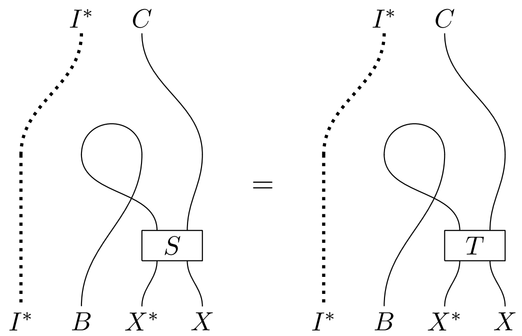
(3.18)

and for each A, B, A', B' one can define a parallel composition supermap by:



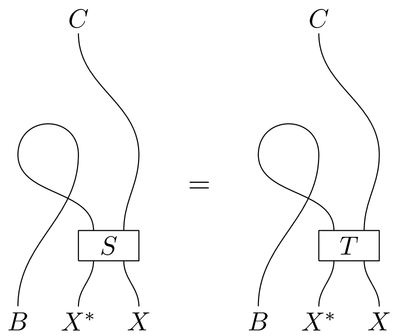
(3.19)

The various laws required, such as the interchange law, just follow from uses of the interchange law in **D** and compact closure of **D**. faithfulness is immediate since the defining premise gives



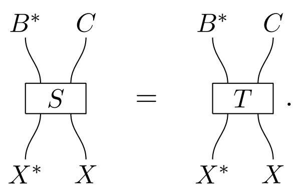
(3.20)

which implies



(3.21)

and so applying cups on the left-hand side of both diagrams gives



(3.22)

While the **D**-supermaps on **C** faithfully enrich **C**, a bare-minimum requirement for the **D**-supermaps on **C** to be a legitimate theory of supermaps, the **Caus** gives a more elaborate example of faithful enrichment. Precisely, one can easily construct a faithful **Caus**[**C**]-smc **Caus**[**C**], a signature of the fact that **Caus**[**C**] is self-contained, meaning that it has all of its own supermaps. The faithful enrichment structure of **Caus**[**C**] in fact follows from its closed monoidal structure as outlined in theorem 4 of the following section, where we will find that under some additional reasonable background assumptions the idea of self-containment is equivalent to the existence of currying and so of closed monoidal structure.

3.1 Currying from Self-Containment

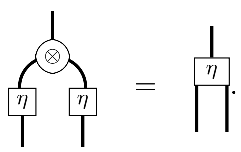
So far we have considered features of one theory **P** which is a theory of supermaps on another theory **C**. In this section we consider the features of the more elaborate construction of higher-order quantum theories (HOQTs) [126–128, 132, 135], where maps, supermaps, and super-supermaps etc all live in the same theory. A key feature of higher-order quantum theory was the property of currying; in this section we will see that currying in HOQTs can be seen as a simple consequence of combining the assumption of enrichment with a few extra axioms:

- All processes in **C** have higher order representations *in* **C** (Self-enrichment).
- There is an equivalence $A \cong [I, A]$ between A and the higher order system $[I, A]$ representing the states of A (linking).
- The usage transformation is faithful.

Conceptually, the first condition models the idea of maps, supermaps, and super-supermaps etc all living in the same theory. We model this with the notion of a **C**-smc **C**. Note a subtlety here, we assume that the symmetric multicategory structure of **C** doing the enriching is that which would be freely constructed from the symmetric monoidal structure being enriched. The second condition is captured by the following:

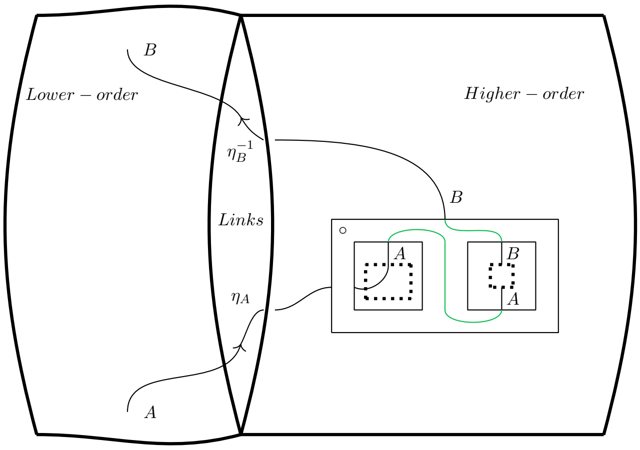
Definition 8. *A* Linked Monoidal Category *is a* **C**-*smc* **C** *equipped with a monoidal natural isomorphism* $\eta_A : A \to [I, A]$[5].

Here monoidal naturality imposes the following for η:



[5] The functor $[i, =] : \mathbf{C} \to \mathbf{P}$ constructed by fixing the left-input of $[-, =]$ as i is monoidal when **P** is monoidal, with natural isomorphism given by the parallel composition supermap [2]. For the general definition of monoidal functor and monoidal natural transformation see [21].

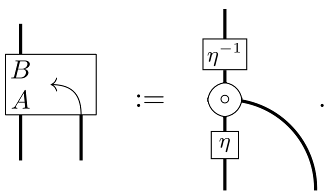
We furthermore say that a linked monoidal category **C** is faithful if it is faithful as a **C**-smc **C**. Intuitively, linked categories have enough structure to define canonical evaluation morphisms (the structural feature of closed symmetric monoidal categories), mixed-order morphisms of type $\epsilon_{AB} : A \otimes [A,B] \to B$ which apply processes to lower order objects, by using link-morphisms and sequential composition morphisms:



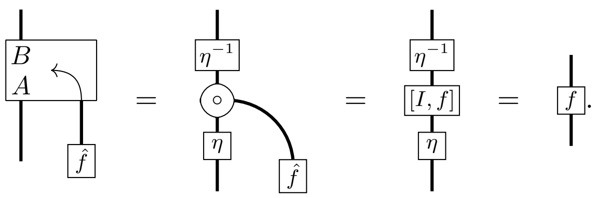
In the above diagram, the available inputs are the bottom wire A and the dotted process input of type $[A,B]$, the output wire is the top wire of type B.

Lemma 2. *Every linked faithful category* **C** *is a closed symmetric monoidal category.*

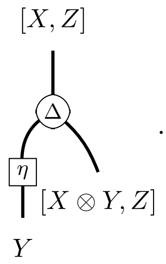
Proof. We begin by showing that the above two bullet points give a closed symmetric monoidal structure. Let the three bullet points be true for **C**; then to each pair $A, B \in \mathbf{C}$ assign the candidate for evaluation

 (3.23)

Since every $\circ$ is completely injective by assumption, so is every ϵ. Since $\eta : A \to [I,A]$ is a natural isomorphism, for any $f \in \mathbf{C}(A,B)$ there exists a morphism $\hat{f}$ such that

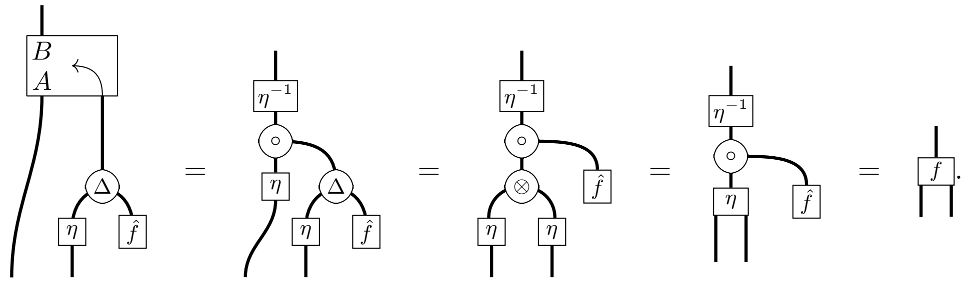
 (3.24)

One can apply the isomorphism η to the partial insertion operation to generate a partial insertion using a lower level type Y as opposed to the higher level type $[I, Y]$



$$. \tag{3.25}$$

This partial insertion operation can be used to construct the curried version of any process f from its static version $\hat{f}$, since



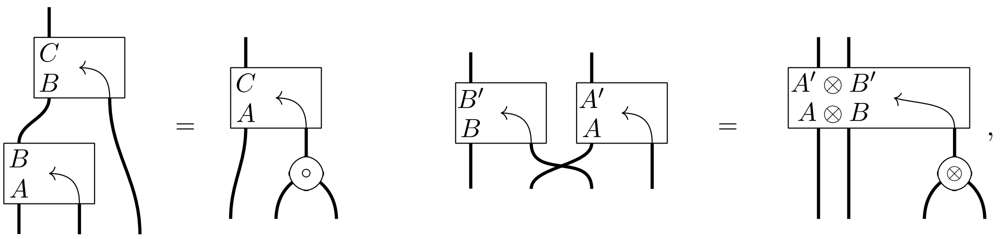
(3.26)

It follows that for every process f its curried version exists, this completes the proof. □

A more well-known property of closed symmetric monoidal categories, is that they are always self-enriched. They are also linked and faithful, meaning that linked faithful monoidal categories are exactly closed symmetric monoidal categories. A proof of this converse direction minus linking, is a standard result of enriched category theory [239].

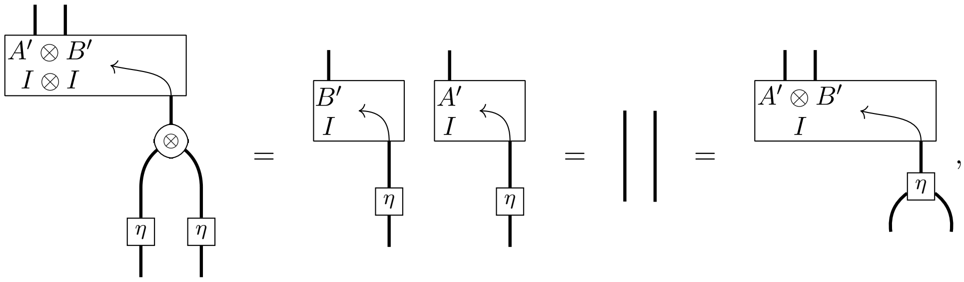
Theorem 4. *There is a one-to-one correspondence between linked faithful monoidal categories and closed symmetric monoidal categories.*

Proof. We give the sketch here for completeness. Let $\mathbf{C}$ be a closed symmetric monoidal category, then there exist sequential and parallel composition morphisms defined as adjuncts to circuits of evaluation morphisms. Concretely the definition of closed symmetric monoidal category enforces that there must exist processes $\otimes$ and $\circ$ satisfying



(3.27)

which satisfy the coherence conditions for a symmetric monoidal category. The uniqueness property for closed symmetric monoidal categories lifts to faithful usage for each sequential composition maps. Finally, a monoidal natural isomorphism $A \cong [I, A]$ for the induced functor $[I, -]$ must be constructed. Indeed, η of $\epsilon_{[I,A]}$ is an isomorphism and a natural transformation of the right type in any closed symmetric monoidal category. Taking η as the inverse, it is immediately also natural, and can be checked to be monoidal using the following steps:



(3.28)

followed by using the uniqueness condition for satisfaction of the defining currying equation for closed symmetric monoidal categories. □

3.2 Causality in Higher-Order Quantum Theory

We have seen that from minimal assumptions one should expect closed symmetric monoidal structure from any construction analogous to higher-order quantum theory of a full higher-order theory over a symmetric monoidal category **C**. In this section, following [1], we see which other properties of categories of the form **Caus**[**C**] can be reconstructed by adding a few additional physics-inspired principles. We will generally refer to categories of this form as higher-order causal categories (HOCCs).

3.2.1 Causality and determinism

To formulate causality in a closed symmetric monoidal category, it is convenient to first define the notion of determinism. We model deterministic theories as those with only a single identity scalar. This generalises affine monoidal categories which have a unique scalar and further a unique effect for each object.

Definition 9 (Deterministic process theory)**.** *A symmetric monoidal category $\mathcal{C}$ is* deterministic *if it contains only one scalar, that is, if $|\mathcal{C}(I, I)| = 1$. The unique scalar in a deterministic theory is denoted by* 1*.*

It is easy enough to check that all HOCCs are deterministic. We now specify the objects of a symmetric monoidal category which are causal.

Definition 10 (Causal object)**.** *An object A is* causal *if it has only one effect, that is, if $|\mathcal{C}(A, I)| = 1$. A symmetric monoidal category $\mathcal{C}$ is* causal *if all the objects $A \in o(\mathcal{C})$ are causal.*

Note that every causal category is automatically deterministic. We conclude the section by showing that, if $\mathcal{C}$ is deterministic, a simple sufficient condition for an object to be causal is that it has "enough states," in the following sense:

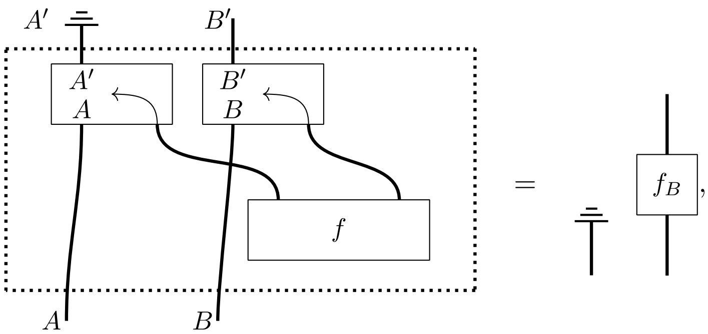
Definition 11 (Enough states)**.** *An object A has enough states if for every object X and for every pair of processes $f, g : A \to X$,*

$$f = g \iff \forall \rho \in \mathcal{C}(I, A) : \; f \circ \rho = g \circ \rho. \tag{3.29}$$

In any deterministic closed symmetric monoidal category if an object A has enough states then it must be causal, i.e. there can be only one effect $A \to I$. Any two effects $e_1, e_2 \in \mathcal{C}(A, I)$ satisfy the condition $e_1 \circ \rho = 1 = e_2 \circ \rho$ for every state $\rho \in \mathcal{C}(I, A)$, and therefore the "enough states" condition implies $e_1 = e_2$.

3.2.2 The no-signalling tensor product

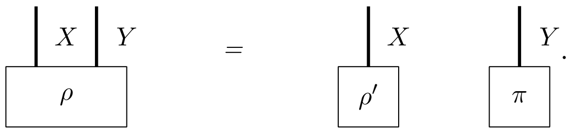
An important insight of [126] is that the tensor product in a higher order causal category does not allow for signalling between tensor factors of process types between causal objects. More specifically, in [126] it was shown that for any first-order objects A, B, A', B' of a HOCC the type $[A, A'] \otimes [B, B']$ represents the space of non-signalling channels, for which the output A' has no dependence on the input B and the output B' has no dependence on the input A. This notion can be expressed in the language of closed symmetric monoidal categories whenever each of A, B, A', B' is causal: a state $f : I \to [A, A'] \otimes [B, B']$ represents a non-signalling channel if there exists a morphism $f_B : B \to B'$ satisfying:



(3.30)

and similarly for discarding B'. In this section we ask whether this property can be derived from additional operational principles on closed symmetric monoidal categories. Indeed, we will find that a simple principle on the kind of correlations which can be formed is sufficient. Intuitively, if a joint state of objects X and Y is interpreted as representing correlations between the states of X and Y, it should not be possible to correlate any auxiliary object X with a single-state object Y. This intuition motivates the following definition:

Definition 12 (No correlation with a single-state object)**.** *A symmetric monoidal category theory $\mathcal{C}$ has* no correlations with single-state objects *if, for any object Y with $|\mathcal{C}(I,Y)| = 1$ and any object $X \in o(\mathcal{C})$, every state $\rho : I \to X \otimes Y$ is of the product form $\rho = \rho' \otimes \pi$ with $\rho' \in \mathcal{C}(I,X)$ and $\pi \in \mathcal{C}(I,Y)$:*



(3.31)

The above condition is satisfied by all HOCCs as defined in [126]:

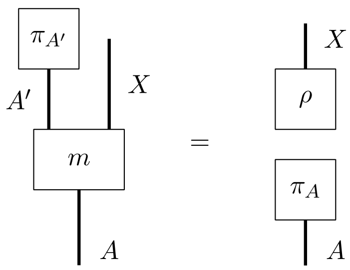
Theorem 5. *Every HOCC has no correlations with single-state objects.*

Proof. A minor generalisation of lemma 6.1 of [126], given for completeness in Appendix F. □

The condition of "no correlation with single-state objects" was crucial to proving that $[A,A'] \otimes [B,B']$ represents a non-signalling channel in [126]. In that context, the statement followed from a specific decomposition of supermaps, as open circuits of causal processes. Here, instead, we take the "no correlation with single-state objects" as a basic operational condition.

We now show that, if there is no correlation with single-state objects, then the tensor product has a no-signalling property. For a given process, we define a generalisation of no-influence conditions, which allows for the existence of many discarding effects:

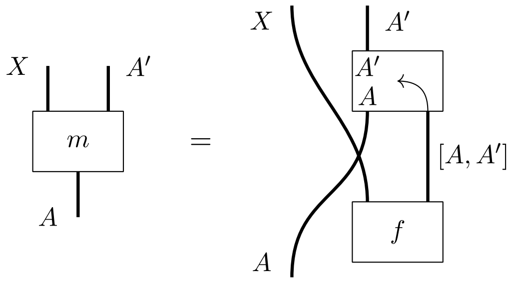
Definition 13 (Non-signalling process)**.** *A process $m : A \to A' \otimes X$ in a deterministic symmetric monoidal category is* non-signalling *from A to X if for every effect $\pi_{A'} : A' \to I$ there exists an effect $\pi_A : A \to I$ and a state $\rho : I \to X$ such that*



(3.32)

The definition expresses the idea that when A' is discarded (in any way) no signal may reach X from A. Note that, in principle, the definition still allows for a notion of signalling from A' to X, because in general the state f' of X could depend on the effect $\pi_{A'}$ used for discarding. Note, however, that signalling from A' to X is not possible if system A' is causal, because in that case the effect $\pi_{A'}$ is unique. In the following, we will restrict our attention to the case where both systems A' and A are causal. We allow for many effects because HOCCs are not causal: for instance on the object $[A,B]$ there are many effects, since there are many supermaps of type $[A, B] \to [I, I]$ given by pre-composition with a state $\rho : I \to A$ and post-composition with an effect $\sigma : B \to I$.

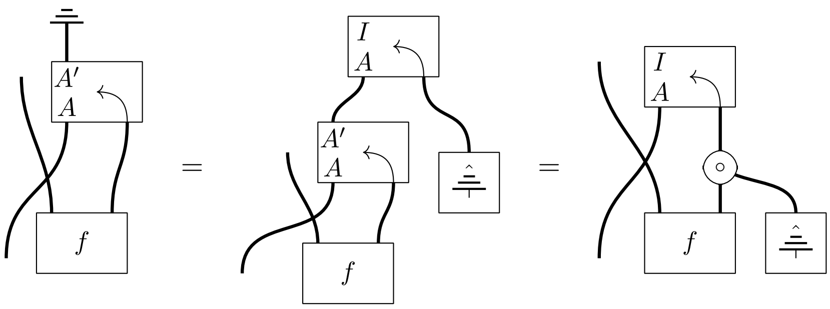
Theorem 6 (Non-signalling processes)**.** *Let $\mathcal{C}$ be a deterministic closed symmetric monoidal category with no correlations with single-state objects, A, A' be two causal objects in $\mathcal{C}$, and $X \in o(\mathcal{C})$ be an arbitrary object. Then, for every state $f : I \to X \otimes [A, A']$ the process m defined by:*



(3.33)

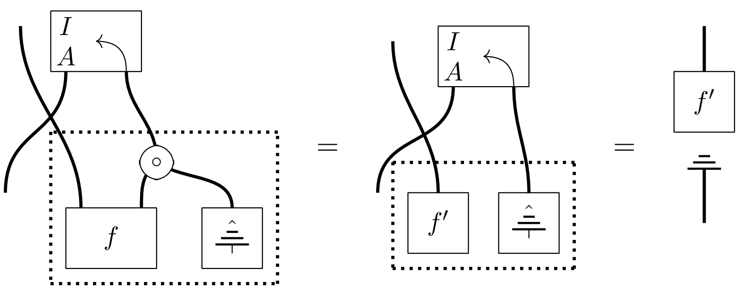
is non-signalling from A to X.

Proof. As in [126], the core of the proof is the "no correlation with single-state objects" property. In the proof, this property is applied to the object $[A, I]$, which is a single-state object because $\mathbf{C}(I, [A, I]) \cong \mathbf{C}(I \otimes A, I) \cong \mathbf{C}(A, I) \implies |\mathbf{C}(I, [A, I])| = 1$. The discarding effect can as a result be pulled through the entire process



(3.34)

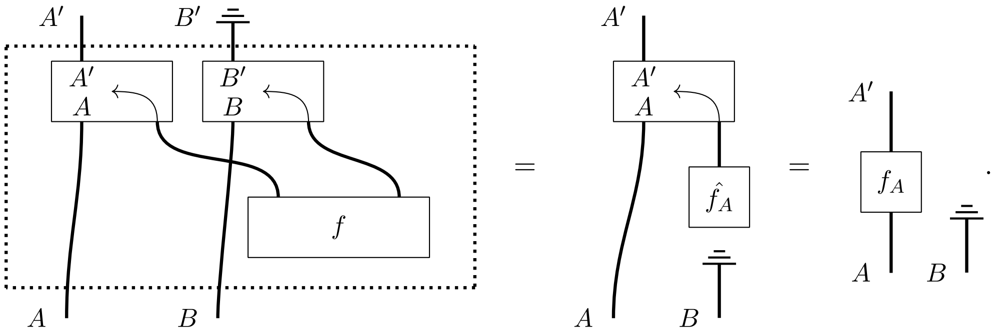
The composition of $\hat{f}$ with the unique discarding effect on A' at the bottom of the diagram gives a state of type $[A, I] \otimes X$, and so "no-correlation with single-state objects" implies that such a state separates as the unique discarding state on $[A, I]$ and a state f' on X:



(3.35)

□

The above immediately entails the fact that states of type $f : I \to [A, A'] \otimes [B, B']$ represent non-signalling channels (when A, A', B, B' are causal) in the sense of [126], since for such a state f,



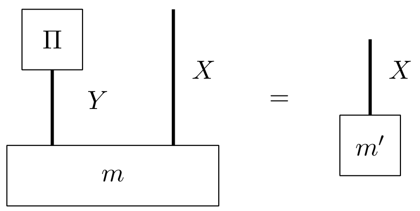
(3.36)

The broad takeaway is that it is the causality of an object A that prevents it from signalling to another object that it is in parallel with.

3.2.3 A Stronger No-Signalling Property from Double Duals

We conclude by showing that a strengthening of this no-signalling property is satisfied by HOCCs. Previously we saw that in a deterministic theory with no correlations with single-state objects, the states of type $(A \Rightarrow A') \otimes X$ represent processes which are non-signalling from A to X whenever A and A' are causal objects. We now show that, in the presence of an additional notion of equivalence to double duals, this no-signalling property can be strengthened: the tensor product $(A \Rightarrow A') \otimes X$ is no-signalling from the whole system $(A \Rightarrow A')$ to X.

Definition 14. *An object Y in a deterministic symmetric monoidal category $\mathcal{C}$ has* no-signalling states *if for every object X and every bipartite state $m : I \to Y \otimes X$ there exists a state $m' : I \to X$ such that for every $\Pi : Y \to I$*



(3.37)

In other words an object Y has no-signalling states if the choice of effect for discarding object Y in a bipartite object $X \otimes Y$ does not affect the marginal state of system X. To reconstruct this feature of HOQT and general HOCCs we will need to lean on another feature, that they are not just closed symmetric monoidal, but also $*$-autonomous [235]. In short, they are equipped with isomorphisms of the form $[[A, I], I] \cong A$ for every object A.

Definition 15 (Equivalence of double duals)**.** *An object A in a closed symmetric monoidal category $\mathcal{C}$ is* canonically equivalent to its double dual *if $d_A : A \to [[A, I], I]$ is an isomorphism.*

Such an isomorphism forces states on A to be nothing other than the effects on effects for A, as is the case in finite dimensional quantum systems. Given two systems A and B that are canonically equivalent to their double duals, it is natural to ask whether equivalence is preserved by the binary operations $(- \otimes =)$ and $[-,=]$, in the following sense:

Definition 16 (Preservation of equivalence of double duals)**.** *A binary operation* $\odot : o(\mathcal{C}) \times o(\mathcal{C}) \to o(\mathcal{C})$ preserves equivalence of double duals *if $d_{A \odot B}$ is an isomorphism whenever d_A and d_B are isomorphisms.*

We can show that the preservation of equivalence by the tensor product $(- \otimes =)$ is enough to guarantee preservation of the equivalence by the higher order composition $[-,=]$.

Theorem 7 (Lifting canonical isomorphisms)**.** *For every closed symmetric monoidal category $\mathcal{C}$, if $(- \otimes -)$ preserves equivalence of double duals then $[-,=]$ preserves equivalence of double duals.*

Proof. Given in the appendix. □

Double duals, when present, for particular objects, entail stronger no-signalling properties for those objects.

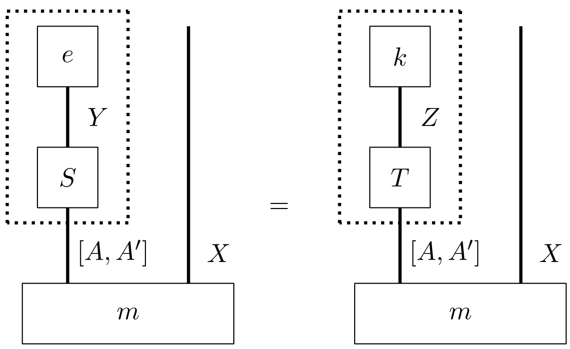
Theorem 8. *Let $\mathcal{C}$ be a deterministic closed symmetric monoidal category with no correlations with single-state objects. If*

- *$\otimes$ preserves equivalence with double duals, and*
- *A and A' are causal and canonically equivalent to their double duals,*

then the object $(A \Rightarrow A')$ has no-signalling states.

Proof. Given in the appendix. □

The theorem shows that, no matter which supermap is applied on the system $A \Rightarrow A'$, and no matter the way a system is discarded, the state of any other system in parallel will be unaffected. Indeed, for every pair of processes $S : [A, A'] \to Y$ and $T : [A, A'] \to Z$, and every pair of effects $e : Y \to I$ and $k : Z \to I$, one has



(3.38)

In other words, the choice of a supermap on system $[A, A']$ cannot signal to any other system X. This can be seen as a generalised causality condition for monoidal categories of supermaps.

3.3 Summary

In this section we began to consider the kinds of features that we should expect from theories of supermaps, and from fully iterated theories of supermaps analogous to higher-order quantum theories and higher-order causal categories. We motivated enriched monoidal categories and closed symmetric monoidal categories as models for each of these kinds of theories, from simple reasonable principles. We then found that some causality features of higher-order causal categories can be deduced by imposing some simple additional axioms onto closed symmetric monoidal categories. In the remainder of this thesis we will turn our focus back to simple theories of supermaps without the iteration of higher-order quantum theory. We will try to construct theories of supermaps which recover the standard physicist's definitions and which further have the expected feature of enrichment when applied to any symmetric monoidal category. Construction of theories of iterated supermaps over any symmetric monoidal category with the more elaborate features of self-enrichment and closed symmetric monoidal structure are beyond the scope of the thesis and suggested in the conclusion as a promising avenue for future research.

Chapter 4

Minimal Behaviour Law for Supermaps: Local Applicability

So far we have developed to key intuitions about the concept of a supermap. We think that theories of supermaps should enrich the structure of the category they act on, and we think that the theory of supermaps on the quantum channels should be the the theory of quantum superchannels. In this chapter, we develop a concrete construction **Lot**[**C**], of a theory of supermaps over any monoidal category, which is satisfactory with respect to these key intuitions for theories of supermaps.

- When applied to any symmetric monoidal category produces a theory which enriches the symmetric monoidal structure of that category
- When applied to the symmetric monoidal category of quantum channels, returns the quantum superchannels

The definition we use is concise, easy to interpret, and we will argue it is clearly a bare-minimum requirement for any theory of supermaps. It is surprising in some way that such an approach is possible, since it means that we will be able to show that the linearity of quantum supermaps is in fact a consequence of purely compositional principles. It is worth stressing that this essentially reverses the story of the definition of supermaps. The usual approach is to take what we call the complete-preservation route as outlined in the introductory material [12, 14, 127]:

- Assume linearity of functions $S : \mathbf{QC}(A, A') \to \mathbf{QC}(B, B')$.
- Use the (compact closure of) the tensor product of linear maps to define the local-application $\widehat{S \otimes i_{\mathbf{QC}(X,X')}}$ of S
- Require complete-preservation of desired input transformations $\widehat{S \otimes i_{T(X,X')}} : \mathbf{QC}(A \otimes X, A' \otimes X') \to \mathbf{QC}(B \otimes X, B' \otimes X')$ by this constructed local-application of S

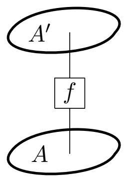
As we have seen, essentially what is being used here is the compact closed structure of the category of linear maps. For general OPTs or Hilbert spaces, without compact closure, such an approach is not available to us. Instead as promised we will reverse the order of proceedings:

- First define a theory-independent notion of local-applicability
- Then show that representation of locally-applicable functions by a linear map S, satisfying complete-preservation, is always possible

As we outlined in the introductory material, standard quantum channels are currently axiomatised in terms of assumption of linearity and complete-preservation [220], so the results of this section suggest that in the future the same kind of script flipping could be applied to the axiomatisation of standard quantum processes.

4.1 Locally-Applicable Transformations

In this section we find some bare minimum behaviour laws for quantum supermaps which can be stated with respect to *any* symmetric monoidal category. Before defining locally-applicable transformations on morphisms of a monoidal category, let us warm up by examining an analogy. The analogous idea we consider, is that morphisms on a monodial category always define locally-applicable transformations on states of that monoidal category. Let us begin by considering any morphism $A \to A'$ in a monoidal category



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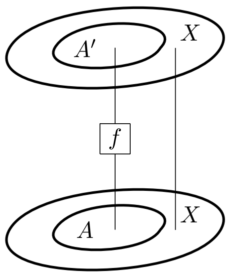
now imagine that there is some auxiliary system X present, formally using the monodial product the morphism $f : A \to A'$ can be extended to a morphism $f \otimes i : A \otimes X \to A' \otimes X'$



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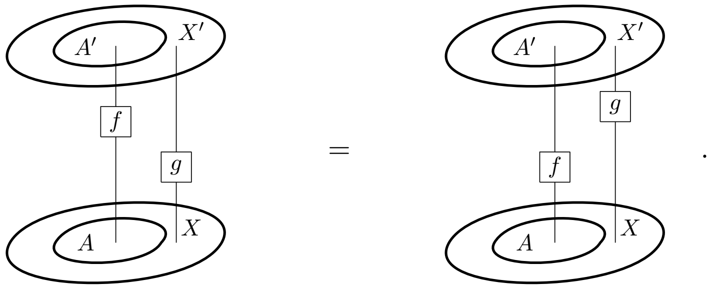
Crucial to the interpretation of locality in $f \otimes id_X$ is that $f \otimes id_X$ commutes with all actions on X, this follows in this case by the interchange law for monoidal categories

$(f \otimes id_{X'}) \circ (id_A \otimes g) = (id_{A'} \otimes g) \circ (f \otimes id_X)$:



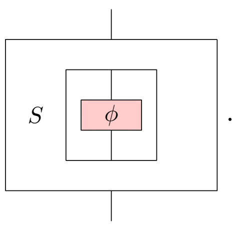
In other words, a general monoidal category gives a collection of morphisms, all of which can be viewed as being locally-applicable, in an informal sense.

A consequence of this local-applicability is the possibility to construct from f a family of functions on states $l(f)_X : \mathbf{C}(I, A \otimes X) \to \mathbf{C}(I, A' \otimes X)$ which exhibit the local-applicability of f. Explicitly, by using tensor extensions with the identity the function $l(f)_X(\rho_{AX}) := (f \otimes id_X)(\rho_{AX})$ can be defined for each X. The abstract functions $l(f)_X$ which represent the action of f on states indeed inherit a notion of local-applicability from f. The functions $l(f)_X$ can be seen to leave the environment system X untouched in the sense that the action of any g on X commutes with the application of the function $l(f)_X$. The above sentence is captured in formal terms by the equation $l(f)_{X'}(id_A \otimes g(\rho_{AX})) = (id_{A'} \otimes g)(l_X(\rho_{AX}))$ which is guaranteed to hold for any $g : X \to X'$ since

$$\begin{aligned} l(f)_{X'}(id_A \otimes g(\rho_{AX})) =& (f \otimes id_{X'}) \circ (id_A \otimes g)(\rho_{AX}) \\ =& (id_{A'} \otimes g) \circ (f \otimes id_X)(\rho_{AX}) \\ =& (id_{A'} \otimes g)(l_X(\rho_{AX})). \end{aligned}$$

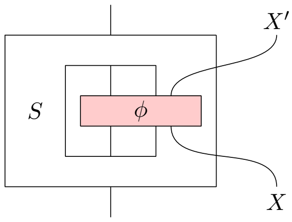
We now give an axiom which re-characterises quantum supermaps by generalising this concept of local-applicability of functions on states to local-applicability of functions on processes. The only instances of locally-applicable transformations on quantum channels will turn out to be those which are simulated by the standard-definition quantum superchannels of [240]. We split the motivations for the definition of locally-applicable transformation into three consecutive principles.

Principle 1: Supermaps are Functions on Processes The kind of picture usually drawn with the aim of capturing diagrammatically the concept of a supermap from the space of processes $\mathbf{C}(A, A')$ to the space of processes $\mathbf{C}(B, B')$ is some variation of the following



As such our first step to characterising supermaps of type $\mathbf{C}(A, A') \to \mathbf{C}(B, B')$ is to consider functions of the same type $\mathbf{C}(A, A') \to \mathbf{C}(B, B')$.

Principle 2: Supermaps Can be Extended to Functions on All Bipartite Processes When we say that we wish for the map $S : \mathbf{C}(A, A') \to \mathbf{C}(B, B')$ to be locally-applicable, we mean that we wish to formalise the following picture:



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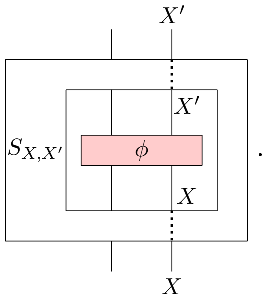
The next step toward such a formalisation is to specify for each X, X' the action of S when applied to the A, A' part of any morphism $\phi \in \mathbf{C}(A \otimes X, A' \otimes X')$. Consequently we say that a locally-applicable transformation must be equipped with a family of extended functions $S_{XX'} : \mathbf{C}(A \otimes X, A' \otimes X') \longrightarrow \mathbf{C}(B \otimes X, B' \otimes X')$ for every X, X'. We now will need to find a way to enforce these extensions behave *as if* they are applied locally. For readability we will from now on notate the action of such a family of functions in the following way

$S_{X,X'}(\phi) \quad :=$



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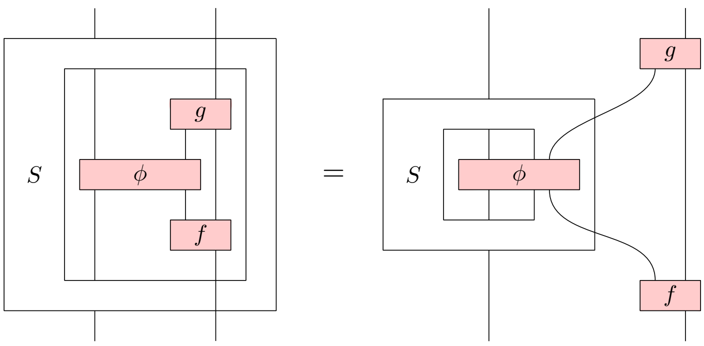
where the dotted lines express the idea that the wires they connect are to be interpreted as auxiliary systems.

Principle 3: Supermaps Commute With Actions on Their Extensions A key feature of a local operation is commutation with operations applied to auxiliary spaces, we generalise this notion of locality to input-output operations, informally we aim to capture the equivalence of the following two pictures:



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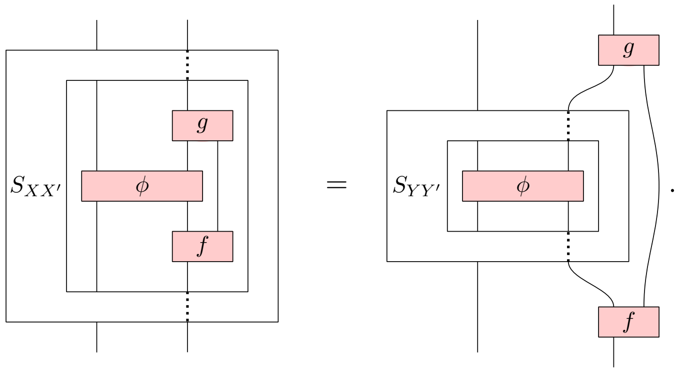
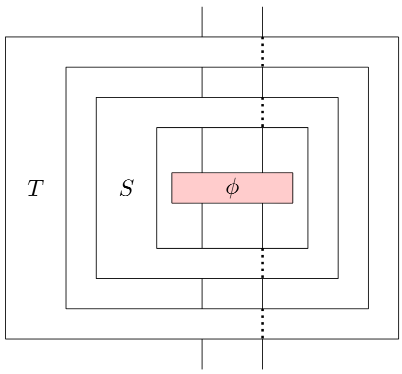
Definition 17 (locally-applicable transformations)**.** *A locally-applicable transformation of type* $S : [A, A'] \longrightarrow [B, B']$ *on a symmetric monoidal category* $\mathbf{C}$ *is a family of functions* $S_{XX'} : \mathbf{C}(A \otimes X, A' \otimes X') \longrightarrow \mathbf{C}(B \otimes X, B' \otimes X')$ *such that for every* $g : Y' \otimes Z \to X'$*,* $f : X \to Y \otimes Z$*, and* $\phi : A \otimes Y \to A' \otimes Y'$ *then*[1]



The same diagrammatic rules have been used for similar purposes in category theory and quantum foundations before. In the former case these rules are some of those required for generalised traces over monoidal categories. In the second case these rules are some of those used to formalise models of time-travel in quantum, theory, where such time-travel operators were even referred to as superoperators. A first compositional principle of supermaps, is that being kinds of transformations, they ought to be composable.

Definition 18 (The category of locally-applicable transformations)**.** *On any monoidal category* $\mathbf{C}$ *a new category* $\mathbf{Lot}_1[\mathbf{C}]$ *can be defined with objects given by pairs of the form* $[A, A']$ *and morphisms of type* $[A, A'] \to [B, B']$ *given by locally-applicable transformations of the same type. The identity morphism* $i^{[A,A']} : [A, A'] \to [A, A']$ *is given by the family of identity functions* $i^{[A,A']}_{X,X'} := i_{\mathbf{C}(AX, A'X')}$*. Composition is given diagrammatically by*



or algebraically by $(S \circ_{\mathbf{Lot}} T)_{X,X'} := S_{X,X'} \circ_{\mathbf{Set}} T_{X,X'}$*. Associativity and identity for composition are inherited from associativity and identity for composition in the category* $\mathbf{Set}$*.*

[1]As discussed in section 5.2, this condition implies that the $S_{XX'}$ form a natural transformation, meaning that we could in standard categorical language have asked for a family of functions *natural in* X and X'.

Note that one must check the proposed composition does indeed returns a new locally-applicable transformation, this is easily seen either diagrammatically or algebraically. This category has a few other properties reminiscent of the category of single-input supermaps. There is a functor $[-,=] : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Lot}_1[\mathbf{C}]$ given on objects by $[A, A']$ and given on morphisms $f : B \to A$ and $g : A' \to B'$ by the morphism

$$[f, g] : [A, A'] \to [B, B'] \tag{4.1}$$

$$[f, g]_{X,X'} :: \phi \mapsto (g \otimes i_{X'}) \circ \phi \circ (f \otimes i_X) \tag{4.2}$$

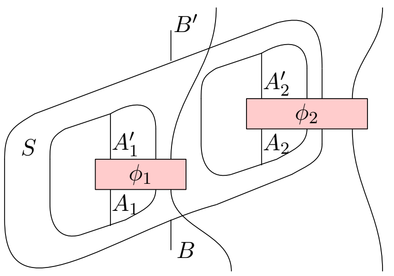
Functorality follows since

$$\begin{aligned}([f, g] \circ [f', g'])_{X,X'}(\phi) =& [f, g]_{X,X'}([f', g']_{X,X'}(\phi)) \\ =& (g' \otimes i_{X'}) \circ (g \otimes i_{X'}) \circ \phi \circ (f \otimes i_X) \circ (f' \otimes i_X) \\ =& [g' \circ g, f \circ f']_{X,X'}(\phi)\end{aligned}$$

Note that the switched order of composition between $g' \circ g$ and $f \circ f'$ is captured by taking the first domain to be $\mathbf{C}^{op}$ rather than $\mathbf{C}$.

4.2 Multi-Party Case

Let us now extend the definition of locally-applicable transformation to that which can be applied not to a single process, but instead to an entire list of processes. In short, we now try to give a bare-minimum behaviour law for any model of the following picture



Clearly this picture represents at the very least a function of type

$$S : \mathbf{C}(A_1, A_1') \times \mathbf{C}(A_2, A_2') \to \mathbf{C}(B, B'),$$

and similarly one can easily imagine a generalisation to N-input supermaps of type $[A_1, A_1'] \dots [A_n, A_n'] \to [B, B']$ which at the very least define functions of type

$$\bigtimes_{i=1}^{n} \mathbf{C}(A_i, A_i') \to \mathbf{C}(B, B').$$

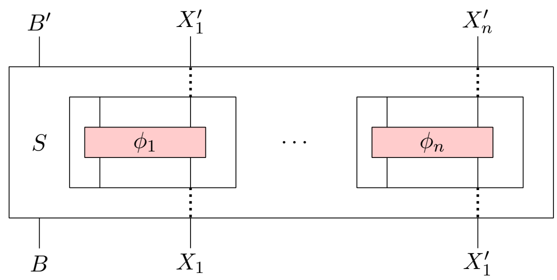
As before, we see that any model of this picture should be able to tell us the functional behaviour when applied to *part* of each of a pair of morphisms. This observation leads us to consider at the least a family of functions of type

$$S_{X_1 \ldots X_n}^{X_1' \ldots X_n'} : \mathbf{C}(A_1X_1, A_1'X_1') \times \mathbf{C}(A_2X_2, A_2'X_2') \to \mathbf{C}(CX_1X_2, C'X_1'X_2'),$$

or more generally we expect that N-input supermaps should at least give us families of functions of type

$$S_{X_1 \ldots X_n}^{X_1' \ldots X_n'} : \bigtimes_{i=1}^{n} \mathbf{C}(A_iX_i, A_i'X_i') \to \mathbf{C}(B \bigotimes_{i=1}^{n} X_i, B' \bigotimes_{j=1}^{n} X_j').$$

Note that when we deal with multiple-input functions we will use bottom indices for inputs auxiliary wires and upper indices for output auxiliary wires when we need to save space. Now, we need to again express locality, the idea that each of these functions does not actually have any effect on the auxiliary systems labelled as X_i, X_i'. We do not take the care to write locality explicitly except using a semi-formal graphical notation which will be sufficient to understand and reason about the imposed law. Let us denote the application of component $S_{X_1 \ldots X_n}^{X_1' \ldots X_n'}$ of a family of functions of the above type again in terms of a function-box notation by:



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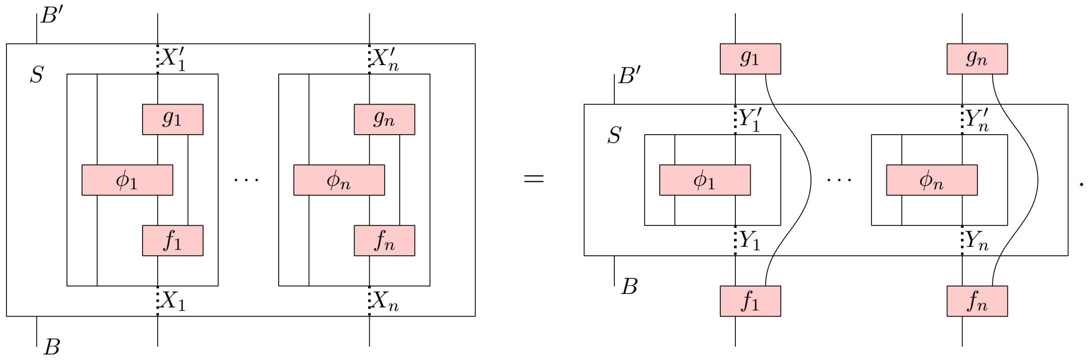
in terms of this picture the required behaviour law for multi-input supermaps is easily stated.

Definition 19. *A locally-applicable transformation of type* $[A_1, A_1'] \ldots [A_n, A_n'] \to [B, B']$ *is a family of functions*

$$S_{X_1 \ldots X_n}^{X_1' \ldots X_n'} : \bigtimes_{i=1}^{n} \mathbf{C}(A_iX_i, A_i'X_i') \to \mathbf{C}(B \bigotimes_{i=1}^{n} X_i, B' \bigotimes_{j=1}^{n} X_j')$$

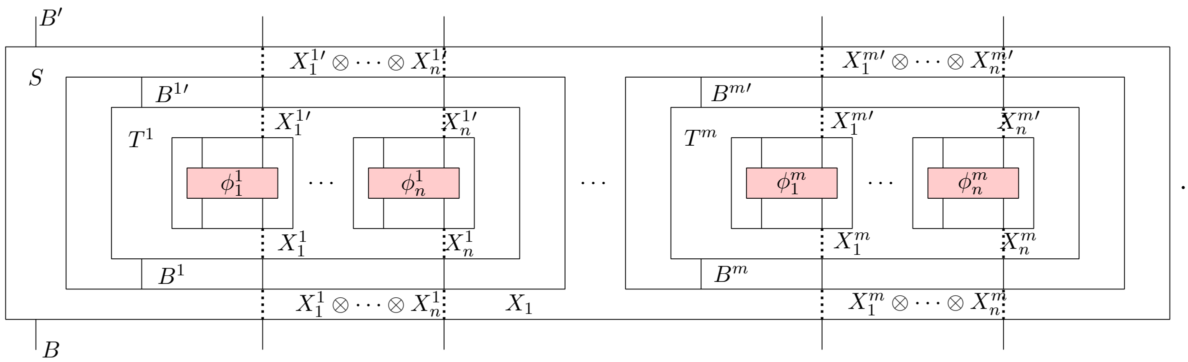
satisfying[2]*:*



There are two notes on this definition which may be obscured by the diagrammatic notation. First, we draw wires as running over the top of S on the right-hand side, to give unambigous meaning to this requires symmetry of the monoidal category $\mathbf{C}$. The second note builds on the first, an alternative way to define the multiparty locally-applicable transformations is inductively, requiring that the induced functions given by applying S to all but the right-most hole returns a locally-applicable transformation of the single input type, and requiring that filling in only the right-most hole returns an $N-1$ locally-applicable transformation after pre and post composition with $\beta_{X_1 \dots X_{N-1}, X_N}$ and $\beta_{X_N, X'_1 \dots X'_{N-1}}$ respectively.

We saw in the preliminary material that supermaps form a multicategory by nesting of boxes. The multi-input locally-applicable transformations can be used to form a multicategory in the same way.

Definition 20. *The symmetric multicategory* $\mathbf{Lot}[\mathbf{C}]$ *has as objects pairs* $[A, A']$ *and as morphisms* $[A_1, A'_1] \dots [A_n, A'_n] \to [B, B']$ *the locally-applicable transformations of the same type. Composition of* $S : [B_1, B'_1] \dots [B_n, B'_n] \to [C, C']$ *with a family of morphisms* $T^i : [A^i_1, A^{i}_1{}'] \dots [A^i_{n_i}, A^i_{n_i}{}'] \to [B^i, B^{i\prime}]$ *is given by the following*



Morphisms of type $\bullet \to [A, A']$ *are given by morphisms of type* $\mathbf{C}(A, A')$ *and composition of* $S : [A_1, A'_1] \dots [A_n, A'_n] \to [B, B']$ *and* $\phi : \bullet \to [A_i, A'_i]$ *is given by inserting* ϕ *into hole*

[2]As discussed for single-input supermaps, in section 5.2 we'll show that this condition implies that the $S_{XX'}$ form a natural transformation, meaning that we could in standard categorical language have asked for a family of functions *natural in* $X_1 \dots X_n$ and $X'_1 \dots X'_n$.

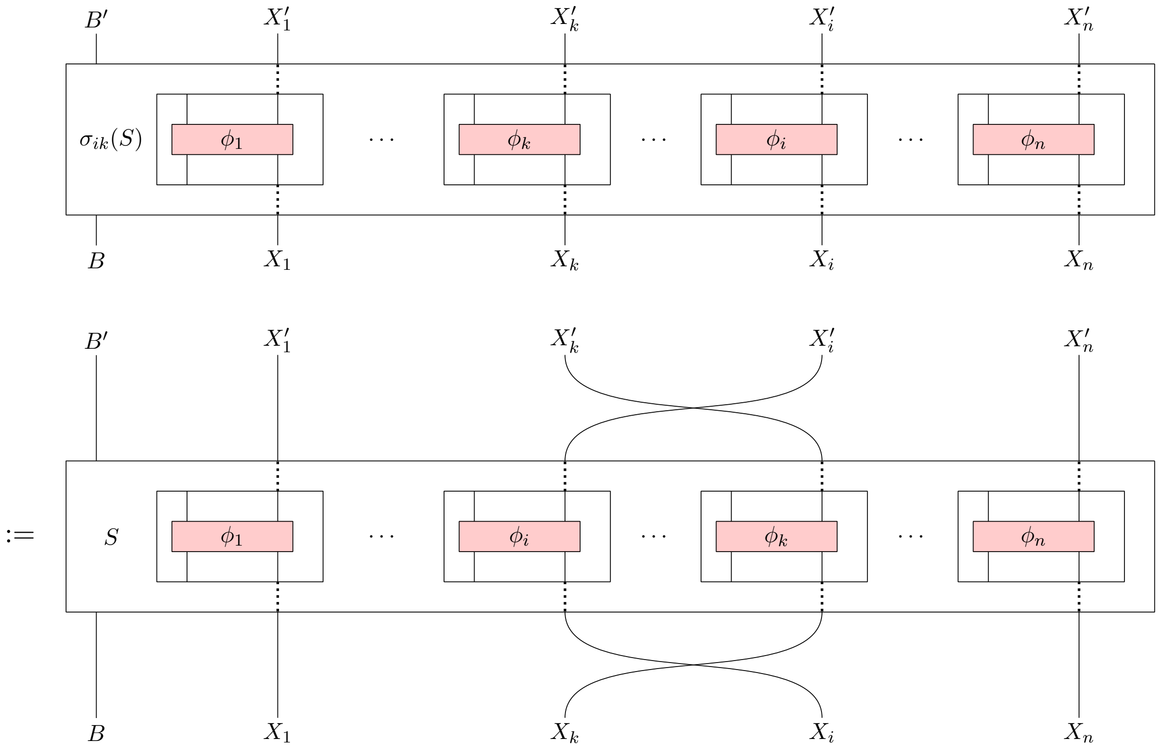
i of S. The action for general permutations of an input list is generated by the action for the swap[3] σ_{ik} *of elements* i, k *the list by taking each*

$$S : [A, A'] \dots [A_i, A_i'] \dots [A_k, A_k'] \dots [A_n, A_n'] \to [B, B']$$

to

$$\sigma_{ik}(S) : [A, A'] \dots [A_k, A_k'] \dots [A_i, A_i'] \dots [A_n, A_n'] \to [B, B']$$

with



A more familiar although less illuminating algebraic presentation of the same composition rule can be given in the following way:

$$S \circ (T^1 \dots T^n)^{X_1^1{}' \dots X_{n_m}^n{}'}_{X_1^1 \dots X_{n_m}^n} := S^{X_1^1{}' \dots X_{n_1}^1{}', \dots, X_1^m{}' \dots X_{n_m}^m{}'}_{X_1^1 \dots X_{n_1}^1, \dots, X_1^m \dots X_{n_m}^m} \circ_{\mathbf{Set}} \times_{k=1}^m T^{X_1^k{}', \dots, X_{n_k}^k{}'}_{X_1^k, \dots, X_{n_k}^k}.$$

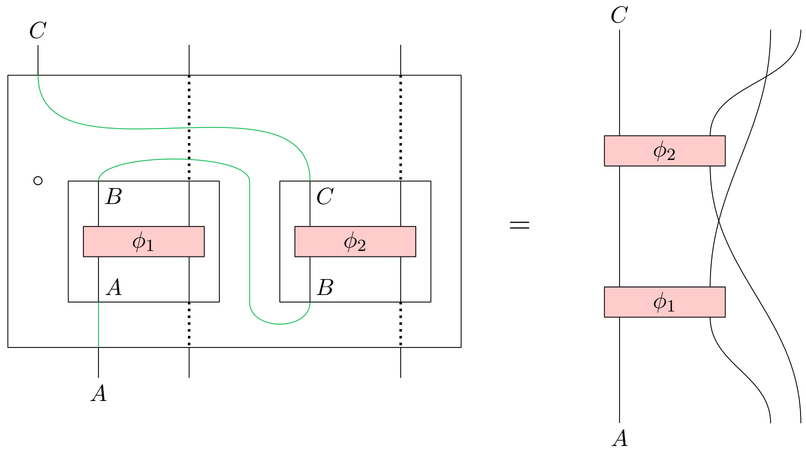
with composition with morphisms from the empty list given by

$$(S \circ \phi)^{X_1' \dots X_{i-1}' X_{i+1}' \dots X_n'}_{X_1 \dots X_{i-1} X_{i+1} \dots X_n}(-, \dots, -) := S^{X_1' \dots X_n'}_{X_1 \dots X_n}(-, \dots, \phi, \dots, -)$$

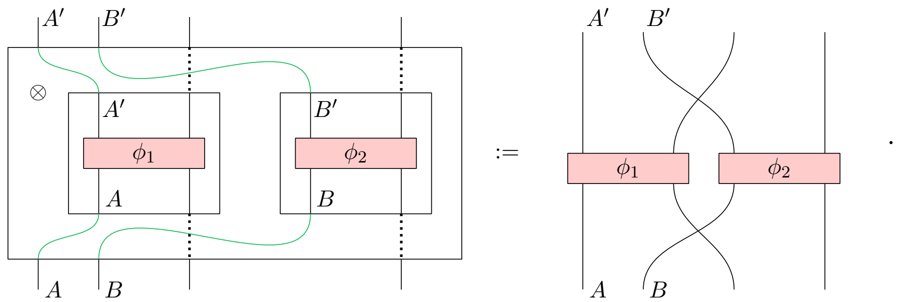
Associativity and interchange laws for multicategory composition are inherited from **Set**.

[3]Each permutation can be decomposed into a sequence of swaps, that the action generated by applying the associated actions for some decomposition into swaps is well-formed and functorial follows from the defining rules for the swaps which make **C** symmetric monoidal and the swaps which make **Set** monoidal.

Definition 21. *The* **Lot**[**C**]*-smc* **C** *can be defined by taking the sequential composition supermap* $\bigcirc_{ABC} : [A, B][B, C] \to [A, C]$ *to be*



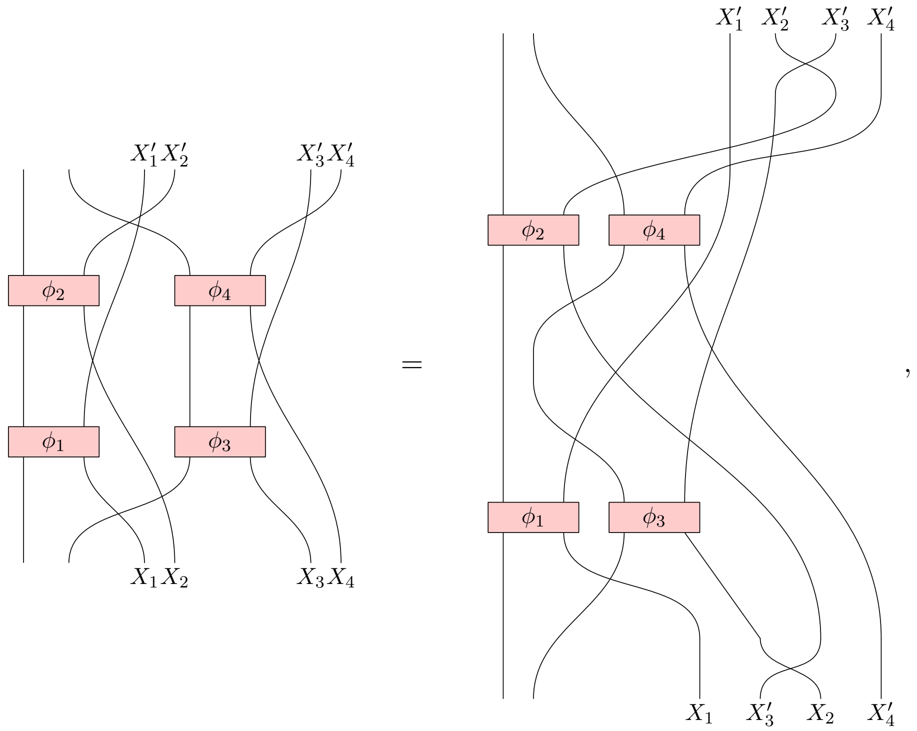
and taking the parallel composition supermap of type $\bigotimes_{AA'BB'} : [A, A'][B, B'] \to [AB, A'B']$ *to be*



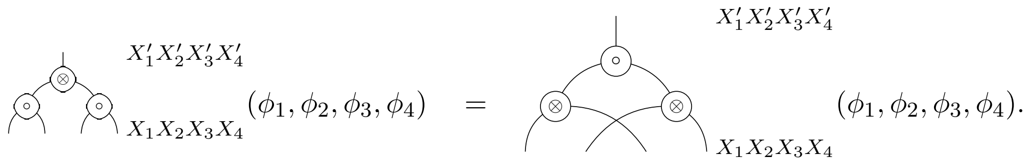
The isomorphism $\mathbf{C}(A, A') \cong \mathbf{Lot}[\mathbf{C}](\bullet, [A, A'])$ *is in this case an equality. It is then immediate that* $\bigcirc_{ABC} \circ_{\mathbf{Lot}[\mathbf{C}]} (\phi_1, \phi_2) = \phi_2 \circ_{\mathbf{C}} \phi_1$ *and similarly for the parallel composition supermap.*

One must check the required laws for these specified morphisms to give enrichment.

As an example let us explicitly prove the enriched interchange law, since



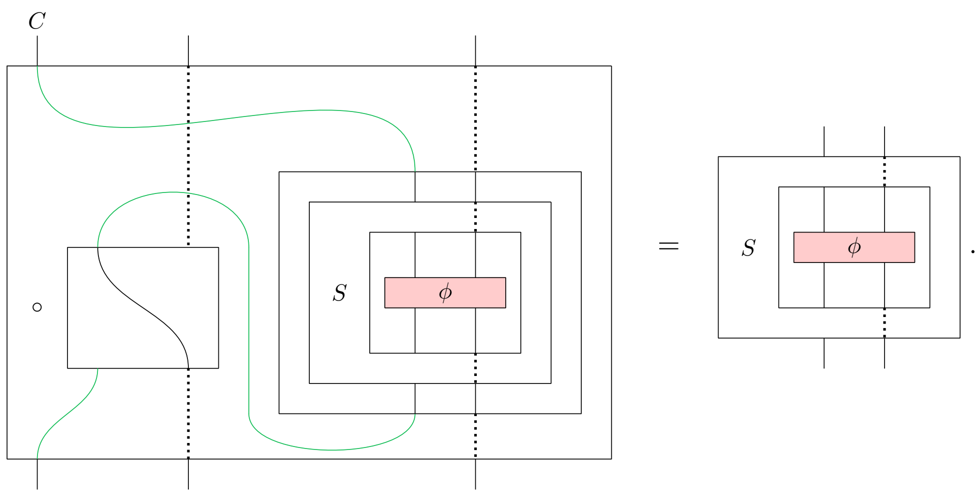
it follows that



Note that the **Lot**[**C**]-smc **C** is furthermore faithful, as defined in the previous chapter.

Lemma 3. *The* **Lot**[**C**]*-smc* **C** *is faithful.*

Proof. This follows by finding a pair of inputs for each to $(\bigcirc_{IAB} \circ (i,S))^{X_1' X_2'}_{X_1 X_2}$ for each X, X' and ϕ which returns $S_{X,X'}(\phi)$. Then it follows that $\bigcirc_{IAB} \circ (i,S) \neq \bigcirc_{IAB} \circ (i,T) \implies S \neq T$. The correct choice of input is most easily seen graphically, by noting that



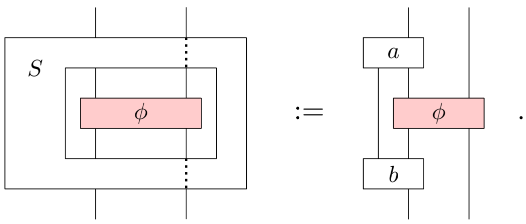
□

In conclusion, a theory of families of functions which represent the basic expected operational behaviour of supermaps, can be defined on any symmetric monoidal category and provides a model of the expected features of theories of supermaps given in the previous chapter.,

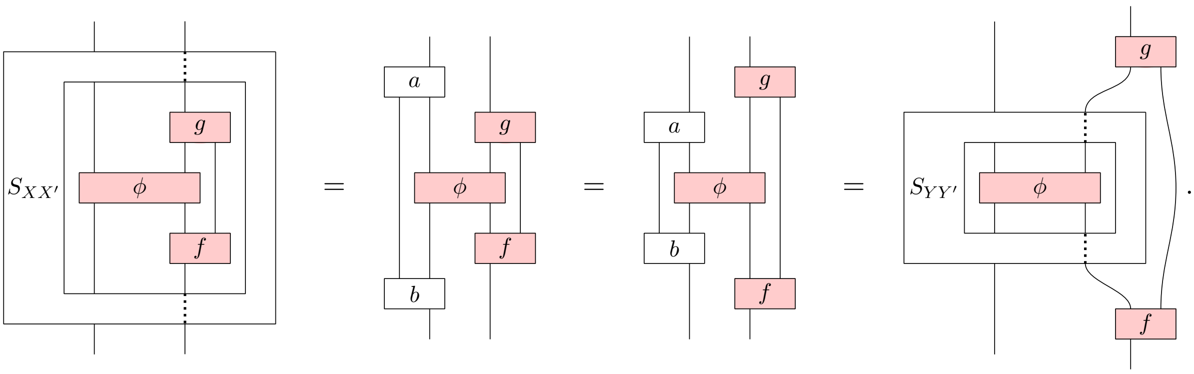
4.3 Examples

Here we examine a series of examples of locally-applicable transformations, of the single-input and multi-input kind. Let us begin with combs [13] on general monoidal categories as defined in [29, 48].

Example 20 (Combs)**.** *For every symmetric monoidal category* $\mathbf{C}$ *and pair of morphisms* $a : A \to E \otimes B$ *and* $b : E \otimes A' \to B'$ *one can define a locally-applicable transformation of type* $[A, A'] \to [B, B']$ *by*

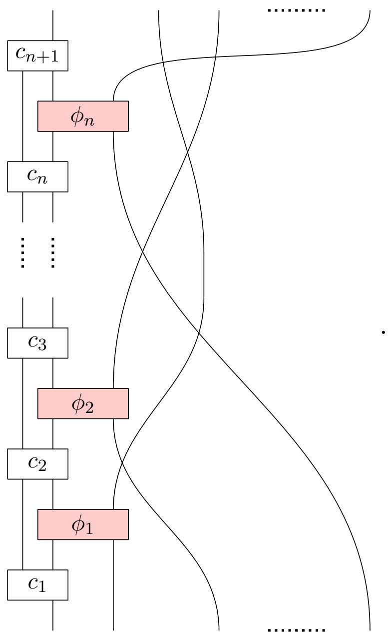


Indeed, note that:



We from now on refer to such a locally-applicable transformation by $\texttt{comb}[a, b]$ and its components by $\texttt{comb}[a, b]_{X,X'}$, such combs form a subcategory of $\mathbf{Lot}[\mathbf{C}]$ which is isomorphic as a category to $\mathbf{comb}[\mathbf{C}]$ as defined in [29].

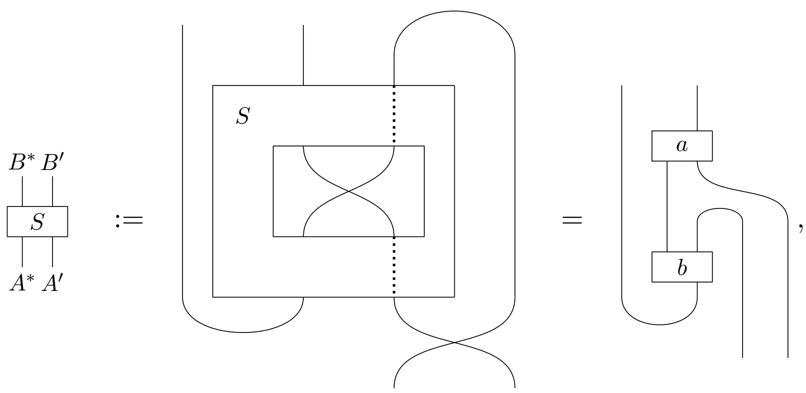
Example 21. *Let* $\mathbf{C}$ *be a symmetric monoidal category, the locally-applicable transformation* $\texttt{comb}[c_1 \dots c_{n+1}]$ *of type* $\texttt{comb}[c_1 \dots c_{n+1}] : [A_1, A_1'] \dots [A_n, A_n'] \to [B, B']$ *is the family* *of functions given by*



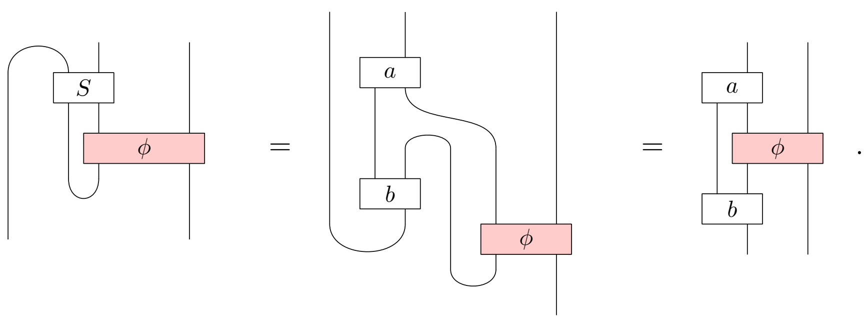
For a compact closed category **D** with **C** $\subseteq$ **D** the notions of **D**-supermap and **D**-comb on **C** are equivalent. Through this equivalence, **D**-supermaps always give examples of locally-applicable transformations.

Lemma 4. *Let* **D** *be a compact closed category and* **C** *be a symmetric monoidal category, there is a one-to-one correspondence between the* **D***-combs on* **C** *and the* **D***-supermaps on* **C**.

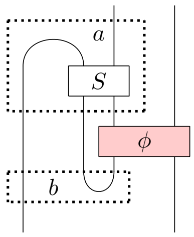
Proof. Let S be a **D**-comb on **C** then one can construct



which indeed is a **D**-supermap since



Instead let S be a $\mathbf{D}$-supermap then one can construct the locally-applicable transformation

$\mathcal{F}(S)_{X,X'} \quad :=$  .

These two constructions are furthermore inverse to each-other. This assignment is also a functor meaning in concrete terms that $\mathcal{F}$ preserves composition and identities. □

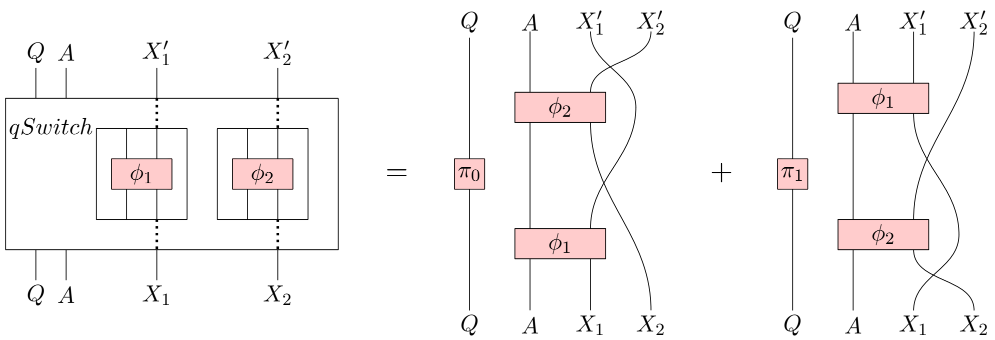
For the same reasons, every $\mathbf{D}$-supermap on $\mathbf{C}$ of type $[A_1, A_1'] \dots [A_n, A_n'] \to [B, B']$ can be used to define a locally-applicable transformation of the same type. This assignment commutes with multicategory composition, meaning that it is a multifunctor. We will denote this multifunctor as $\mathcal{F}_{\mathbf{D},\mathbf{C}} : \mathbf{Dsup}[\mathbf{C}] \to \mathbf{Lot}[\mathbf{C}]$.

Now we have a general theory-neutral definition, we can look for examples which are not $\mathbf{D}$-supermaps for some compact closed category $\mathbf{D}$.

Definition 22. *The category* **Hilb** *has objects given by Hilbert spaces and morphisms given by bounded linear operators [202].*

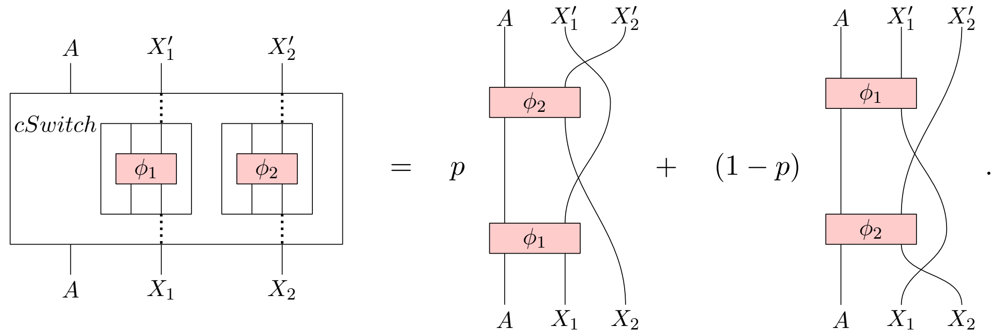
Note that we do not require the Hilbert spaces to be finite dimensional or even separable here. The category **Hilb** is a symmetric dagger monoidal category and furthermore, one can show that it is enriched in the category of commutative monoids, meaning concretely that sequential composition, parallel composition, and daggers commute with sums. We refer to the symmetric monoidal subcategory of unitaries between general Hilbert spaces as $\mathbf{UHilb} \subseteq \mathbf{Hilb}$.

Example 22 (The Quantum Switch for Arbitrary Hilbert Spaces)**.** *We call any pair of morphisms* $\{\pi_0, \pi_1\}$ *in* **UHilb** *such that* $\pi_i \circ \pi_j = \delta_{ij}$ *a control. The quantum switch on* **UHilb** *with control* $\{\pi_0, \pi_1\}$ *is defined as the locally-applicable transformation of type* `Switch` $: [A, A][A, A] \to [Q \otimes A, Q \otimes A]$ *given by:*

 (4.3)

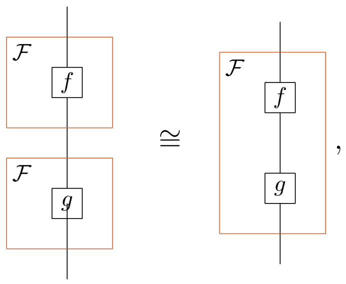
Orthogonality of π_0 *and* π_1 *and compatibility between daggers and sums implies unitarity of this linear map, local-applicability follows by compatibility of sequential composition and parallel composition with sums.*

Example 23 (Convex Switch)**.** *Let us now consider the symmetric monoidal category* **QO** *of quantum operations [241]. between arbitrary Hilbert spaces. In this category morphisms* $f, g : A \to B$ *can be combined using convex combinations* $pf + (1-p)g : A \to B$ *where* $p \in [0,1]$*. Using this structure we can construct a generalisation of the convex switch, choosing one order of composition with the probability* p *and the other with probability* $(1-p)$

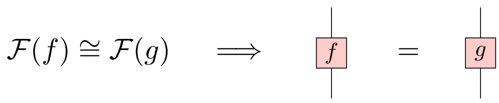


To see that this convex switch satisfies local-applicability, note that sequential and parallel composition commute with convex combinations.

For our next example we will introduce functor box notation for weak symmetric monoidal functors, the interested reader may read the details of [188, 242–244] for more details, however what we will need to iunderstand to follow the thesis is quite minimal. Whilst $\mathcal{F}(f)$ will be notated as before, for a weak monoidal functor, functorality is only-up-to ismorphism so that we may write:



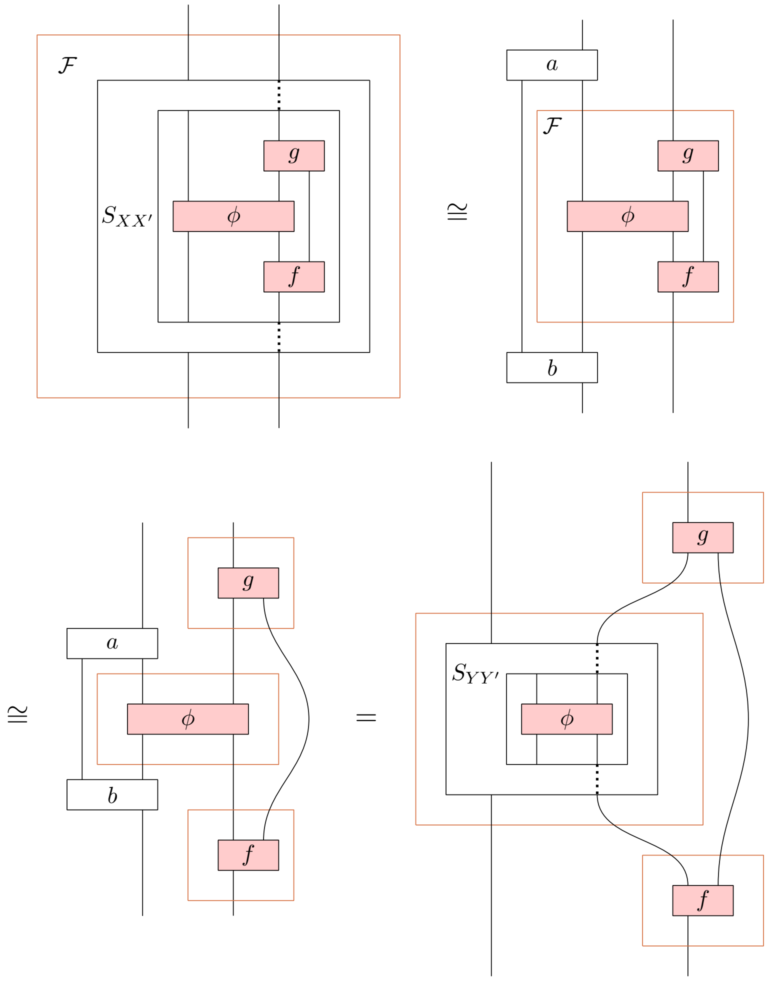
where $\cong$ is such that $f \cong g$ and $f' \cong g'$ implies $f \circ g \cong f' \circ g'$ and simillarly for parallel composition. We will say that a functor is 2-faithful if



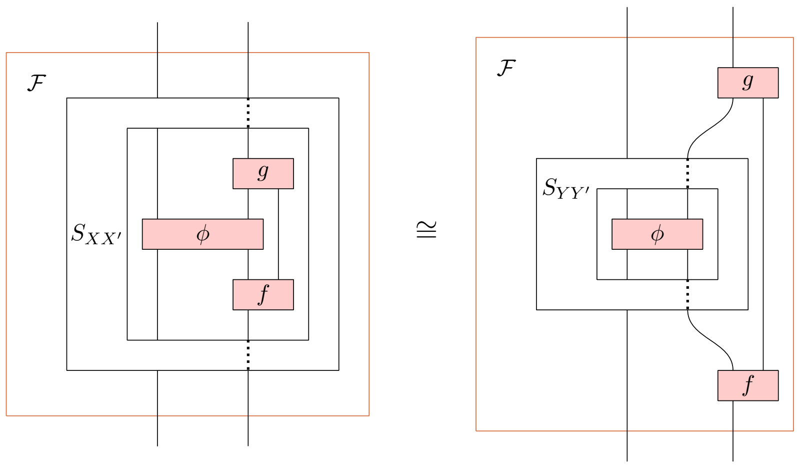
The above allows us to generalise **D**-representable supermaps to a setting which allows us to use compact closure when defining superunitaries on separable Hilbert spaces.

Lemma 5. *Let* **C** *be a symmetric monoidal* 2*-category with trivial* 2*-morphisms and* **D** *be a symmetric monoidal* 2*-category with a weak* 2*-faithful symmetric monoidal* 2*-functor* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$*. Any* `comb`$[a, b]$ *with* $a, b \in \mathbf{D}$ *such that for all* $\phi \in \mathbf{C}(AX, A'X')$ *there* *exists $\psi \in \mathbf{C}(BX, B'X')$ such that $\mathcal{F}(\psi) \cong \texttt{comb}[a,b]_{\mathcal{G}X\mathcal{G}X'}(\phi)$ defines a locally-applicable transformation by taking $S_{X,X'}(\phi)$ to be the unique ψ s.t $\mathcal{G}(\psi) = \texttt{comb}[a,b](\phi)$. Here uniqueness follows since $\mathcal{F}$ is 2-faithful. Such a locally-applicable transformation is termed a $\mathcal{G}$ representable supermap on $\mathbf{C}$ of type $[A, A'] \to [B, B']$.*

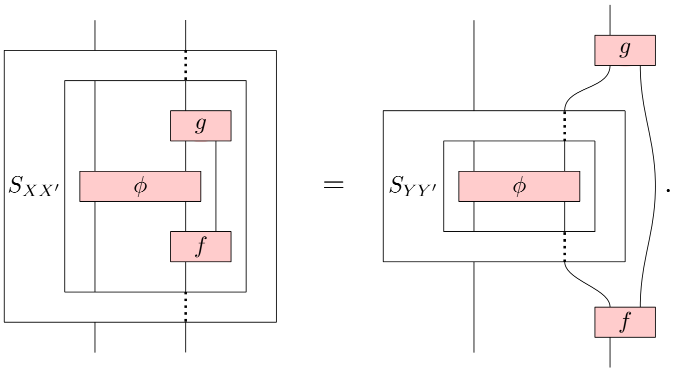
Proof. Note that



and so



which by 2-faithful-ness of $\mathcal{F}$ gives



□

This gives a way to construct and represent examples of supermaps on the category **sepU** $\subseteq$ **sepHilb** of unitaries between separable Hilbert spaces by using the embedding of the category **sepHilb** of bounded linear maps between separable Hilbert spaces into the category $*$**Hilb** of non-standard Hilbert spaces [242].

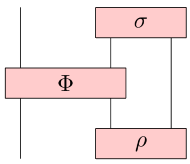
Example 24. *There is a 2-faithful weak symmetric monoidal 2-functor* $\mathcal{G}$: **sepU** $\to$ ***Hilb** *given by composition of the embedding* **sepU** $\subseteq$ **sepHilb** *and the truncation functor* $\texttt{trunc}[-]_w$: **sepHilb** $\to$ ***Hilb** *[242]. The induced supermaps are then termed* $\texttt{trunc}[-]$ *representable supermaps on* **C**.

One can straight-forwardly generalise the above construction to define $\mathcal{G}$-representable supermaps with multiple inputs, and so in particular define $\texttt{trunc}[-]$ representable supermaps on **sepU** with multiple inputs.

4.4 Locally-Applicable Transformations Between Constrained Sets

As seen in the preliminary material, supermaps are also often defined on subsets of sets of channels [17], let us now generalise the dilation extensions of subsets so that they can be formulated in any symmetric monoidal category[4].

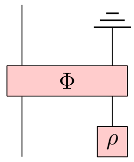
Definition 23. *For each* $K \subseteq \mathbf{C}(A, A')$ *and pair* (X, X') *the dilation extension by* X, X' *denoted* $\texttt{dext}_{X,X'}(K)$ *is the subset of* $\mathbf{C}(A \otimes X, A' \otimes X')$ *given by:*

$$\Phi \in \texttt{dext}_{X,X'}(K) \iff \forall \rho, \sigma :$$  $$\in \quad K.$$

[4]Note that the definition we give of dilation extension trivialises when categories have no states and effects, such as the category of unitaries. In such a category a more general notion of extension set is more suitable as outlined in [3].

Note that for the example $K = \mathbf{C}(A, A')$ then the extension $\texttt{dext}_{X,X'}(K)$ returns the entire set of bipartite morphisms $\mathbf{C}(A \otimes X, A' \otimes X')$. For a causal symmetric monoidal category the extended set can be rephrased in the following way:

$$\phi \in \texttt{dext}_{X,X'}[K] \iff \forall \rho :$$

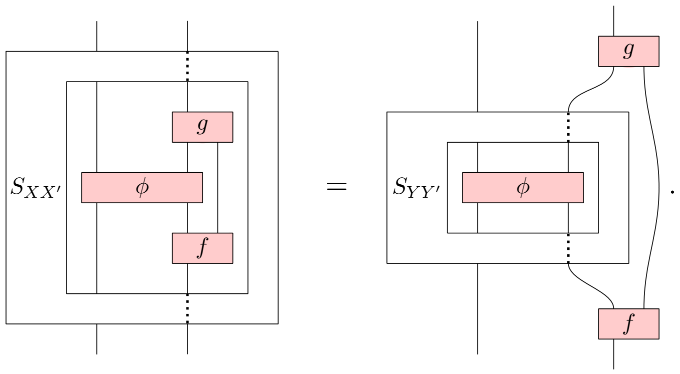


$$\in \quad K,$$

which is the form used in the preliminary material and in [14, 17] to define extensions to subsets of quantum channels. Whenever a subset $K \subseteq \mathbf{QC}(A, A')$ is closed under convex combinations then it follows that $\texttt{dExt}_{X,X'}(K)$ is closed under convex combinations, we will from now on rephrase the statement that a set K be closed under convex combinations as simply the statement that K be convex.

We will now define locally-applicable transformations on dilation extensions of subsets of processes.

Definition 24 (locally-applicable transformations)**.** *A locally-applicable transformation of type $S : K \longrightarrow M$ on a symmetric monoidal category $\mathbf{C}$ is a family of functions $S_{XX'} : \texttt{dext}_{X,X'}(K) \to \texttt{dext}_{X,X'}(M)$ such that for every $g : Y' \otimes Z \to X'$, $f : X \to Y \otimes Z$, and $\phi : A \otimes X \to A' \otimes X'$ then*

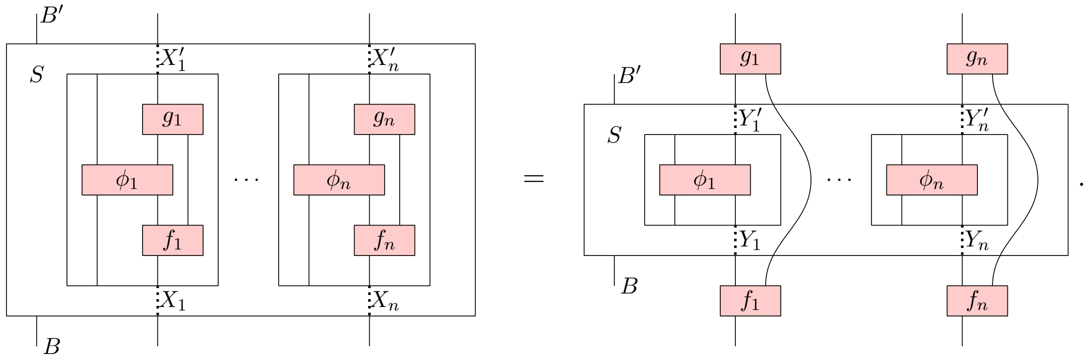


Naturally, this definition can be extended to the multi-party setting using the exact same diagrams as before.

Definition 25. *A locally-applicable transformation of type $K_1 \dots K_n \longrightarrow M$ is a family of functions*

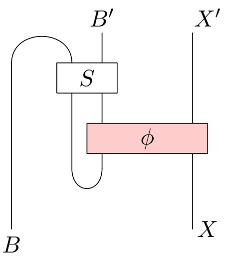
$$S^{X'_1 \dots X'_n}_{X_1 \dots X_n} : \bigtimes_{i=1}^{n} \texttt{dext}_{X_i,X'_i}(K_i) \longrightarrow \texttt{dext}_{\bigotimes_{i=1}^{n} X_i, \bigotimes_{j=1}^{n} X'_j}(M)$$

satisfying:



As for the case of unconstrained types, we can construct examples using embeddings into compact closed categories.

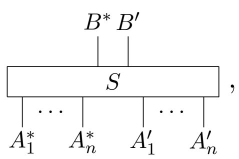
Definition 26. *Let* $\mathbf{C} \subseteq \mathbf{D}$ *be an inclusion of a symmetric monoidal category* $\mathbf{C}$ *into a compact closed category* $\mathbf{D}$ *and let* $K \subseteq \mathbf{C}(A, A')$ *and* $M \subseteq \mathbf{C}(B, B')$ *. A* $\mathbf{D}$*-supermap on* $\mathbf{C}$ *of type* $S : K \to M$ *is a morphism in* $\mathbf{D}$ *of type* $S : A^* \otimes A' \to B^* \otimes B'$ *such that for every* $\phi \in \texttt{dext}_{X,X'}(K)$ *then*



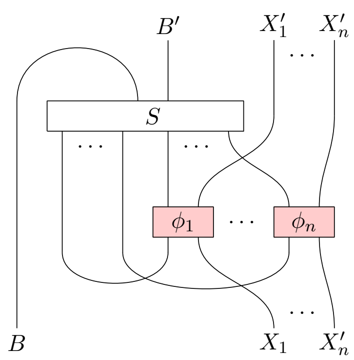
$$\in \quad \texttt{dext}_{X,X'}(M).$$

Again this definition can be generalised to the multi-input case.

Definition 27. *Let* $\mathbf{C} \subseteq \mathbf{D}$ *be an inclusion of a symmetric monoidal category* $\mathbf{C}$ *into a compact closed category* $\mathbf{D}$*. A* $\mathbf{D}$*-supermap on* $\mathbf{C}$ *of type* $S : K_1 \dots K_n \to M$ *is a morphism*



in $\mathbf{D}$ *such that for every family* $\phi_{(i)} \in \bigtimes_{i=1}^{n} \texttt{dext}_{X_i,X_i'}(K_i)$ *then*



$$\in \quad \texttt{dext}_{\bigotimes_{i=1^n} X_i, \bigotimes_{i=1}^{n} X_i'}(M).$$

For any inclusion $\mathbf{C} \subseteq \mathbf{D}$ of a symmetric monoidal category into a compact closed category, every $\mathbf{D}$-supermap on $\mathbf{C}$ of type $K \to M$ can used to construct a locally-applicable transformations of the same type, using the same method as for the case of locally-applicable transformations between unconstrained types. Note that when $\mathbf{C} = \mathbf{QC}$ and $\mathbf{D} = \mathbf{CP}$ this definition of $\mathbf{D}$-supermap on $\mathbf{C}$ of type $K \to M$ is the previously defined notion of a $\mathbf{CP}$-supermap on $\mathbf{QC}$ of type $K \to M$, which we will often refer to as simply a quantum supermap of type $K \to M$. This entire discussion naturally extends to the multi-input setting.

4.5 Locally-applicable transformations from axioms for theories of supermaps

So far we have established three key concepts in the study of supermaps on symmetric monoidal categories. Two of those concepts have been phrased in terms of the properties of entire compositional theories of supermaps, the final concept has instead been phrased in terms of the expected properties of individual supermaps with respect to the compositionality of the theory they act on:

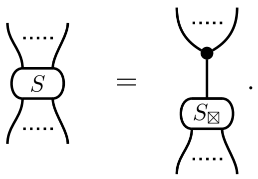
- Theories of supermaps ought to define polycategories.
- Theories of supermaps ought to have sequential and parallel composition supermaps.
- The operational behaviour of a supermap ought to be described by a locally-applicable transformation.

We have seen that if we take supermaps to be locally-applicable transformations then there are indeed always sequential and parallel composition supermaps. Furthermore, in the next chapter we will see that locally-applicable transformations can be strengthened to define polyslots, which can be equipped with a polycategorical composition rule. In this section we work in the opposite direction, we will recover the local-applicability of individual supermaps from axioms on the behaviour of entire categories of supermaps. More concretely, we will show that any theory which satisfies the first two bullet points along with a few extra conditions can always be mapped into to the multicategory of locally-applicable transformations. We will split our required axioms for theories of supermaps into two parts, with the first axiom collecting together the previous two bullet points.

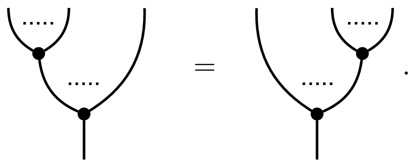
Axiom 1. *A theory of supermaps is a* $\mathbf{P}$*-smc* $\mathbf{C}$ *with* $\mathbf{P}$ *a symmetric polycategory.*

When we ask for a $\mathbf{P}$-smc $\mathbf{C}$ with $\mathbf{P}$ a symmetric polycategory, we mean more carefully that we ask for a $\mathbf{M}[\mathbf{P}]$-smc $\mathbf{C}$ as defined in 3 with $\mathbf{P}$ a symmetric polycategory and $\mathbf{M}[\mathbf{P}]$ the symmetric multicategory freely constructed from the symmetric polycategory $\mathbf{P}$ by only keeping morphisms with output lists of length 1. In the second axiom we note that so far our constructions of supermaps with have treated multiple-output lists such as $[B_1, B'_1] \dots [B_m, B'_m]$ as equivalent to single composite output types such as $[B_1, B'_1] \boxtimes \cdots \boxtimes [B_m, B'_m] := [B_1 \otimes \cdots \otimes B_m, B'_1 \otimes \cdots \otimes B_m]$. This equivalence, when formalised is the requirement that the outputs of a polycategory of supermaps should have a cotensor [216, 229] given by $\boxtimes$, or more precisely that the outputs of a polycategory should be representable.

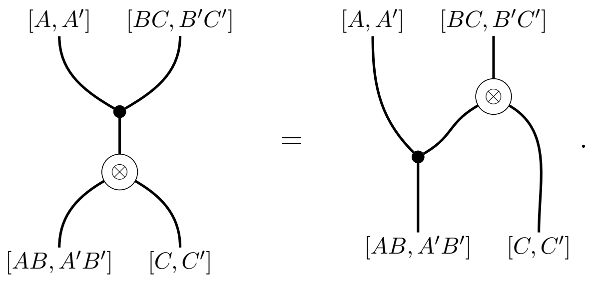
Axiom 2. *The outputs of the theory of supermaps are representable by a cotensor compatible with composition in* $\mathbf{C}$*. Concretely for each tuple* $X_1 \dots X_n$ *of objects of* $\mathbf{P}$ *there is an object* $X_1 \boxtimes \cdots \boxtimes X_n$ *and morphism* $\pi : X_1 \boxtimes \cdots \boxtimes X_n \to X_1 \dots X_n$ *such that for every* $S : \Theta \to X_1 \dots X_n$ *there exists a unique morphism* $S_{\boxtimes} : \Theta \to X_1 \boxtimes \cdots \boxtimes X_n$ *such that* $\pi \circ S_{\boxtimes} = S$*. We will refer to the morphisms* $\boxtimes$ *exhibiting representability as cotensors [], and will diagrammatically represent cotensors with black dots. The defining equation of the cotensor can then be written as:*



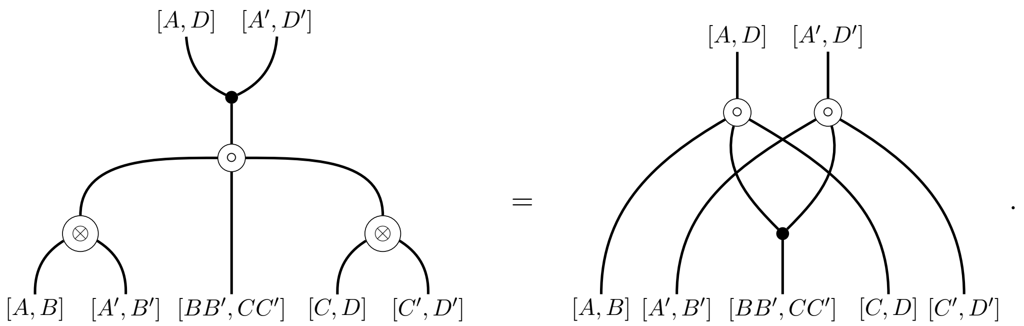
Furthermore, every well-typed composition of such dots is required to be equal, such a requirement can be phrased diagrammatically as an associativity or spider law:



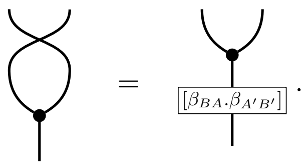
We also require these splitting maps to play nicely with the enriched symmetric monoidal structure of $\mathbf{C}$*, meaning that* $[A, A'] \boxtimes [B, B'] = [A \otimes B, A' \otimes B']$ *along with three compatibility laws. The first law required says that the order of putting systems together with enriching maps and splitting them apart with representability maps does not matter:*



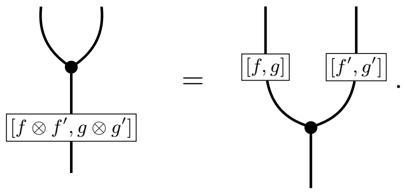
The second law says that pre and post composition on morphisms of $\mathbf{C}$ *can be performed before or after splitting using representability maps:*



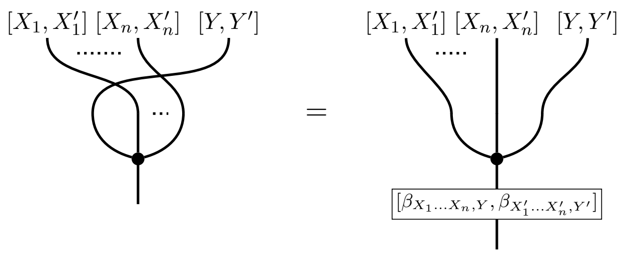
Finally, a third law requires that swapping of two halves of a bipartite process can be performed before or after splitting:



These laws say that for all intents and purposes, π does nothing except to let us examine subsystems. Note that when states are inserted into the point-free naturality law, we recover the following:

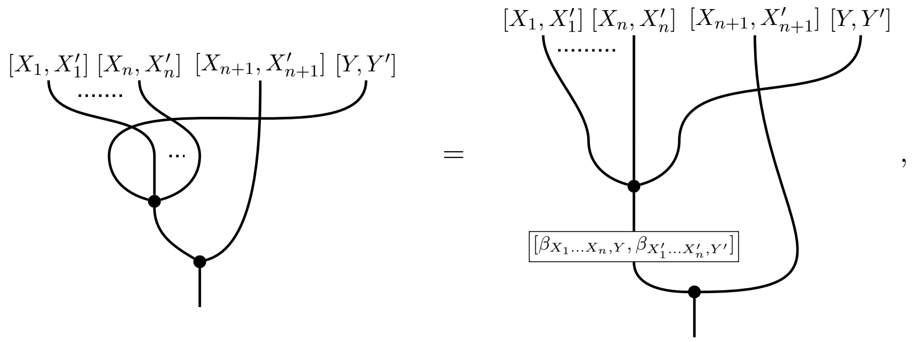


Note furthermore that the braid law implies the following more elaborate result for braids.

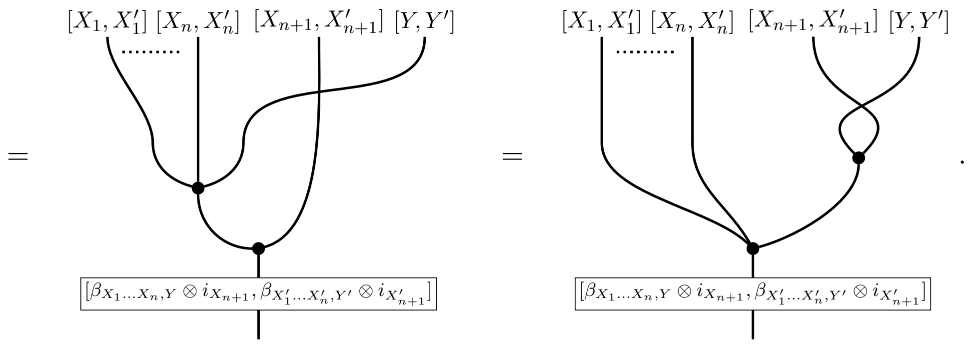


Indeed, this is true by assumption for $n = 1$ and can be proven for general n by induction. Imagine that the property were true for some n, then when checking the property for $n+1$

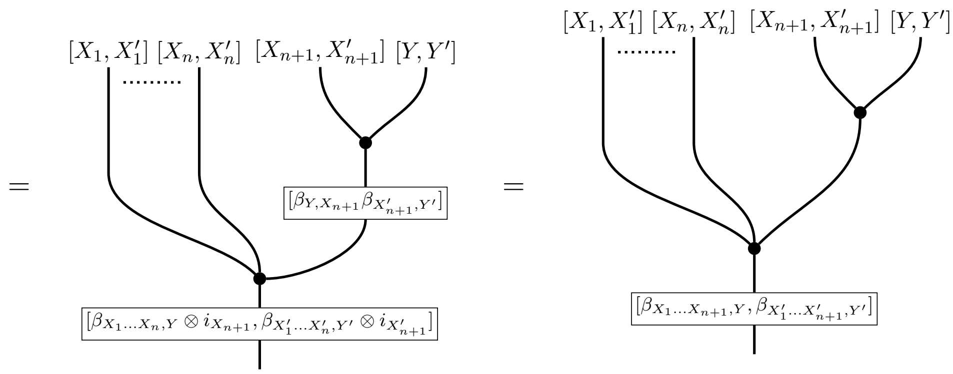
we find that



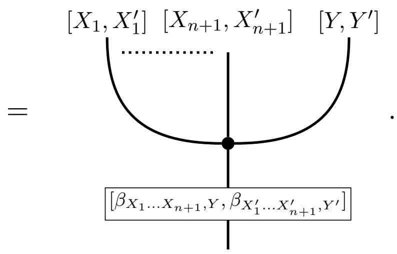
which after using naturality and associativity gives



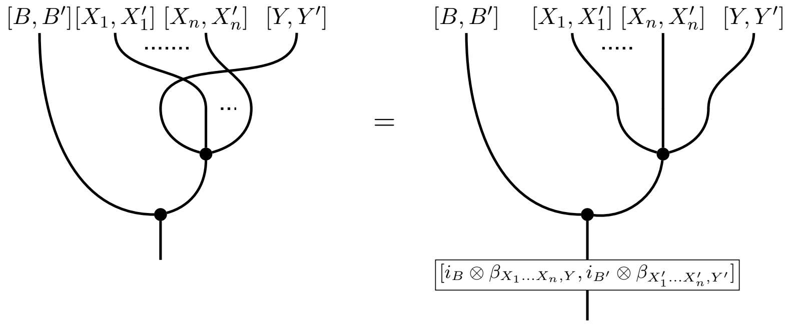
Next, using the property for $n = 1$ along with naturality and rules for the composition of swap morphisms in **C** gives



which by associativity gives the required form

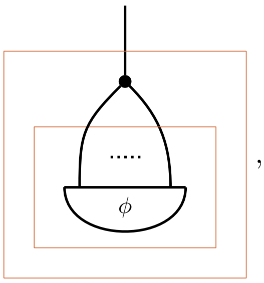


Note that as a consequence of these property we can also see that

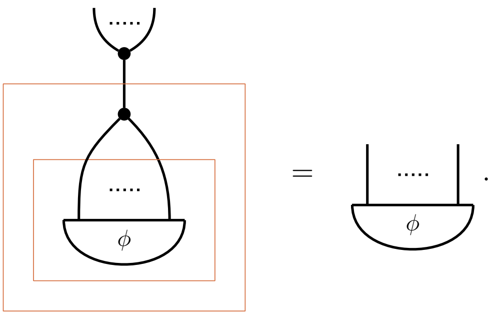


One may wonder why we do not analogously ask for representability of the inputs of **P** by tensors. Such a property would be convenient, however, there are good reasons to omit this form of representability. First, our simple example of **D**-supermaps on **C** on simple types, is representable in its output structure but not in its input structure. This is because supermaps outputs $[A, A'][B, B']$ are representable by another simple type $[A \otimes B, A' \otimes B']$. Inputs, on the other hand, we expect would be representable not on the space of all bipartite maps but just some abstraction of the non-signalling ones. So, without working with supermaps on constrained spaces, we would not expect to see representability for inputs. This issue also arises in [121], in which circuits with holes are observed to be *promonoidal* [245] rather than fully monoidal.

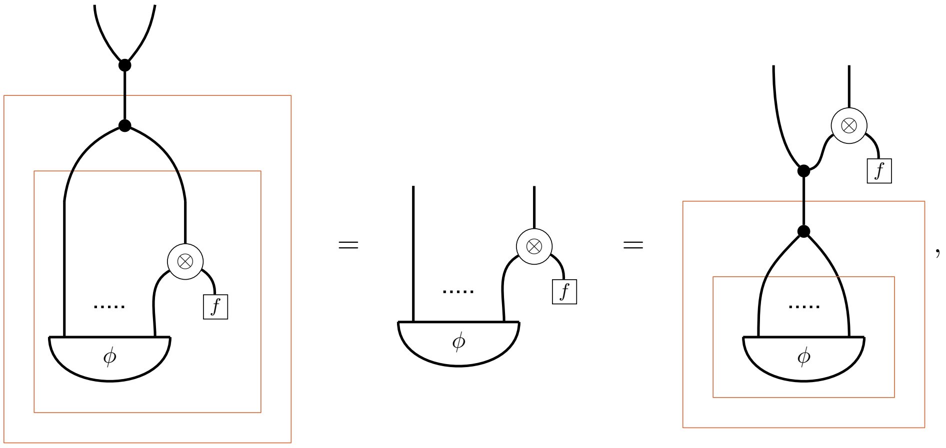
We will now introduce an additional diagrammatic function-box style notation which will make it easier for us to work with polycategories satisfying our two axioms, which we will tentatively refer to as *theories of supermaps*. The key idea is to write the inverse of the isomorphism given by representability diagrammatically as a merging map:



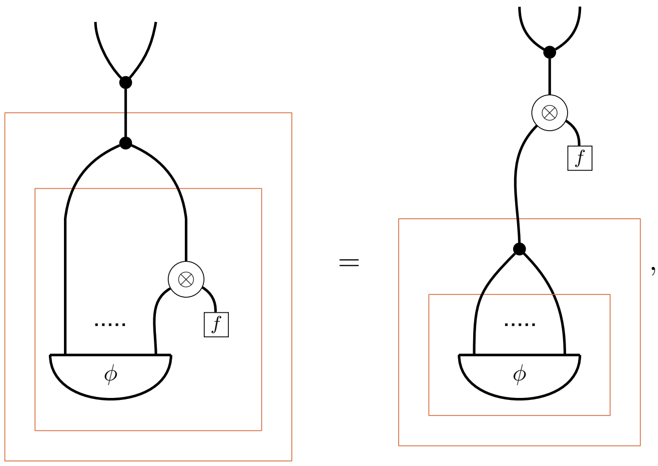
with the defining diagramamtic condition being that merging and then splitting does nothing as follows:



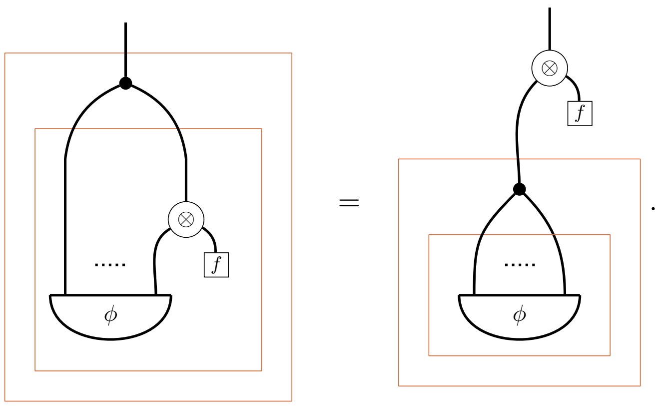
It is likely, that such diagrams are related to, or more easily represented by, proof nets for linearly distributive categories [246]. There are a few key diagrammatic laws which can then be directly inferred, and which will be used in the main theorem of this section, we will collect them together now to streamline the presentation of this theorem. First of all, since



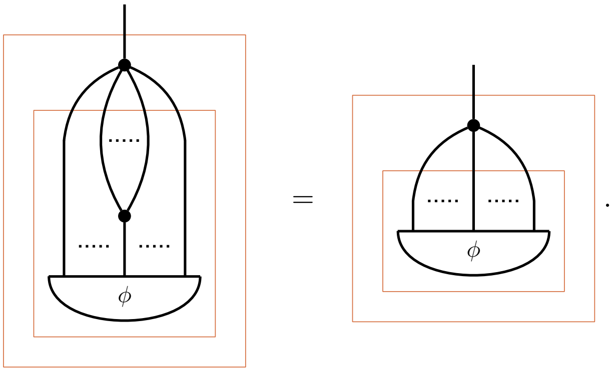
then by the Frobenius law



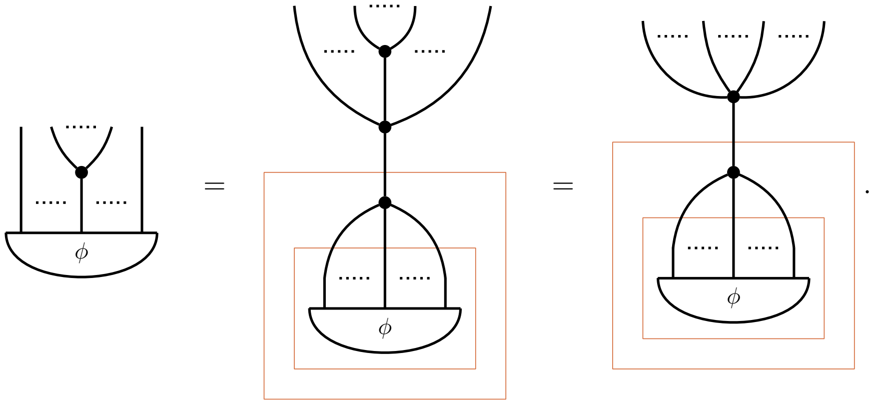
and so



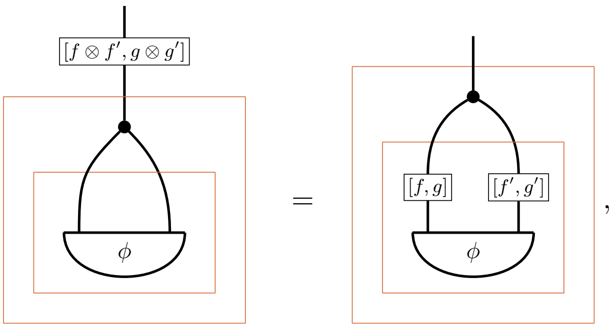
The next rule, shows that black dots can be merged in the following sense:



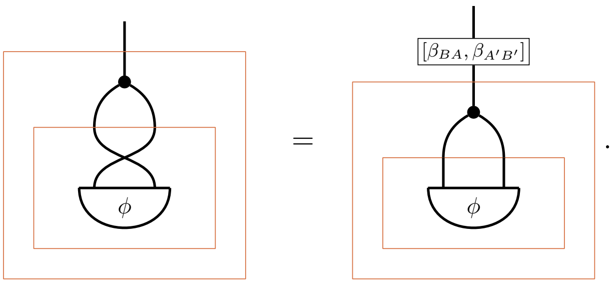
Indeed, this property follows immediately from the fact that



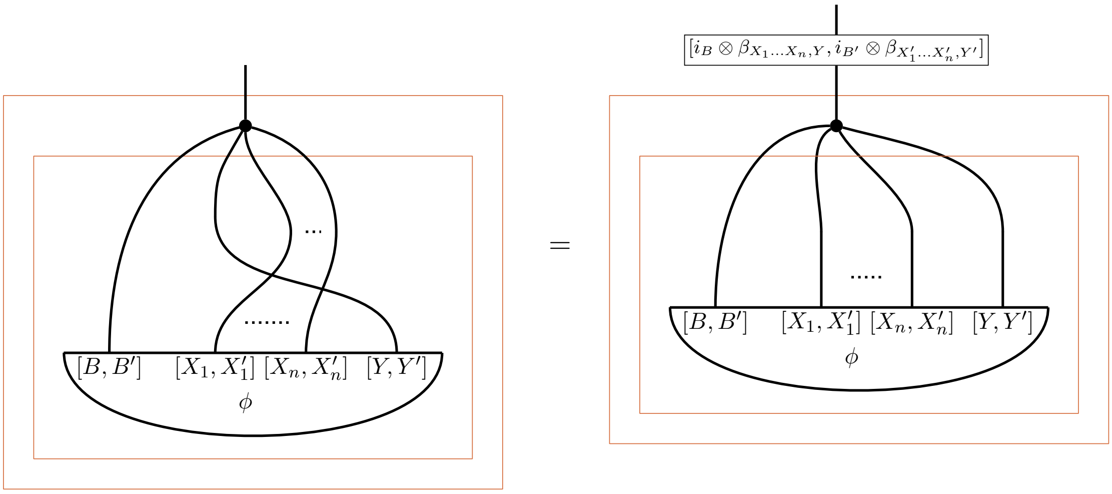
Using similar techniques, one can derive a naturality property for function-box dots



and furthermore one can derive a symmetry law for function-box dots:



The generalisation of this law which we proved for splitting maps, is also inherited to the corresponding function-box dots to give



.

Using these diagrammatic notations will greatly simplify the presentation of our main result, where we will see that in any theory of supermaps, the morphisms of type $[A, A'] \to [B, B']$ define locally-applicable transformations of the same type. No theorem, on the soundness for even string diagrams for symmetric polycategories prior to introduction of functions boxes for representability, is known to the author. As a result, the proofs and calculations of this chapter can only be seen as efficient instructions for constructing more elaborate algebraic proofs.

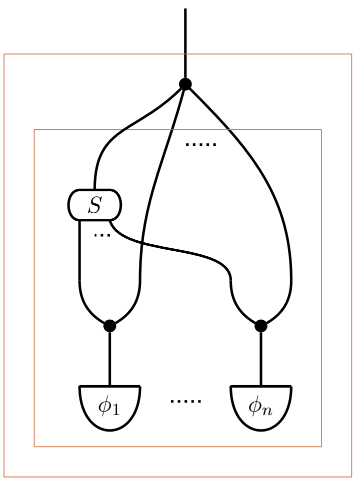
To phrase the main result of this section we will define $\mathbf{P}_{[-,=]}$ to be the sub-polycategory of $\mathbf{P}$ with only objects of the form $[A, A']$.

Theorem 9 (Theories of Supermaps Define Theories of Locally-Applicable Transformations)**.** *For every* $\mathbf{P}$*-smc* $\mathbf{C}$ *which is a theory of supermaps, there is a multifunctor* $\mathcal{F} : \mathbf{P}_{[-,=]} \to \mathbf{Lot}[\mathbf{C}]$.

Proof. On objects we take $\mathcal{F}[A, A'] = [A, A']$ and on morphisms we take

$$\mathcal{F}(S : [A_1, A_1'] \dots [A_n, A_n'] \to [B, B'])_{E_1 \dots E_n, E_1' \dots E_n'}(\phi_1, \dots, \phi_n)$$

to be given by:



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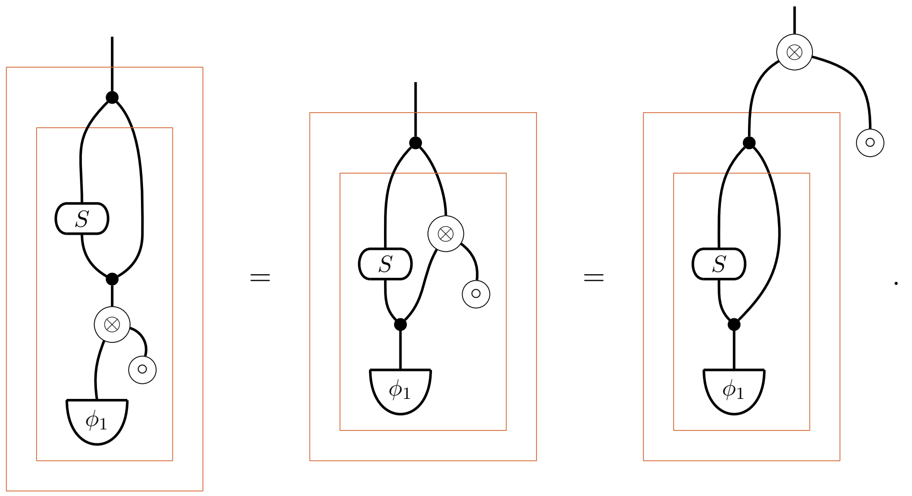
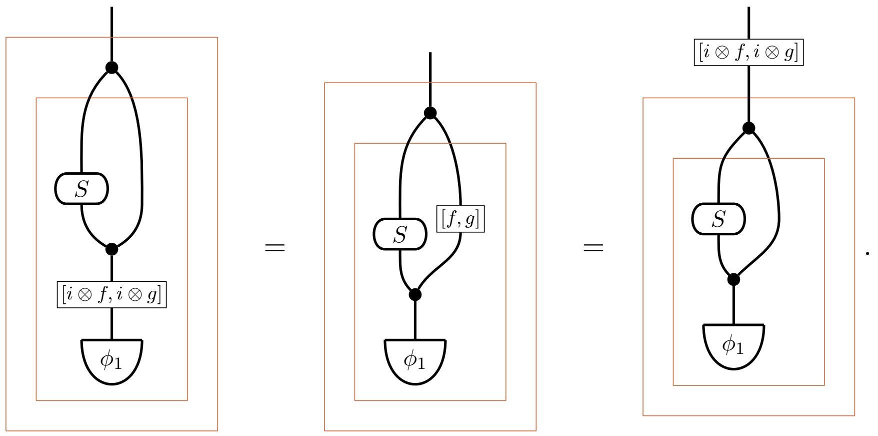
where we use cotensors to split each $\phi_i \in \mathbf{P}(\bullet, [A_i E_i, A'_i E'_i])$ into a state with two outputs of type $\mathbf{P}(\bullet, [A_i, A'_i][E_i, E'_i])$ so that we can apply S to the $[A_i, A'_i]$ part.

We now check the result is indeed a locally-applicable transformation of the same type, to do so we check commutation with all combs by separately checking commutation with sequential and parallel compositions[5]. We give the proof for one-input supermaps for simplicity, the multi-input case can then be proven by induction. In the single-input case, we have for the dragging law



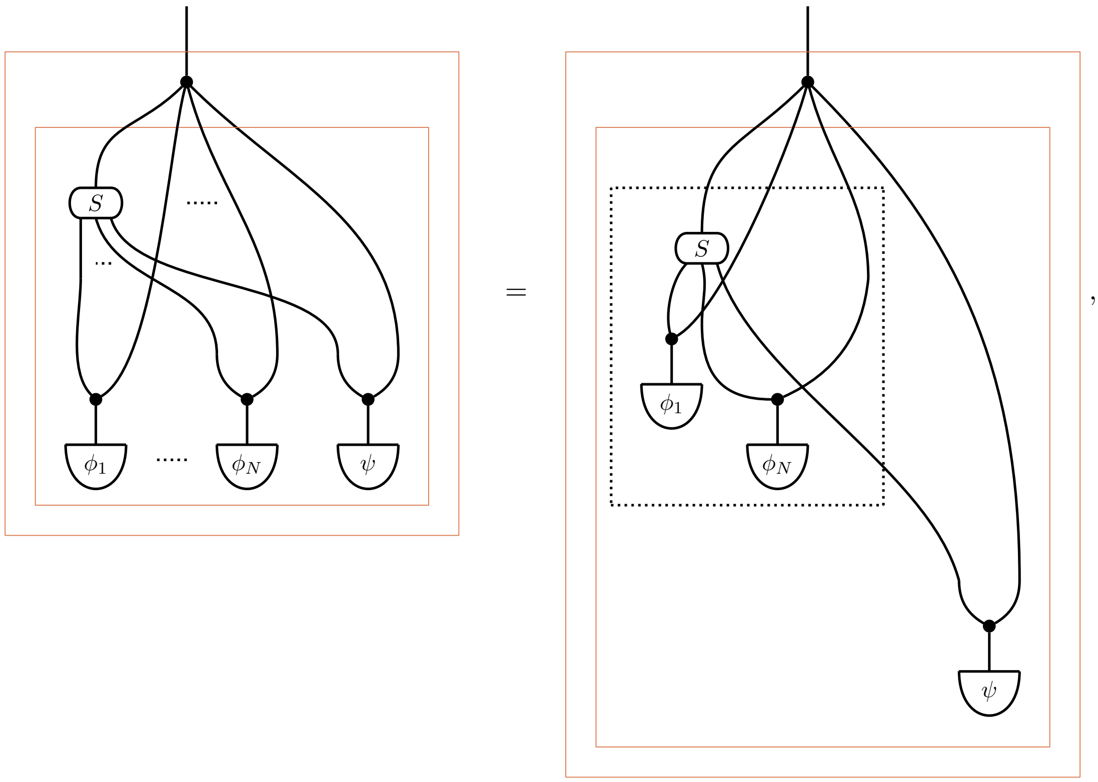
For the sliding law we have



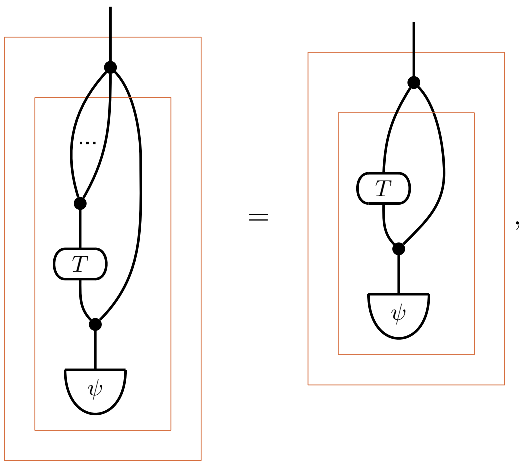
Let us now verify the multi-input case by induction, assume that the hypothesis is true for the N-input case, now consider the $N+1$ input case. A family of functions is an $N+1$-input locally-applicable transformation if filling in the first N entries gives an $N=1$ input locally-applicable transformation and filling in the last entry gives an N-input locally applicable transformation (up to applying swaps). Indeed, for the former case see that

[5] See the proof of theorem 14 for a fuller discussion of sliding and dragging rules.

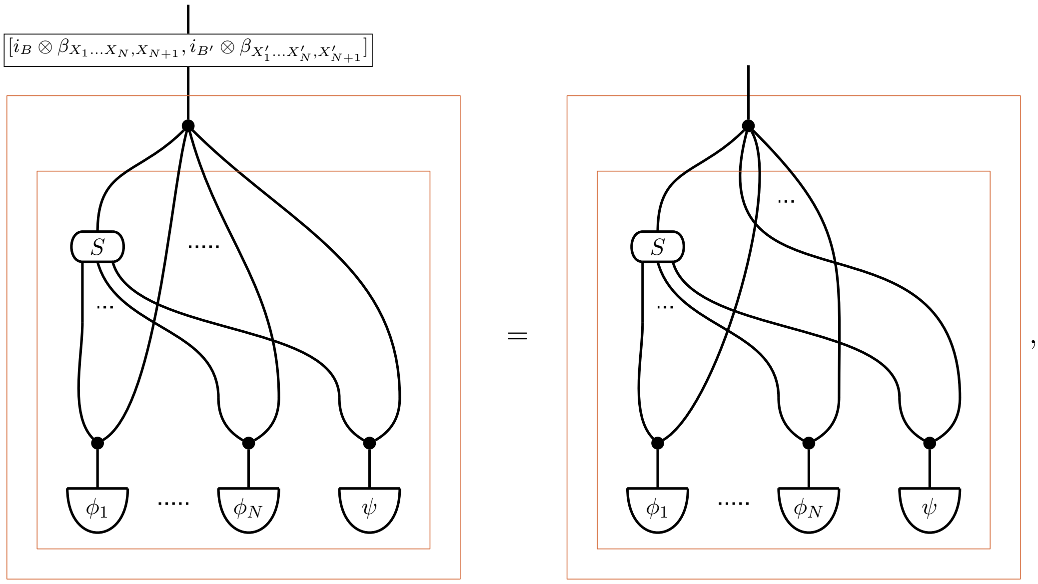
since



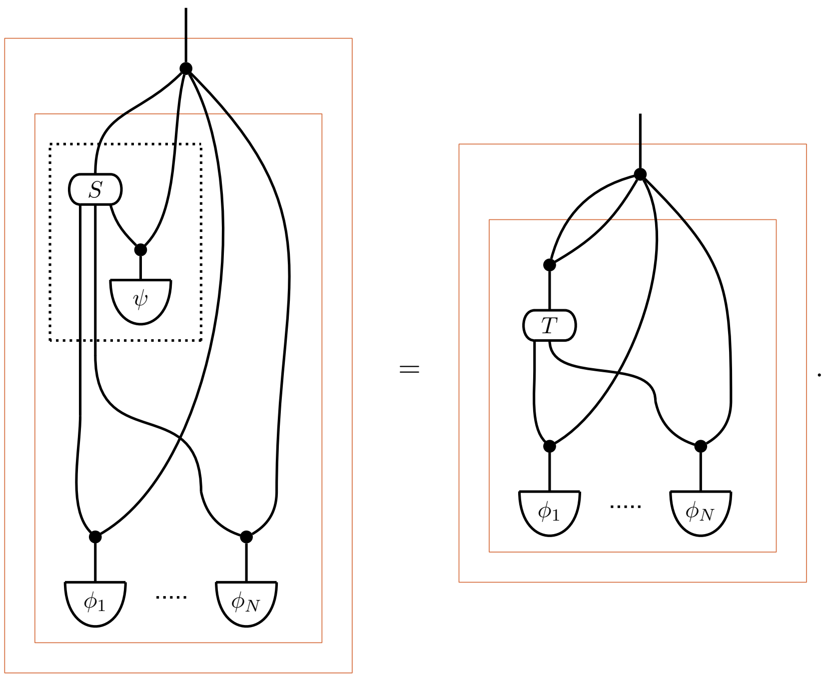
we can use representability to recover the required form



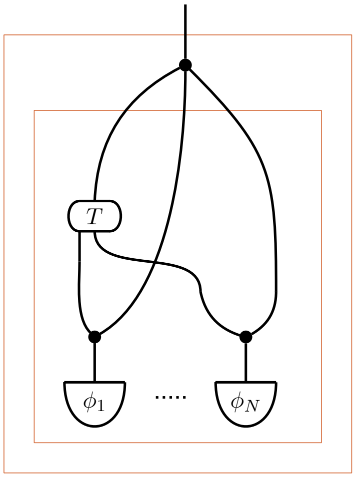
and so filling in the first N-inputs gives a 1-input locally applicable transformation. For the latter case



which by the interchange law can be rewritten as



After using the merging rule this gives



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and so filling input $N+1$ gives up to swaps an N-input locally applicable transformation by assumption of the induction hypothesis, completing the proof. Multifunctorality follows from the key diagrammatic rule for representability, merging then splitting returns the identity. □

4.6 Summary

In this section we defined locally-applicable transformations, families of functions satisfying a simple set of axioms for describing the action of a supermap on part of a bipartite process. The locally-applicable transformations on **C** furthermore form a faithful multicategory which enriches **C**, and so provides a candidate theory of supermaps with the basic features we identified as key in the previous chapter. It's not clear in general that locally-applicable transformations can be used to construct theories of supermaps, meaning enrichments into polycategories with a few extra axioms, however, we discovered that theories of supermaps always can be mapped into the theory of locally-applicable transformations. For now though, there is another way to confirm whether **Lot**[**C**] is a good model for supermaps on **C**, apply it to cases of interest and see if it recovers the original physicists' definitions. This will be the topic of the next chapter, and the main technical contribution of the thesis.

Chapter 5

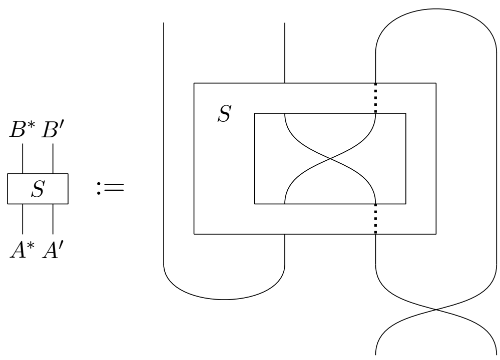
Quantum Superchannels are Characterized by Local-Applicability

We have now established a minimal requirement for supermaps, they must define locally-applicable transformations. In this chapter we find that locally-applicable transformations on the monoidal category of quantum channels are in a one-to-one correspondence with the quantum superchannels. In other words, we find that this minimum behaviour law is all that is needed to reconstruct the most commonly used notion of supermap.

Before we begin, let us see the crux of the proof in a simplified form. The crux is in-fact rather trivial, every locally-applicable transformation on the completely positive maps is representable by a completely positive map, because the completely positive maps are compact closed. The key technical difficulties in this section will be concerned with showing that every locally-applicable transformation on the quantum channels can be uniquely *extended* to a locally applicable transformation on the completely positive maps.

Theorem 10. *Locally-applicable transformations on a compact closed category* $\mathbf{C}$ *of type* $[A, A'] \to [B, B']$ *are in one-to-one correspondence with the morphisms in* $\mathbf{C}$ *of type* $A^* \otimes A \to B^* \otimes B$.

Proof. The key trick is to apply the locally-applicable transformation to the swap, to convert the intuitive functional wires into genuine wires of $\mathbf{C}$:



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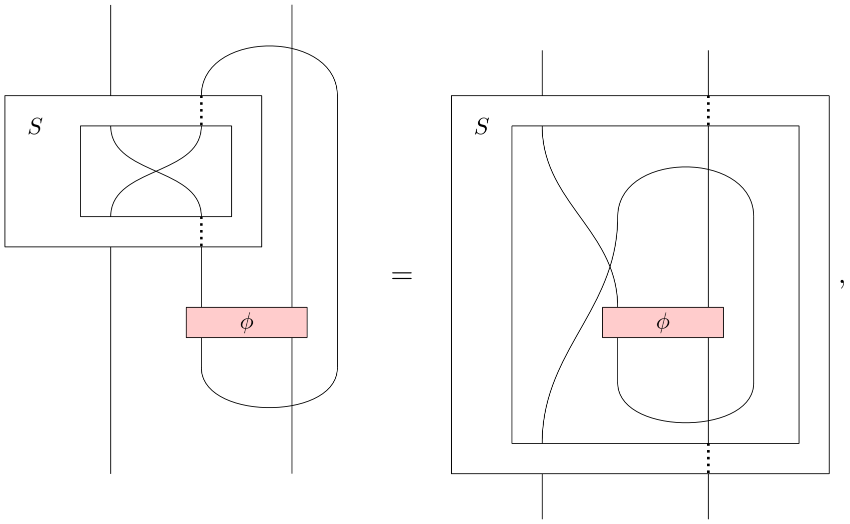
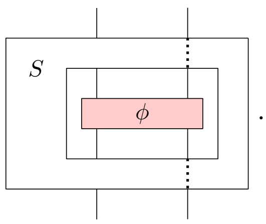
thereby sending each locally-applicable transformation to a morphism of **C**. We also know that every morphism of **C** can be sent to a locally applicable transformation on **C**, but we do not yet know if these two mappings are inverse to each-other. We confirm that they are, indeed by local-applicability:



which by compact closure gives



□

This theorem extends to multi-party locally-applicable transformations by induction.

Theorem 11. *Let* **C** *be a compact closed category, there is a one-to-one correspondence*

$$\mathbf{Lot}[\mathbf{C}]([A_1, A_1'] \dots [A_n, A_n'], [B, B']) \cong \mathbf{C}(A_1^* \otimes A_1' \dots A_n^* \otimes A_n', B^* \otimes B')$$

So, we can see two pieces of a puzzle coming together, locally-applicable transformations on compact closed categories internalise, and the quantum channels embed into a compact closed category of completely positive maps. Our goal will be to put these pieces together, showing that every locally-applicable transformation on quantum channels extends to the completely positive maps in a unique way, so that we can then use the above observations on the internalisation of locally-applicable transformations on compact closed categories.

5.1 Characterisation of Superchannels

In this section we prove that the principle of local-applicability is sufficient to characterise quantum supermaps. Concretely we show that for any locally-applicable transformation S on $\mathbf{QC}$ of type $K \to M$ with K, M satisfying two additional properties, there exists a $\mathbf{CP}$-supermap S_Q of type $K \to M$ which implements it. Formal meaning of implementation is given by existence of some S_Q such that $S = \mathcal{F}_{\mathbf{CP},\mathbf{QC}}(S_Q)$ where $\mathcal{F}_{\mathbf{CP},\mathbf{QC}}$ is the embedding from $\mathbf{CP}$-supermaps into locally-applicable transformations previously defined in the proof of lemma 4.

To show the characterisation of locally-applicable transformations as quantum supermaps we will need to establish a few preliminary lemmas on the kinds of sets of channels for which our characterisation will work.

Definition 28. *A set $K \subseteq QC(A, A')$ is termed normal if for every effects $d : A \to I$ and state $\rho : I \to A$ then the process $\rho \circ d : A \to A'$ is in K.*

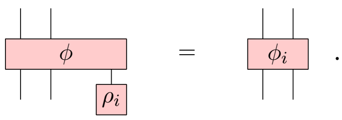
The reason that such sets are of importance to us is that the swap morphism is guaranteed to be within the dilation extension.

Lemma 6. *Let $K \subseteq \mathbf{QC}(A, A')$ be a normal set, then $\mathtt{SWAP}_{A,A'} \in \mathtt{dext}_{A',A}(K)$.*

Proof. Every process of $\mathbf{QC}(A, A')$ given by applying a state and effect to half of the swap channel, is a discard-prepare channel. □

We will also make use of the notion of convexity, we say that a subset $K \subseteq \mathbf{QC}(A, A')$ is convex if for every $f, g \in K$ and $p \in [0, 1]$ then $pf + (1-p)g \in K$.

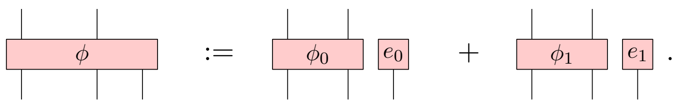
Definition 29 (Control)**.** *A set K has control if for every pair $\phi_0, \phi_1 \in \mathtt{dext}_{X,X'}(K)$ there exists $\Phi \in \mathtt{dext}_{X\otimes Y, X'\otimes Y'}(S)$ and a pair of states ρ_0, ρ_1 such that:*



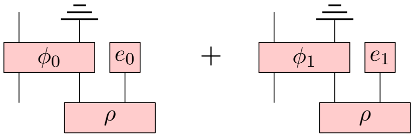
The existence of control is equivalent to asking for closure under convex combinations.

Lemma 7. *A set $K \subseteq \mathbf{QC}(A, A')$ has control if and only if it is convex.*

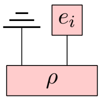
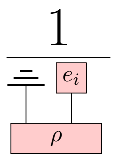
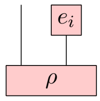
Proof. We begin by showing that whenever K is convex it has control. Note that whenever K is convex then $\mathtt{dext}_{X,X'}(K)$ is convex for every choice of X and X'. Now choose a pair of states $\rho_0, \rho_1 \in \mathbf{QC}(I, Y)$ on an object Y which are distinguishable in the sense that there exist effects $e_0, e_1 \in \mathbf{CP}(Y, I)$ satisfying $e_i \circ \rho_j = \delta_{ij}$ and then construct the following process Φ



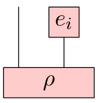
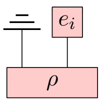
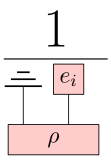
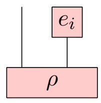
Certainly by inserting ρ_0, ρ_1 into the rightmost wire of the above process the channels ϕ_0, ϕ_1 are recovered, what remains is to show that Φ is in $\texttt{dext}_{X,X'}(K)$. Indeed consider checking the reduction of Φ given by applying an arbitrary state and effect of **QC** to its auxiliary wires, given that **QC** is causal this is given by:



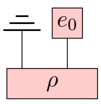
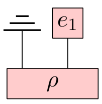
for some ρ. Now note that each of the post-selected states is a normalised state multiplied by a probability, indeed defining:

$p(e_i|\rho)$:=  and $\rho_{|i}$:=  $\times$ 

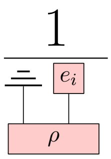
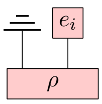
then

 =  $\times$  $\times$  = $p(e_i|\rho) \times \rho_{|i}$

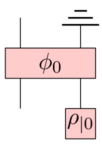
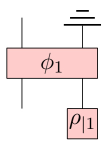
where the $p(e_i|\rho)$ are probabilities since

$p(e_0|\rho) + p(e_1|\rho)$ =  +  = 1.

Note that each $\rho_{|i}$ is a normalised density matrix since

$Tr[\rho_{|i}]$ =  $\times$  = 1.

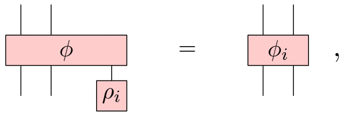
Since ϕ_0, ϕ_1 are elements of $\texttt{dext}_{X,X'}(K)$ it then follows from the above that the reduction of Φ is a convex combination of elements of K, explicitly the application of arbitrary state and effect to Φ returns:

$p(e_0|\rho)$  + $p(e_1|\rho)$ 

where $p(e_0|\rho) + p(e_1|\rho) = 1$. We now check the converse, that when K has control it is convex. Indeed, for any $\phi_0, \phi_1 \in K$ choose their control operation $\Phi \in \texttt{dext}_{Y,I}(K)$. Consider an arbitrary convex combination $p\phi_0 + (1-p)\phi_1$, this combination is given by inserting $\sigma := p\rho_0 + (1-p)\rho_1$ into the wire Y of Φ. Since Φ is in $\texttt{dext}_{Y,I}$ then insertion of σ into Φ must return an element of K and so it follows that the convex combination $p\phi_0 + (1-p)\phi_1$ is an element of K. □

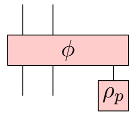
Lemma 8 (Convex linearity)**.** *Let K, M be convex, then every locally-applicable transformation of type $S : K \to M$ on* **QC** *preserves convex combinations.*

Proof. Consider a pair $\phi_i \in \texttt{dext}_{X,X'}(K)$ of channels, since K is convex it has control, there exists $\phi \in \texttt{dext}_{X,X'}(K)$ such that



and so an arbitrary convex combination $p\phi_0 + (1-p)\phi_1$ can be written as

$$p\phi_0 + (1-p)\phi_1 \quad = $$



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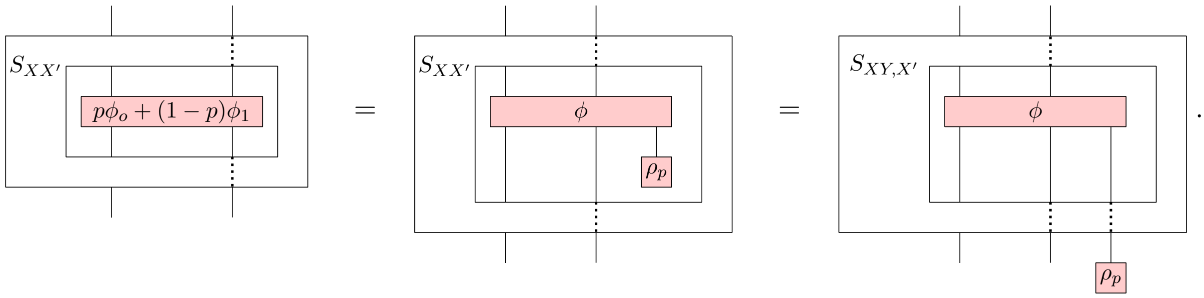
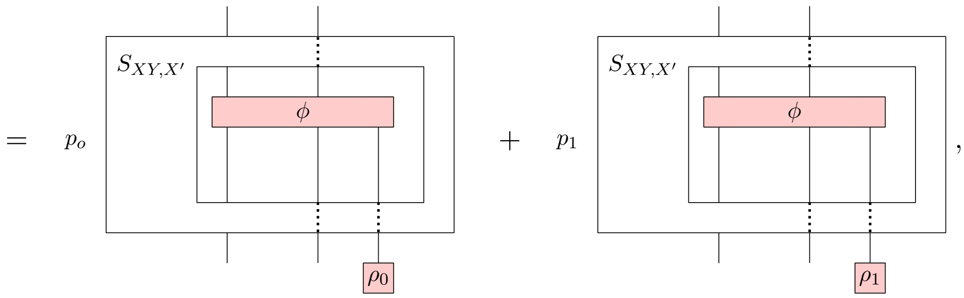
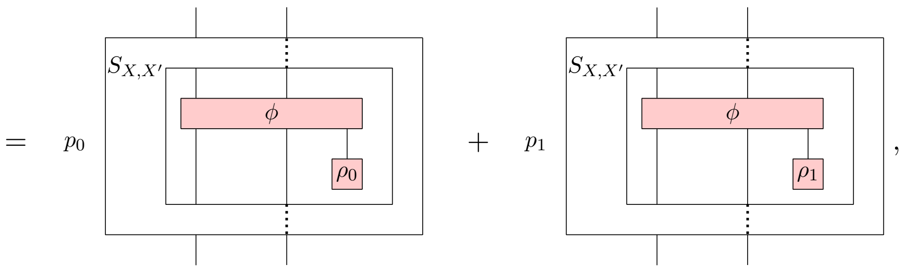
with $\rho_p := p\rho_0 + (1-p)\rho_1$. Writing $p_0 = p$ and $p_1 = (1-p)$, then $S_{X,X'}(p_0\phi_0 + p_1\phi_1)$ is given using naturality by:



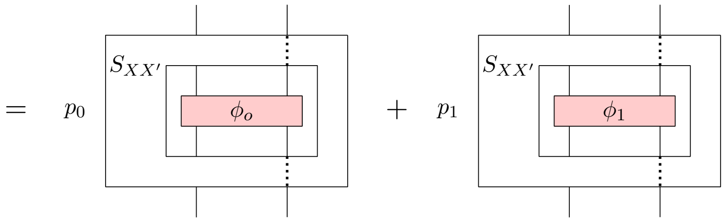
Rewriting ρ in terms of ρ_0, ρ_1 gives



then using naturality again



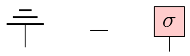
and finally using the definition of control recovers the result



$$= p_0 S_{X,X'}(\phi_0) + p_1 S_{X,X'}(\phi_1).$$

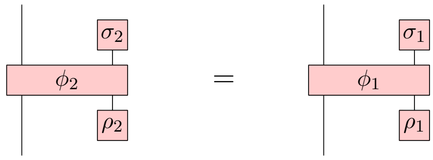
∎

From this point follows a technical lemma, intuitively concerned with the existence of a unique extension of each $S_{XX'}$ to action on all completely positive maps. We will use the notation $\mathbf{CP}_{\leq}(X', I)$ for the set of all $\sigma \in \mathbf{CP}(X', I)$ such that

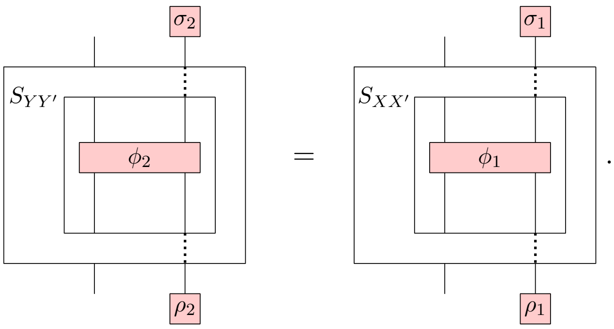


$$\in \mathbf{CP}(X', I).$$

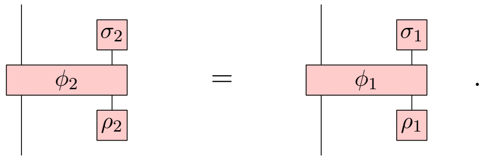
Lemma 9 (Extension to Completely Positive maps)**.** *Let K, M be convex, for any locally-applicable transformation of type $S : K \to M$ and pair of triples $(\rho_1, \phi_1, \sigma_1) \in \mathbf{QC}(I, X) \times K(X, X') \times \mathbf{CP}_{\geq}(X', I)$, $(\rho_2, \phi_2, \sigma_2) \in \mathbf{QC}(I, Y) \times K(Y, Y') \times \mathbf{CP}_{\geq}(Y', I)$ such that*



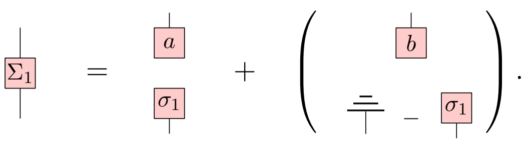
then



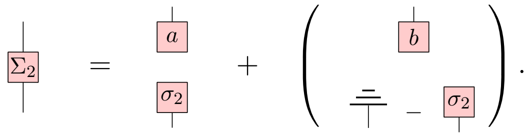
Proof. Let



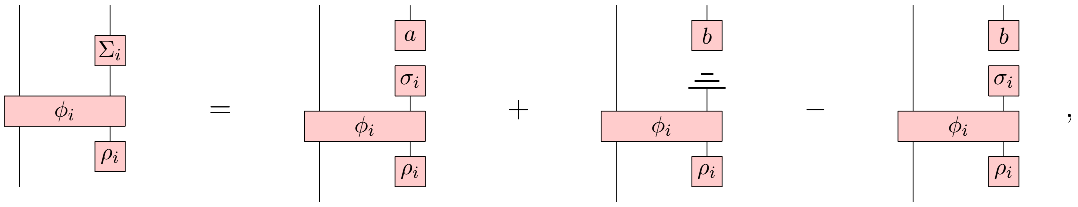
Consider some Hilbert space Q of dimension at least 2 so that there exists two orthogonal states $|a\rangle, |b\rangle \in Q$, then define the process $\Sigma_1 : X' \to Q$ by



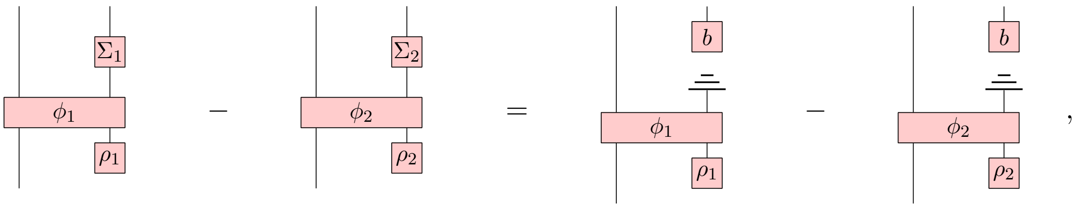
A process $\Sigma_2 : Y' \to Q$ can be defined similarly as



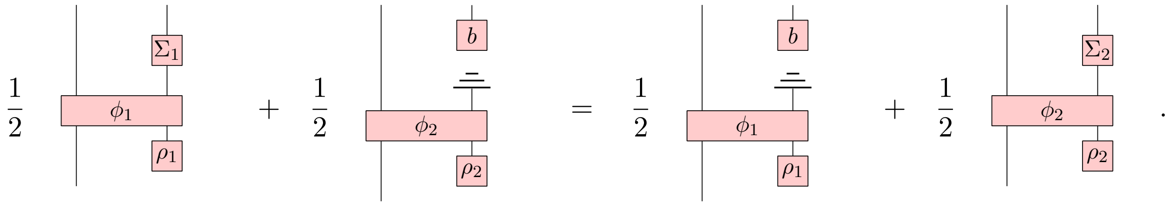
Note that both Σ, Σ' are trace preserving and so are members of **QC**, meaning that they can be slid along dotted wires. We now consider the result of applying them to ϕ_1, ϕ_2:



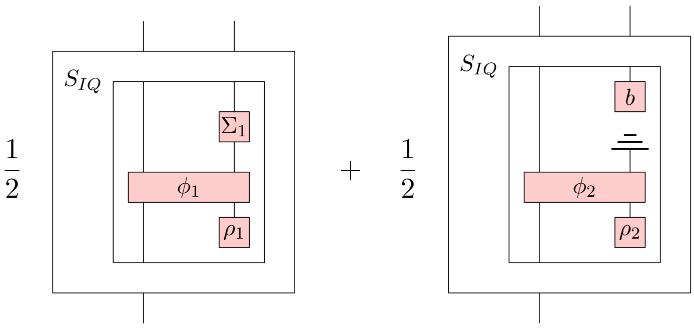
this in turn implies that



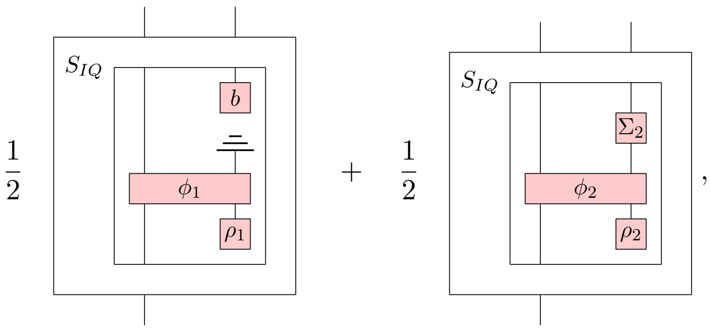
and so that



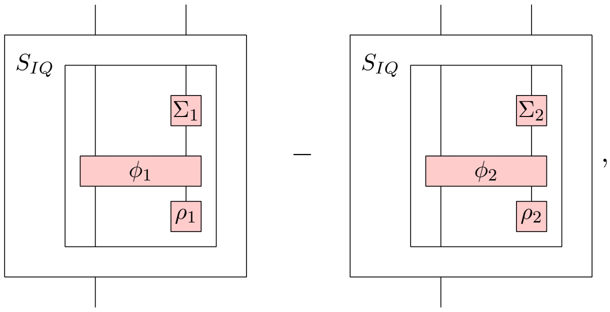
By convex linearity of each $S_{ZZ'}$ we can then conclude that



is equal to

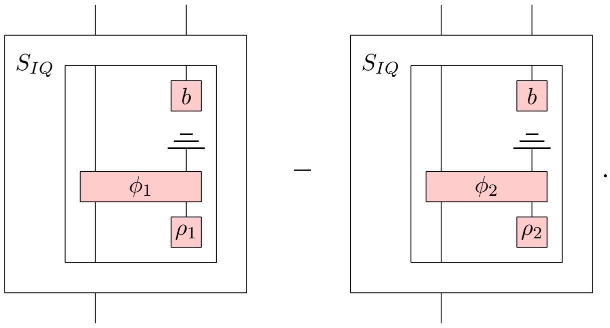


and so that



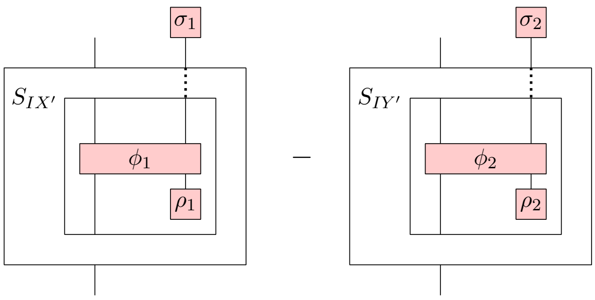
(5.1)

is equal to

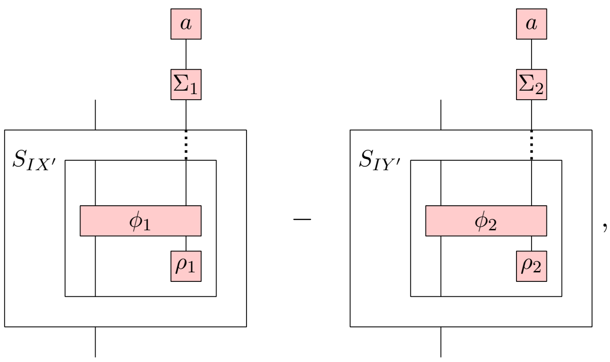


(5.2)

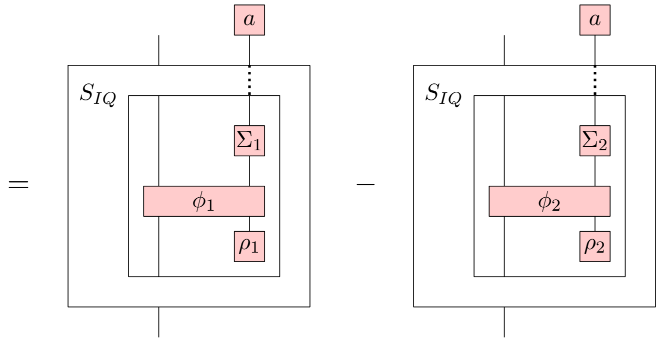
We will make use of the above to show that the following expression:



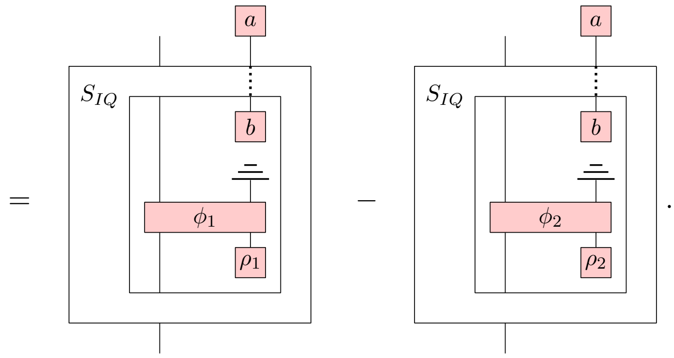
is 0. Indeed by definition of Σ_1 and Σ_2 the above expression is equal to



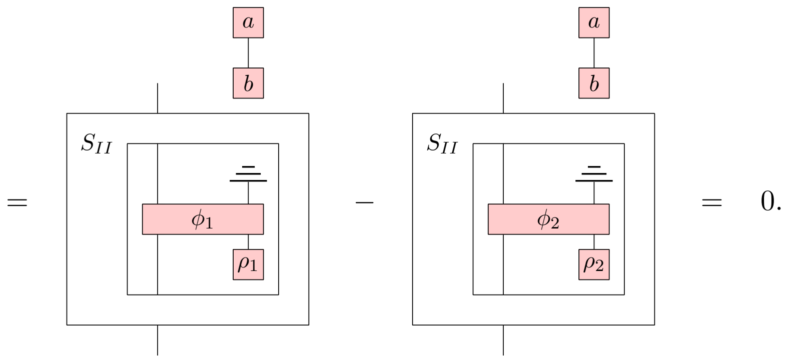
and since Σ_1, Σ_2 *are* quantum channels they can be pulled through dotted lines



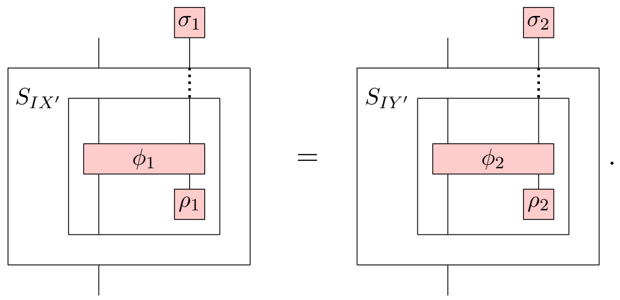
after which we make use of the equality of expressions 5.1 and 5.2:



Finally, since the preparation $b : I \to X'$ is a quantum channel, it can be pulled back through dotted wires, after which orthogonality implies that the difference has to be 0:



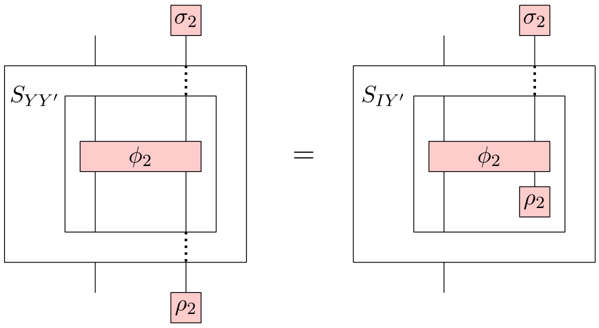
Since the difference is 0 then it follows that



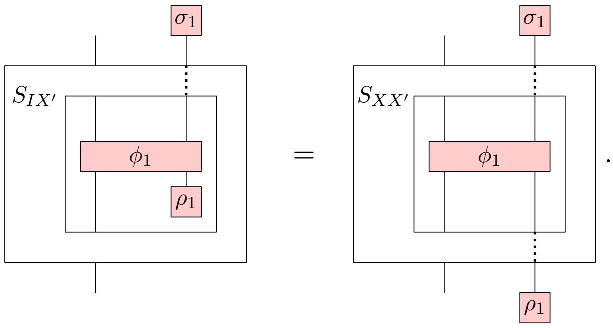
(5.3)

We can now consider the bottom side which is easier to reason with. Since every $\rho \in$

$\mathbf{QC}(I, X)$ we can write



which by using equation 5.3 is equal to

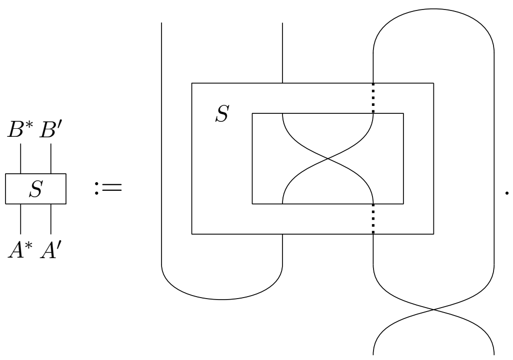


□

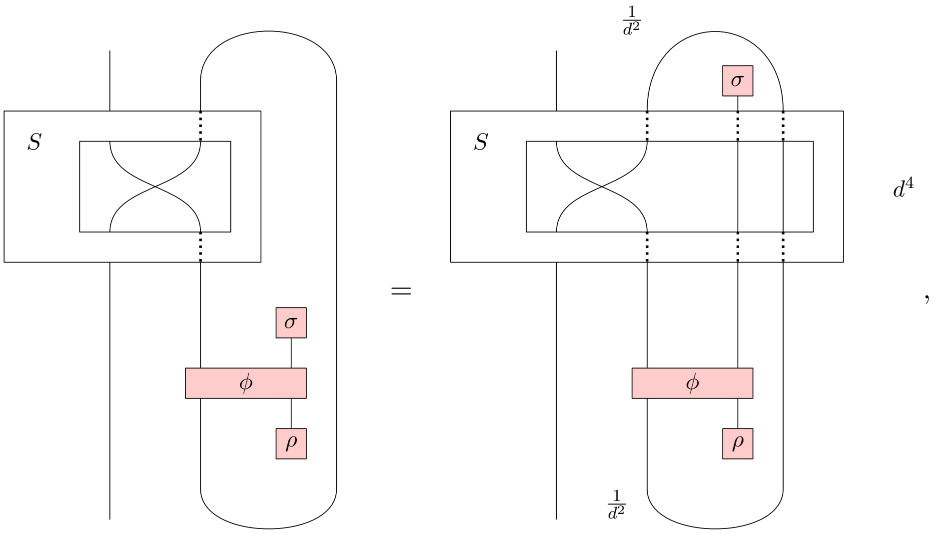
The above is the key to our result, we now are ready to construct a candidate quantum supermap for simulating the action of our locally-applicable transformation $S_{X,X'}$ by tensor extension with identities on X, X'. To do so we apply our locally-applicable transformation S to the swap-morphism, the intuition being that the swap gives a way to noiselessly extract information about the input behaviour of a higher order map, by converting its input into a pair of lower-order objects.

Theorem 12 (Re-characterisation of supermaps)**.** *Let K, M be normal convex subsets of channels of* $\mathbf{QC}$*, there is a one-to-one correspondence between quantum supermaps of type $K \to M$ and locally-applicable transformations of the same type.*

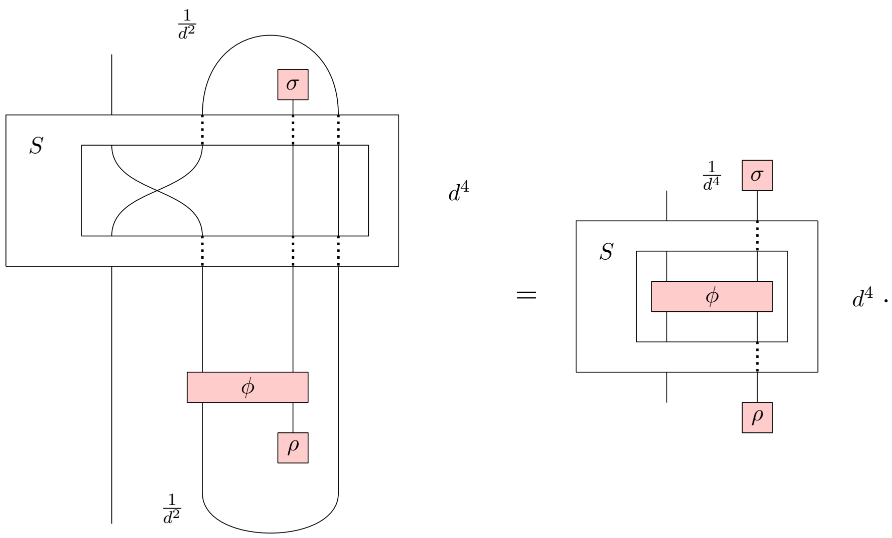
Proof. Given a locally-applicable transformation S of type $K \to M$ on $\mathbf{QC}$ with $K \subseteq \mathbf{QC}(A, A')$ and $M \subseteq \mathbf{QC}(B, B')$ we define $S_Q : A^* \otimes A' \to B^* \otimes B'$ by:



In other words we apply S to the swap in $\mathbf{QC}$ and then embed into $\mathbf{CP}$ so that we may apply caps and cups, note that normality of K, M is required here to ensure that the swap lives within their dilation extensions. We now consider the application of arbitrary states and effects ρ, σ in $\mathbf{CP}$ to the auxiliary wires with ρ a normalised state and σ a sub-normalised effect in $\mathbf{CP}_{\geq}(X', I)$. First a direct use of local applicability gives us

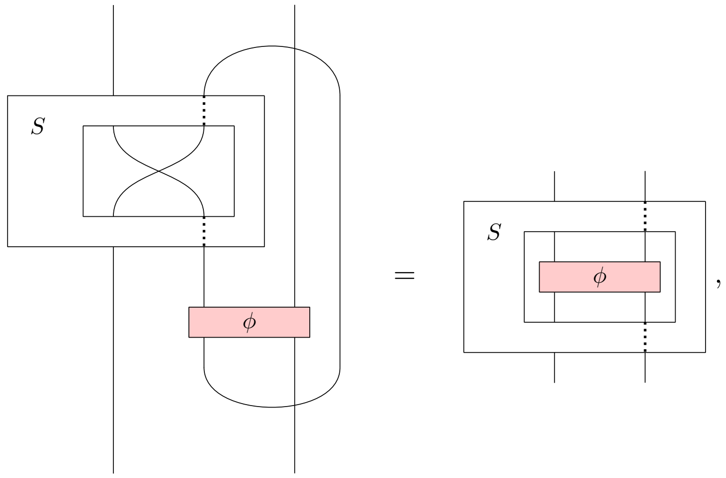


after-which using lemma 9 we can say that



Since this is true for all $\rho \in \mathbf{QC}(I, X), \sigma \in \mathbf{CP}_{\leq}(X', I)$ it is true for all $\rho \in \mathbf{CP}(I, X)$ and

$\sigma \in \mathbf{CP}(X', I)$, from here it follows that:



where we have made use of the local well-pointedness of **CP** which follows from combining its well-pointedness and compact closure as in [126]. To conclude, there indeed exists a quantum supermap of type $S_Q : K \to M$, such that $\mathcal{F}_{\mathbf{CP}}(S_Q) = S$ where $\mathcal{F}_{\mathbf{CP}}$ is the embedding of lemma 4 from **CP**-supermaps on **QC** into locally-applicable transformations on **QC**. □

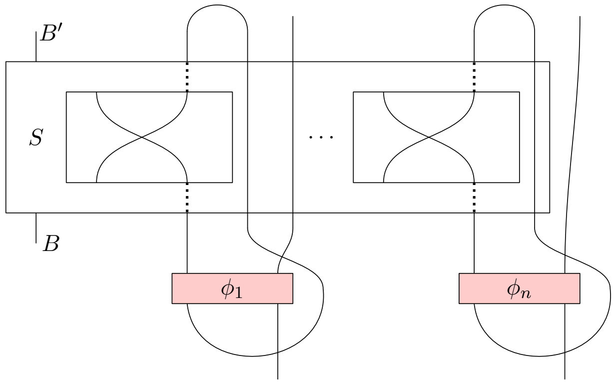
Let us now turn our attention to the multi-party case. Rather than going from the beginning we can work inductively by noting that any multi-input locally-applicable transformation defines a single-input locally-applicable transformation when all but one of its variables are fixed.

Corollary 1. *let $K_1, \ldots, K_n, M$ be convex sets of morphisms of* **QC**, *there is a one-to-one correspondence between* **CP**-*supermaps of type $K_1 \ldots K_n \to M$ and locally-applicable transformations of the same type.*

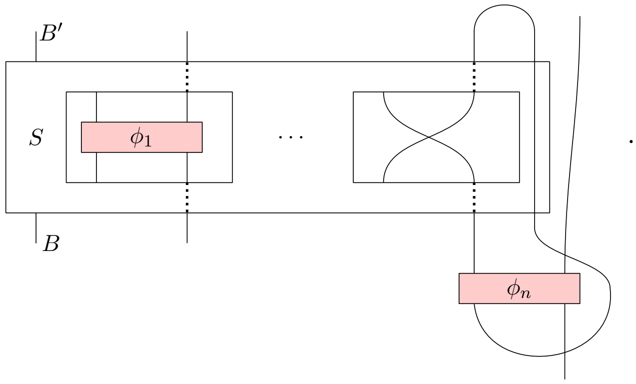
Proof. That **CP**-supermaps still give locally-applicable transformations follows from multiple uses of the interchange law for symmetric monoidal categories. What remains is to prove that every locally-applicable transformation is implemented by a **CP**-supermap. Up to braiding the family of functions given by

$$S((-)\psi_2 \ldots \psi_n)_{X_1, X_1'}(\psi_1) := S(\psi_1 \ldots \psi_n)_{X_1 \ldots X_n, X_1' \ldots X_n'}$$

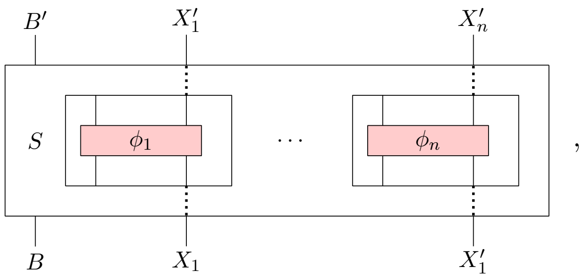
is a locally-applicable transformation with one-input, consequently we can use our main theorem to show that



is equal to



Repeating this step for each consecutive input from 2 to n returns



which completes the proof. □

5.1.1 Characterisation of Classical Supermaps

We will now see that the basic structural features of the category of quantum channels which we used to characterise the locally-applicable transformations, are also present in the category of classical quantum channels, or stochastic matrices.

Lemma 10. *A subset $K \in$ **Stoch**(A, A') is convex if and only if it has control*

Proof. All that was required to construct the proof was the existence of an object Y with a pair of distinguishable states in the sense that $e_i \circ \rho_j = \delta_{ij}$ and the possibility to take positive sums. Sums are taken care of by **Mat**$[\mathbb{R}_+]$ and Y may be taken to be 2. Indeed one can define $\rho_i : 1 \to 2$ by taking the k^{th} component of the column vector ρ_i to be δ_{ik} and similarly for the effects $e_j : 2 \to 1$. □

Theorem 13. *For K, M convex in* **Stoch** *there is a one-to-one correspondence between the* $\mathbf{Mat}[\mathbb{R}_+]$*-supermaps of type $K \to M$ on* **Stoch** *and the locally-applicable transformations of type $K \to M$ on* **Stoch**.

Proof. The proof is a direct copy of the proof for convex subsets of quantum channels. Let us briefly outline the required commonalities between **Stoch** and **QC** and between **CP** and $\mathbf{Mat}[\mathbb{R}_+]$ needed to get the proof to work. First, the equivalent between convexity and control implies convexity of locally-applicable transformations. Second, $\mathbf{Mat}[\mathbb{R}_+]$ embeds into $\mathbf{Mat}[\mathbb{R}]$ so that subtractions can be defined, and that for every effect $\sigma \in \mathbf{Mat}[\mathbb{R}_+](A, I)$ there exists $\lambda \in \mathbb{R}_+$ and $\sigma' \in \mathbf{Mat}$ such that $\lambda\sigma + \sigma'$ is in **Stoch**. Finally, $\mathbf{Mat}[\mathbb{R}_+]$ is compact closed and well-pointed. □

5.2 Quantum Supermaps are Natural Transformations

In this section we will show that the space of quantum superchannels corresponds to one of the most fundamental categorical concepts, they are the space natural transformations between well-chosen functors. In the preliminary material we introduced the hom-functor $\mathbf{C}(-, =) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$. When $\mathbf{C}$ is a monoidal category, the hom-functor can be generalised to a functor of type

$$\mathbf{C}(A \otimes -, A' \otimes =) : \mathbf{C}^{op} \times \mathbf{C} \longrightarrow \mathbf{Set},$$

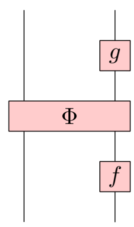
which assigns to each (X, X') the object $\mathbf{C}(A \otimes X, A' \otimes X')$ and to each morphism $f : Y \to X$ and each morphism $g : X' \to Y'$ the function

$$\mathbf{C}(A \otimes f, A' \otimes g) : \mathbf{C}(A \otimes X, A' \otimes X') \to \mathbf{C}(A \otimes Y, A' \otimes Y')$$
$$\mathbf{C}(A \otimes f, A' \otimes g) :: \phi \mapsto (i \otimes g) \circ \phi \circ (i \otimes f).$$

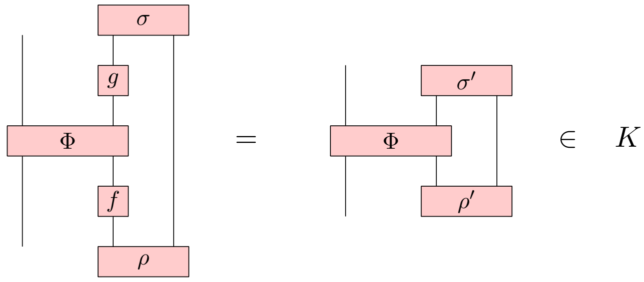
This functor can be further generalised to a functor $\mathtt{dext}(K)(-, =)$, which we now define.

Definition 30 (Extension functor)**.** *For every $K \subseteq \mathbf{C}(A, A')$ in a symmetric monoidal category $\mathbf{C}$ one can define a functor $\mathtt{dext}(K)(-, =) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ given by*

- $\mathtt{dext}(K)(X, X') := \mathtt{dext}_{X,X'}(K)$
- $\mathtt{dext}(K)(f, g) : \mathtt{dext}_{X,X'}(K) \to \mathtt{dext}_{Y,Y'}(K)$ *defined by*

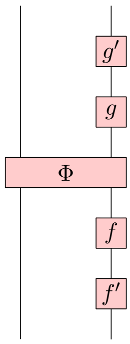
$\mathtt{dext}(K)(f, g)(\phi) :=$ 

The functor $\mathbf{C}(A \otimes -, A' \otimes =)$ can be defined as the special case given by $\texttt{dext}(\mathbf{C}(A, A'))$. $\texttt{dext}$ is well defined, whenever $\phi \in \texttt{dext}_{X,X'}(S)$ then $\texttt{dext}(S)(f,g)(\phi) \in \texttt{dext}_{Y,Y'}(S)$ since for each f, g and ρ, σ then



since $\phi \in \texttt{dext}_{X,X'}(K)$. Furthermore the assignment $\texttt{dext}(S)(f,g)$ is functorial since

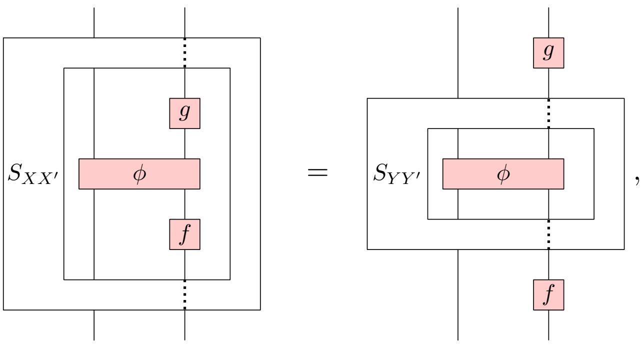
$\texttt{dext}(S)(f',g')(\texttt{dext}(S)(f,g)(\phi)) \quad =$



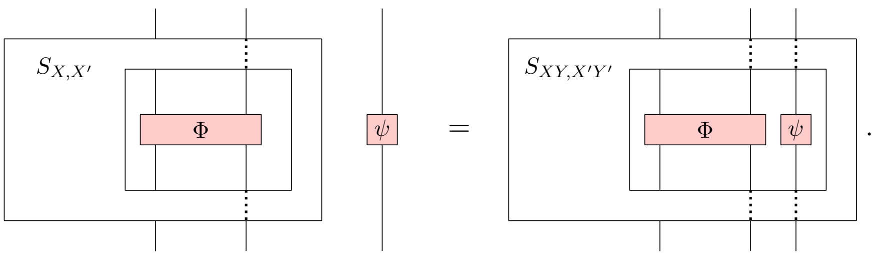
$= \quad \texttt{dext}(S)(f \circ f', g' \circ g)(\phi)$

Theorem 14 (Quantum Superchannels are Equivalent to Natural Transformations)**.** *There is a one-to-one correspondence between quantum supermaps of type* $[A, A'] \to [B, B']$ *and natural transformations of type* $\mathbf{C}(A \otimes -, A' \otimes =) \to \mathbf{C}(B \otimes -, B' \otimes =)$.

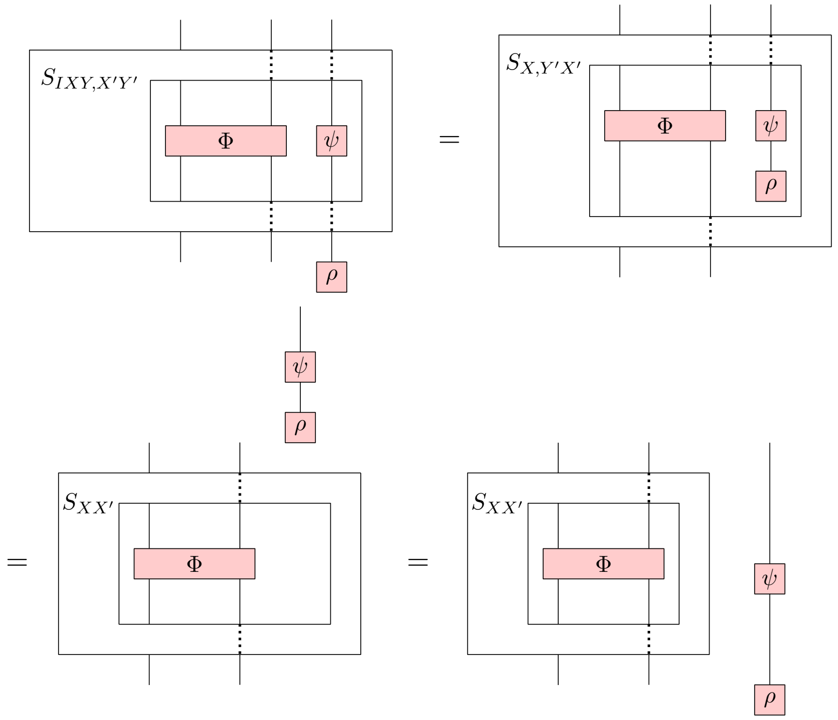
Proof. The definition of locally-applicable transformation can be split into two separate parts which we will call *sliding*



and *dragging*



In the monoidal category of quantum channels, sliding implies dragging, this is because for all causal ρ assuming sliding gives:



Together these entail box-dragging since quantum theory has *enough causal states* [126]. We will now show that a natural transformation of type $S : \mathbf{C}(A \otimes -, A' \otimes =) \longrightarrow \mathbf{C}(B \otimes -, B' \otimes =)$ is exactly a family of functions which satisfies sliding. Indeed, a natural transformation $S : \mathbf{C}(A \otimes -, A' \otimes =) \longrightarrow \mathbf{C}(B \otimes -, B' \otimes =)$ will be any family of functions $S_{XX'}$ making the following diagram commute for all f, g:

$$\begin{array}{ccc} \mathbf{C}(A \otimes X, A' \otimes X') & \xrightarrow{S_{XX'}} & \mathbf{C}(B \otimes X, B' \otimes X') \\ {\scriptstyle \mathbf{C}(A \otimes f, A' \otimes g)} \downarrow & & \downarrow {\scriptstyle \mathbf{C}(B \otimes f, B' \otimes g)} \\ \mathbf{C}(A \otimes Y, A' \otimes Y') & \xrightarrow[S_{YY'}]{} & \mathbf{C}(B \otimes Y, B' \otimes Y') \end{array}$$

In other words such that $S_{Y,Y'} \circ \mathbf{C}(B \otimes f, B' \otimes g) = \mathbf{C}(B \otimes f, B' \otimes g) \circ S_{X,X'}$. Evaluated on processes ϕ, this condition reads $S_{Y,Y'}(\mathbf{C}(B \otimes f, B' \otimes g)(\phi)) = \mathbf{C}(B \otimes f, B' \otimes g)(S_{X,X'}(\phi))$, which unpacking the definition of $\mathbf{C}(B \otimes f, B' \otimes g)$ is precisely the *sliding* rule. □

This observation extends to general subsets, so that a locally-applicable transformation of type $S : K \to M$ in $\mathbf{QC}$ is exactly a natural transformation of type $S_{X,X'} : \mathtt{dext}_{X,X'}(K) \to \mathtt{dext}_{X,X'}(M)$. Locally applicable transformations of type $K^1 \dots K^n \to M$ on $\mathbf{QC}$ can similarly be phrased as natural transformations of type

$$\bigtimes_{i=1}^{n} \mathtt{dext}_{-_i,=_i}(K^i) \longrightarrow \mathtt{dext}_{\bigotimes_{i=1}^{n} -_i, \cdots \bigotimes_{j=1}^{n} =_i}(M)$$

where for any $\mathcal{F} : \mathbf{C}_1 \to \mathbf{C}_2$ and $\mathcal{G} : \mathbf{D}_1 \to \mathbf{D}_2$ the product functor $\mathcal{F} \times \mathcal{G} : \mathbf{C}_1 \times \mathbf{D}_1 \to \mathbf{C}_2 \times \mathbf{D}_2$ is defined by $(\mathcal{F} \times \mathcal{G})(c, d) = (\mathcal{F}(c), \mathcal{G}(d))$ and similarly on morphisms.

5.3 Summary

A construction **Lot**$[-]$ exists which:

- Is motivated in terms of an easy to argue bare-minimum requirement for supermaps.
- Always has sequential and parallel composition supermaps.
- Sends the symmetric monoidal category **QC** of quantum channels to the superchannels.

In short, it appears that supermaps have been generalised to all symmetric monoidal categories.

However, some pesky questions rear their heads. First, can locally-applicable transformations be given multiple outputs and a polycategorical composition rule, as the superchannels and superunitaries can, and as is suggested by the intuitive picture of a supermap? Second, what happens when **Lot**$[-]$ is applied to the category of unitaries, do we recover the superunitaries? It turns out the answer to both questions is no, and so, the story is not over yet.

Chapter 6

How to Construct Theories of Black-Box Supermaps

Throughout this thesis we have considered two kinds of goals for definitions of supermaps as outlined in the following bullet points.

- **Goal 1:** When applied to symmetric monoidal categories of interest in quantum theory, they recover the standard-usage quantum supermaps [17] for those categories
- **Goal 2:** When applied to any symmetric monoidal category, they can be equipped with some key compositional structures

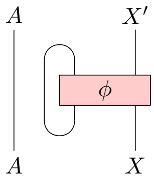
For the first goal, we would like our theories to recover the superchannels when applied to the quantum channels, and recover the superunitaries when applied to the unitaries. For the second goal, we would like to see polycategorical and enriched structures. Previously, we made some progress towards these goals using locally-applicable transformations. We found that the construction $\mathbf{Lot}[-]$ sends each symmetric monoidal category $\mathbf{C}$ to a symmetric *multicategory* which enriches $\mathbf{C}$. We also found that when applied to the symmetric monoidal category $\mathbf{QC}$ of quantum channels then $\mathbf{Lot}[\mathbf{QC}]$ returns the superchannels.

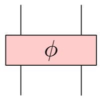
In this chapter we will begin by arguing that to make further progress we need to adapt locally-applicable transformations to define *polyslots* on symmetric monoidal categories. We will begin by identifying some problematic locally-applicable transformations on the symmetric monoidal category of unitaries, and then move on to defining polyslots and their polycategorical composition rules, finishing by proving characterisation of the superunitaries.

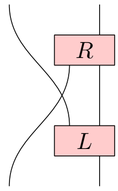
6.1 Locally-applicable Transformations which are not Superunitaries

Here we are going to write down two locally-applicable transformations on the unitaries which cannot be constructed from superunitaries and prevent us from constructing a polycategorical composition rule which satisfies the interchange law. Our first example is one which applies a trace [247] to unitaries conditional on their decompositional structure.

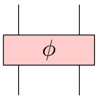
Definition 31. *The locally-applicable transformation $S^{loop} : [A, A] \to [A, A]$ on the symmetric monoidal category* $\mathbf{U}$ *is defined by taking $S^{loop}_{XX'}(\phi)$ to be*

:= 

if ∃ unitaries R, L s.t 

= 

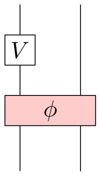
(6.1)

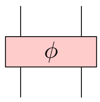
:= 

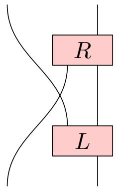
if else (6.2)

Our second example is one which applies a post-processing conditional on the same decompositional structure.

Definition 32. *The locally-applicable transformation $S^{V} : [A, A] \to [A, A]$ on the symmetric monoidal category* $\mathbf{U}$ *is defined by taking $S^{V}_{XX'}(\phi)$ to be*

:= 

if ∃ unitaries R, L s.t 

= 

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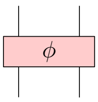
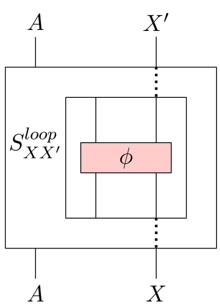
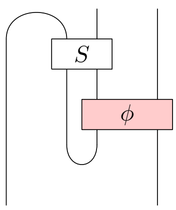
:= 

if else (6.4)

As we suggested in the introduction, these locally-applicable transformations are not superunitaries, formally we mean that they are not in the image of the functor $\mathcal{F}_{\mathbf{U}}$: $\mathbf{uQS} \to \mathbf{Lot}[\mathbf{U}]$ which assigns a locally-applicable transformation to each superunitary.

Lemma 11. *Let $S : [A, A] \to [A, A]$ be a quantum superunitary such that*

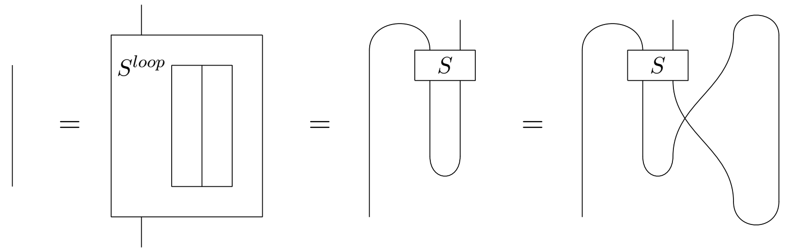


= 

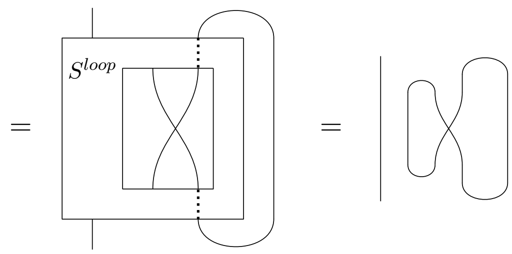
,

then $A \cong I$.

Proof. Assuming the premise of the lemma, consider an arbitrary object A and its associated identity morphism id_A, then:



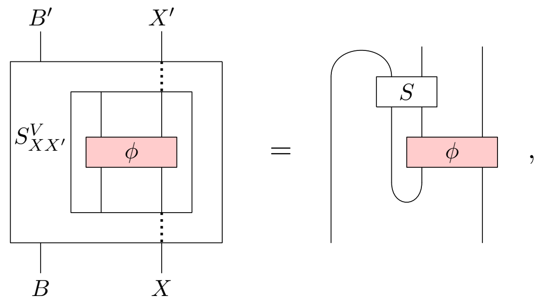
(6.5)



(6.6)

The loop in **fHilb** is the dimension d of the Hilbert space, which gives a contradiction unless $d_A = 1$ [22]. Consequently, $A \cong \mathbb{C}$. □

Lemma 12. *Let $S : [A, A] \to [A, A]$ be a quantum superunitary such that*

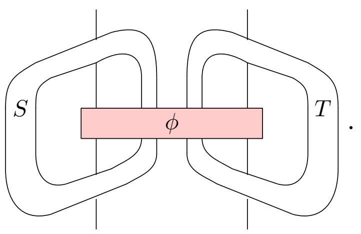


then $V = id_A$.

Proof. This follows from the same sequence of steps used to prove lemma 11. □

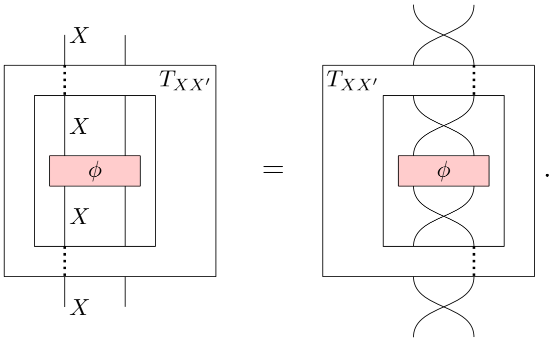
Let us now consider the issue with trying to construct a polycategorical composition rule. The issue is a failure of interchange law and can be seen even when considering single-input maps, in which case the issue is really one of trying to construct a monoidal category of single-input locally-applicable transformations.

Intuitively we imagine that given access to a bipartite process $\phi : A \otimes B \to A' \otimes B'$, one could imagine applying some supermap $S \boxtimes T$ which represents acting with $S : [A_1, A'_1] \to [A_2, A'_2]$ on the left hand side and with $T : [B_1, B'_1] \to [B_2, B'_2]$ on the right hand side:

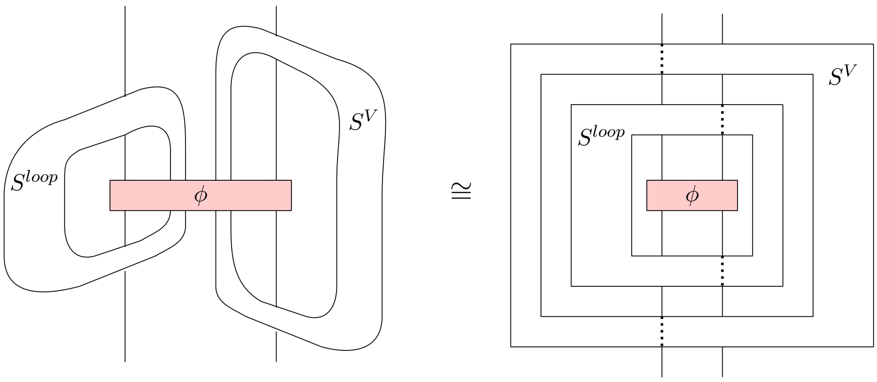


.

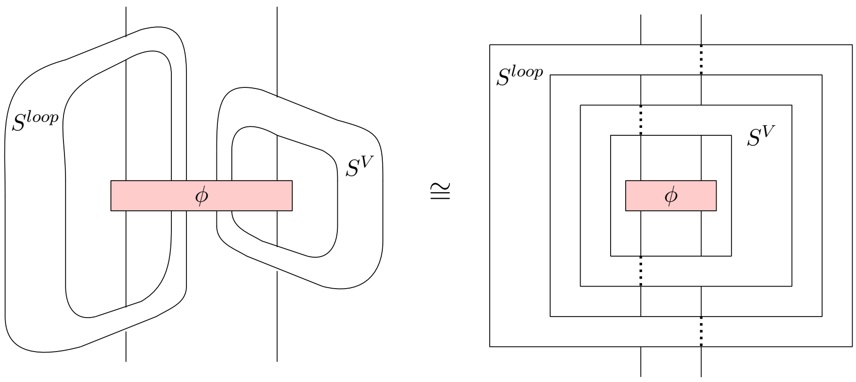
To see the approximate issues concerning the interchange law for composition of locally-applicable transformations, imagine defining the application on the right hand side for T by



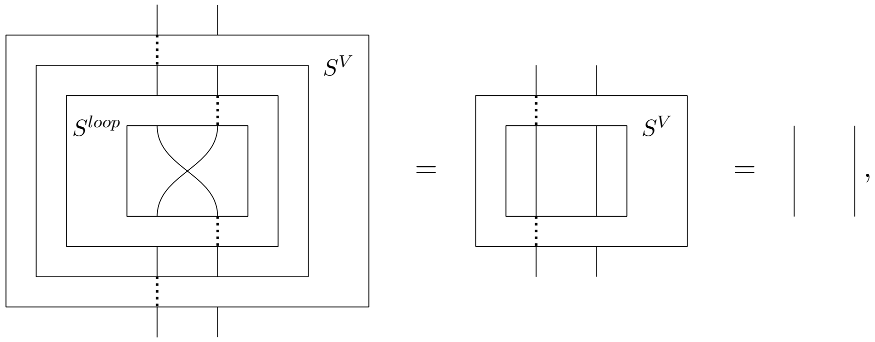
One could hope to give meaning to the picture representing some notion of $(id \otimes S^V) \circ (S^{loop} \otimes id)$ by



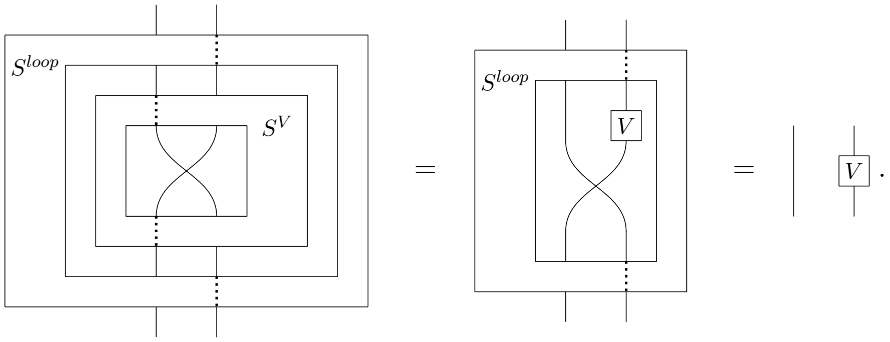
Analogously we can write what we would hope to be the diagram representing $(S^{loop} \otimes id) \circ (id \otimes S^V)$:



In a monoidal category these two terms would need to be the same, however, for the specific locally-applicable transformations S^V and S^{loop} we can see that this will not be the case. Consider the action of each term on the swap morphism, the first order of composition gives



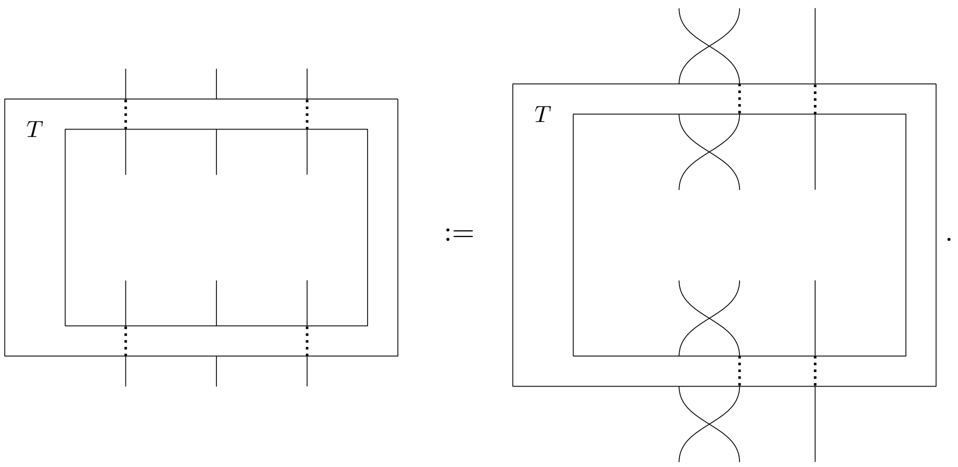
whereas the second order of composition gives:



In short, the locally-applicable transformations on **U** which are *not* superunitaries are those which break the interchange law.

6.2 Solution: Slots

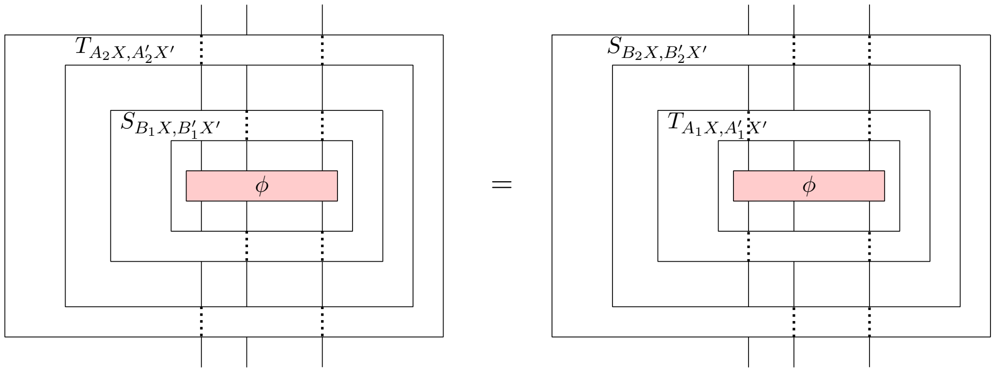
We now strengthen the construction of locally-applicable transformations over monoidal categories, to do so we will need to introduce some new diagrammatic notation:



(6.7)

We will now define **S**trongly **LO**cally-applicable **T**ransformations, for short *slots*, as those locally-applicable transformations which are so local that they commute with any locally-applicable transformations applied to auxiliary wires.

Definition 33. *A slot of type* $S : [A_1, A_1'] \to [A_2, A_2']$ *is a locally-applicable transformation of the same type such that for every locally-applicable transformation* $T : [B_1, B_1'] \to [B_2, B_2']$ *and* $\phi \in \mathbf{C}(A_1 \otimes B_1 \otimes X, A_1' \otimes B_1' \otimes X')$ *then:*



. (6.8)

Thinking of slots in this way makes it clear that they form a monoidal category with tensor product $[A, A'] \boxtimes [B, B'] = [A \otimes B, A' \otimes B']$.

So in intuitive terms, slots are those functions that are so local, that they commute not only with combs but with all other functions which commute with combs. A more-categorical way to think of slots is as the central morphisms of the pre-monoidal category of locally-applicable transformations [248][1].

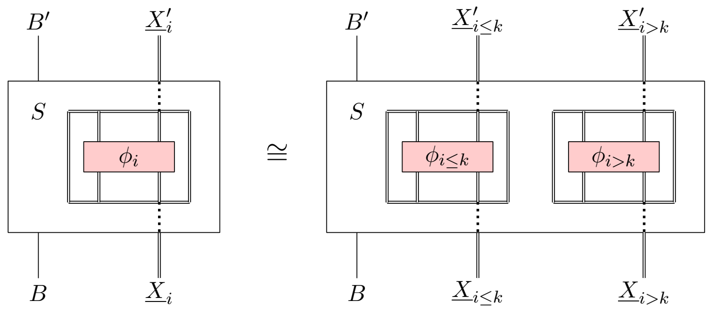
6.3 The Multi-Input Case: Polyslots

We now consider a multi-party generalisation of slots to polyslots. The key idea is that each individual component of a polyslot acts as a slot. We will use the notation $\|\underline{A}\|$ to denote the set of natural numbers $\{1 \ldots |\underline{A}|\}$ with $|\underline{A}|$ the length of list $\underline{A}$. In the diagrammatic representation of locally-applicable transformations, we will use the symbol $\underline{X}$ to denote the monoidal product of the elements of a list $\underline{X}$ of objects of $\mathbf{C}$ whenever the meaning is clear from context. Using these notations, we will use a shorthand for families of functions of type

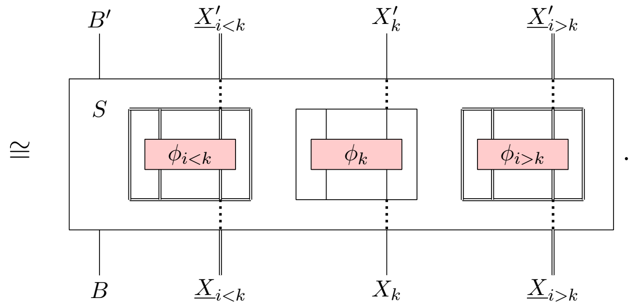
$$S^{X_1' \ldots X_{\|\underline{A}\|}'}_{X_1 \ldots X_{\|\underline{A}\|}} : \bigtimes_{i \in \|\underline{A}\|} \mathbf{C}(A_i \otimes X_i, A_i' \otimes X_i') \longrightarrow \mathbf{C}(B \bigotimes_{i \in \|\underline{A}\|} X_i, B' \bigotimes_{j \in \|\underline{A}\|} X_j'),$$

[1]The author is grateful to James Hefford for this observation.

by moving freely between the following diagrammatic representations of the same family:



(6.9)



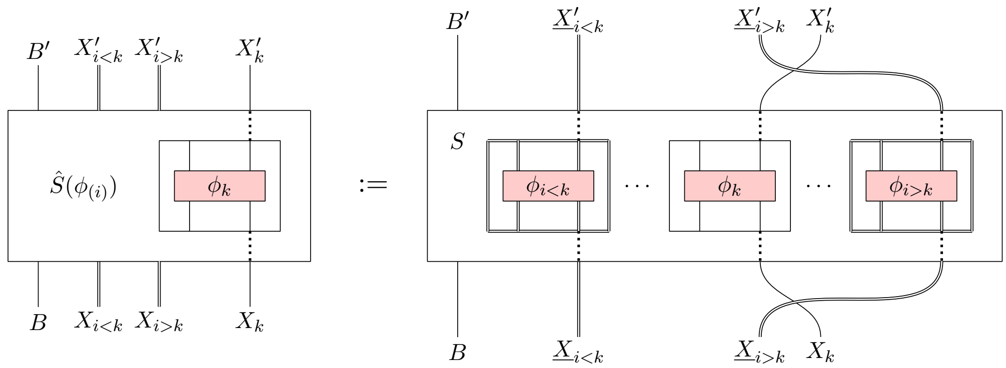
. (6.10)

In terms of this compacted notation, we can very cleanly define polsylots.

Definition 34 (Polyslots)**.** *Let* $\underline{\mathbf{A}}$ *be a list with each element of the form* $\mathbf{A}_i = [A_i, A_i']$ *for some objects* A_i, A_i' *of* $\mathbf{C}$*. A poly-slot of type* $S : \underline{\mathbf{A}} \to [B, B']$ *is a locally-applicable transformation of type* $\underline{\mathbf{A}} \to [B, B']$ *such that for every* k *and every element*

$$\phi_{(i)} \in \bigtimes_{i \in \frac{\|A\|}{k}} \mathbf{C}(A_i \otimes X_i, A_i' \otimes X_i'),$$

then the family of functions given by



, (6.11)

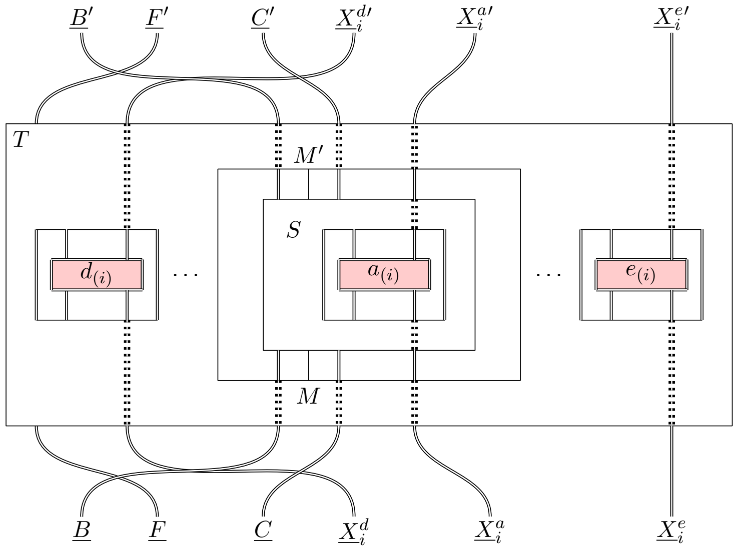
is a slot of type

$$[A_k, A_k'] \to [B \bigotimes_{i \in \frac{\|A\|}{k}} X_i, B' \bigotimes_{j \in \frac{\|A\|}{k}} X_j']. \tag{6.12}$$

By defining polyslots as functions which behave locally as slots, we guarantee that they will satisfy the required associativity/interchange laws for polycategories.

Theorem 15. *The polyslots on* $\mathbf{C}$ *define a symmetric polycategory* $\mathbf{pslot}[\mathbf{C}]$ *with:*

- *Objects given by pairs* $[A, A']$ *with* A, A' *objects of* $\mathbf{C}$.
- *Morphisms of type* $S : \underline{\mathbf{A}} \to \underline{\mathbf{B}}$ *given by polyslots of type* $S : [A_1, A'_1] \dots [A_n, A'_n] \to [B_1 \otimes \dots \otimes B_m, B'_1 \otimes \dots \otimes B'_m]$.
- *Composition* $T \circ_M S$ *of* $S : \underline{\mathbf{A}} \to \underline{\mathbf{B}}\mathbf{M}\underline{\mathbf{C}}$ *and* $T : \underline{\mathbf{D}}\mathbf{M}\underline{\mathbf{E}} \to \underline{\mathbf{F}}$ *given by taking* $T \circ_M S(d_{(i)}, a_{(j)}, e_{(k)})$ *to be*



. (6.13)

Proof. Given in Appendix *B*. □

Theorem 16. *The polyslot construction* **pslot**$[\mathbf{C}]$ *returns a theory of supermaps*

Proof. The sequential and parallel composition locally-applicable transformations introduced to see **Lot**$[\mathbf{C}]$ as an enrichment of $\mathbf{C}$ can easily seen to be polyslots. Defining $[A, A'] \boxtimes [B, B'] := [A \otimes B, A' \otimes B']$ the cotensor polyslots $\boxtimes : [A, A'] \boxtimes [B, B'] \to [A, A'][B, B']$ which turn the outputs of the polycategory representable are given by defining each $\boxtimes_{X,X'} : \mathbf{C}(A \otimes B \otimes X, A' \otimes B' \otimes X') \to \mathbf{C}(A \otimes B \otimes X, A' \otimes B' \otimes X')$ to be the identity function. By virtue of being the identity, unique representability of each multiple output morphism by the cotensor and compatibility with the symmetric monoidal enrichment of $\mathbf{C}$ are immediate. □

Taking a step back here, what we appear to have done is to take a generalisation of polycategories in which the interchange law is not assumed to hold, and have then simply forced the interchange law. This is yet again reminiscent of taking the centre of a pre-monoidal category. Such a notion has been generalised to taking the centre of a pre-multicategory [249], and it is likely that the polycategory of polyslots arises from taking the centre of some further generalised notion of pre-polycategory.

6.3.1 Single-Party-Representable Supermaps

Just as a polyslot is one which behaves as a slot at each input, we can define a sub-theory of single-party representable maps meaning those which behave as a comb at each input[2]. Note, that we do not require the entire higher-order process to be a circuit with many holes, instead we are requiring that when all but one input is filled-in, what remains is a circuit with a single hole. It turns out that by the decomposition theorem for single-party superchannels and superunitaries [12] that this construction will also be sufficient to reconstruct the superchannels and superunitaries. We however prefer to focus on the polyslots, since they do not assume apriori such a decomposition into combs, and are strictly more general, instead purely being phrased in terms of strength of locality.

Definition 35. *A single-party representable supermap of type*

$$S : [A_1, A_1'] \dots [A_N, A_N'] \to [B, B']$$

is a family of functions

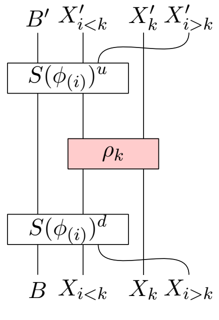
$$S_{X_1 \dots X_N, X_1' \dots X_N'} : \bigtimes_{l \in \frac{\|A\|}{k}} \mathbf{C}(A_l \otimes X_l, A_l' \otimes X_l') \to \mathbf{C}(B \bigotimes_{i \in \frac{\|A\|}{k}} X_i, B_1' \bigotimes_{j \in \frac{\|A\|}{k}} X_j')$$

such that for every k and every element

$$\phi_{(i)} \in \bigtimes_{i \in \frac{\|A\|}{k}} \mathbf{C}(A_i \otimes X_i, A_i' \otimes X_i'),$$

there exists $S(\phi_{(i)})^u$ and $S(\phi_{(i)})^d$ satisfying

$$S_{X_1 \dots X_N, X_1 \dots X_N'}(\phi_1 \dots \rho_i \dots \phi_N) \quad =$$

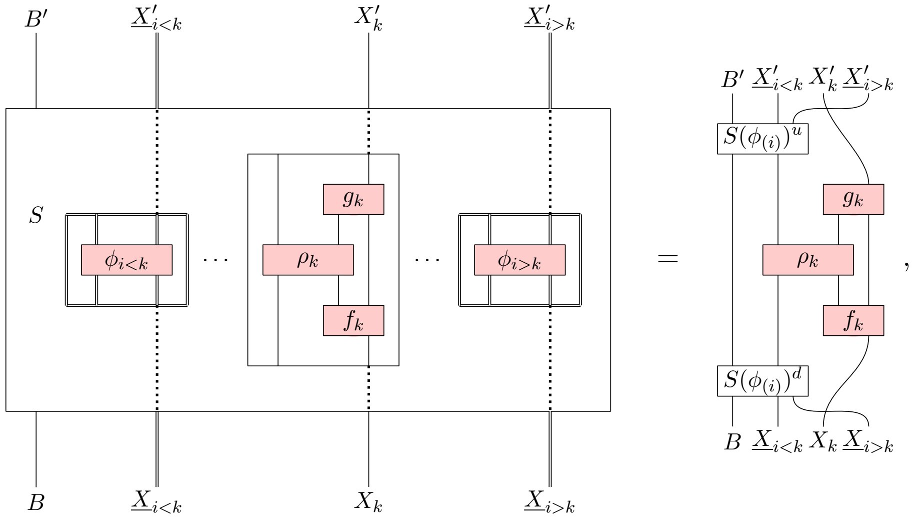


.

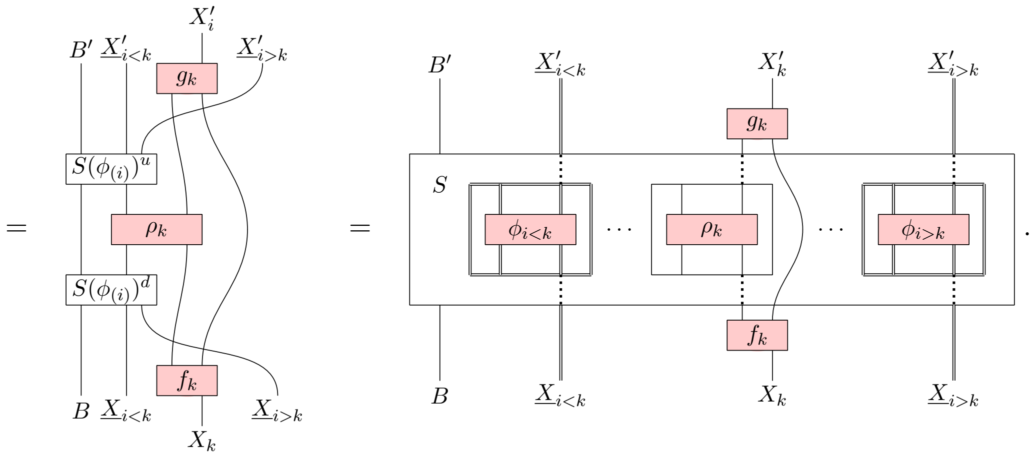
Lemma 13. *Single-party representable supermaps of type $S : [A_1, A_1'] \dots [A_N, A_N'] \to [B, B']$ are locally-applicable transformations of the same type.*

[2]The author is grateful to Augustin Vanrietvelde for this observation

Proof. Note that



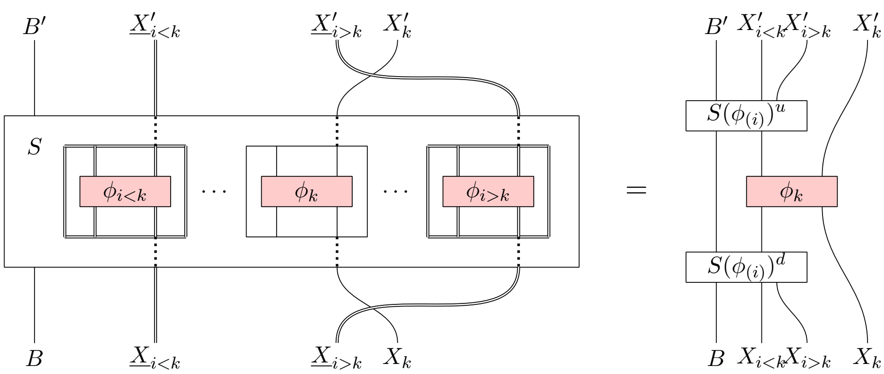
and so using the interchange law for symmetric monoidal categories we find:



Going through the same steps for every k completes the proof. □

Lemma 14. *The single-party representable supermaps are polyslots.*

Proof. By re-ordering the wires in the defining equation for S to be a single-party representable supermap,



(6.14)

we see that S is a polyslot since every comb is a slot. □

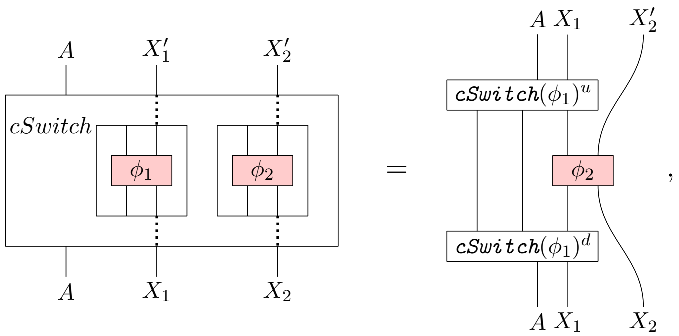
Corollary 2. *The single-party representable supermaps on* **C** *define a polycategory* **srep**[**C**].

Proof. Since single-party representable supermaps are polyslots, all we need to check is that they are preserved under polyslot composition, which follows from the observation that combs are closed under composition. This construction in-fact returns a theory of supermaps, with enrichment and representability inherited from the polyslots. □

6.4 Examples of Polyslots

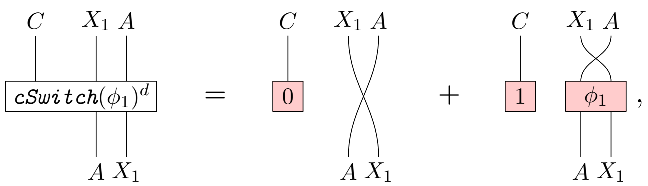
The slot and polyslot conditions work for constructing well-behaved composition rules for supermaps, but one might worry that they rule out some supermaps of interest. Here we check that polyslots include the generalisations of the convex switch to infinite dimensional stochastic matrices and the quantum switch to arbitrary Hilbert spaces.

Example 25. *The generalisation of the convex switch [17] to the category of quantum operations is a polyslot since its action (with ϕ_1 fixed) on ϕ_2 can be written as a comb:*



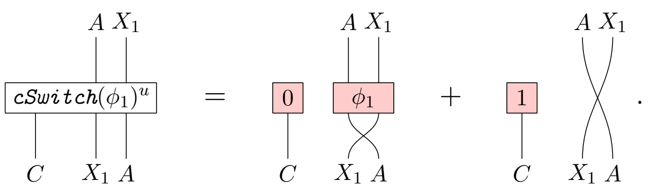
(6.15)

where defining C to be any 2-dimensional Hilbert space, and $0, 1$ to be any orthonormal basis for C, then:



(6.16)

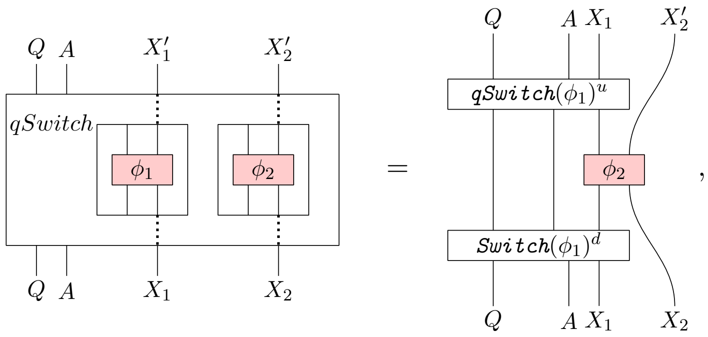
and



(6.17)

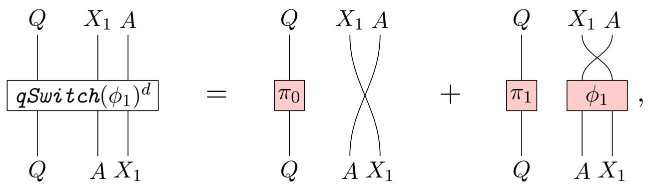
and similarly for the action on ϕ_1. This definition naturally extends to N-party convex switches of type $[A, A] \ldots [A, A] \to [A, A]$.

Example 26. *The generalisation of the quantum switch [17, 136] to arbitrary Hilbert spaces is a polyslot since its action (with ϕ_1 fixed) on ϕ_2 can be written as a comb:*



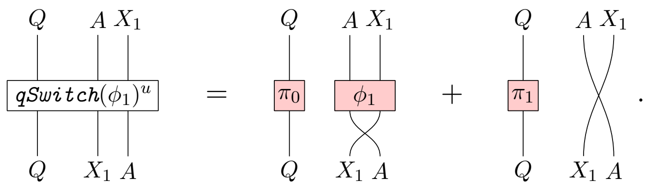
(6.18)

where



(6.19)

and



(6.20)

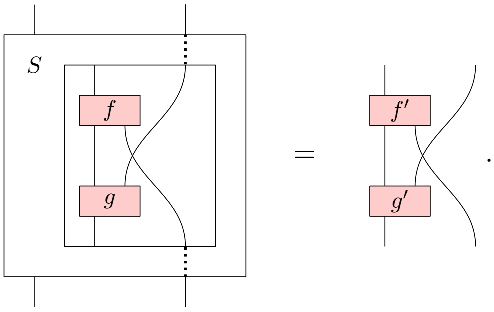
and similarly for the action on ϕ_1. This definition naturally extends to N-party switches of type $[A, A] \dots [A, A] \to [Q \otimes A, Q \otimes A]$.

Note that both of these examples are furthermore single-party representable, in-fact, it was this that we proved to check that these maps are polyslots.

6.5 Characterisation of the Superunitaries by Strong Local-Applicability

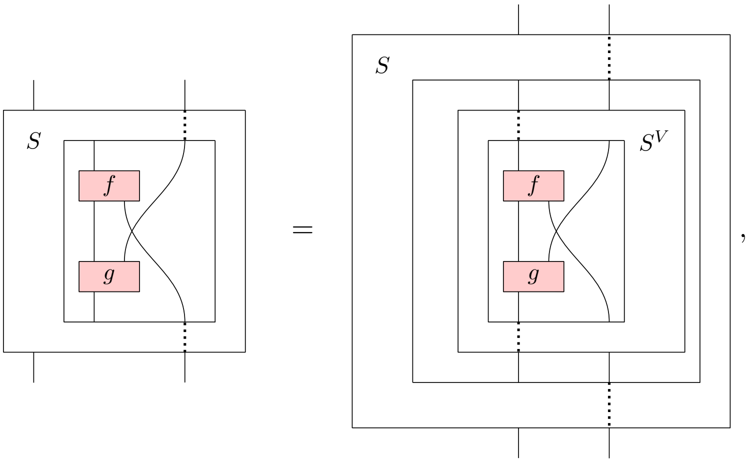
In this final section, we show that our most general free-construction of a theory of supermaps over monoidal categories exactly recovers the standard physics approach to defining supermaps [12, 17]. More concretely, we show that the polyslots on the unitaries are exactly the superunitaries, and, we use the characterisation theorem of section 5 to check that the polyslots on the quantum channels are still the superchannels. We begin with some basic lemmas, writing $\mathbf{C} \subseteq_{\otimes} \mathbf{D}$ to express the condition that $\mathbf{C}$ be a symmetric monoidal sub-category of $\mathbf{D}$.

Lemma 15. *Let $\mathbf{C} \subseteq_{\otimes} \mathbf{D}$ with $\mathbf{C}$ a groupoid and $\mathbf{D}$ compact closed, every slot of type $S : [A_1, A_1'] \to [A_2, A_2']$ on $\mathbf{C}$ preserves no-pathing shapes. Explicitly, for every f, g then* *there exists f', g' such that*



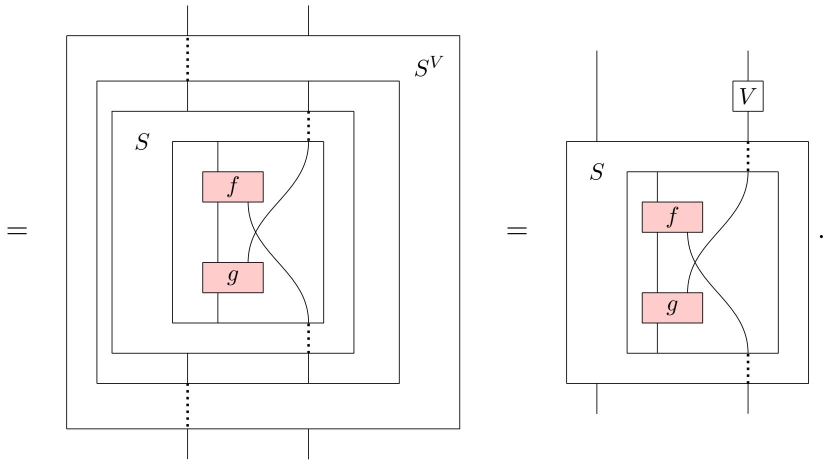
(6.21)

Proof. Assuming the converse, we begin by noting that any S^V acts trivially on the right-hand side of such a process:



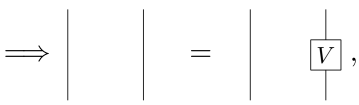
(6.22)

then using commutativity of S with any S^V with $V \neq id$



(6.23)

Using the fact that every morphism in **C** is an isomorphism:



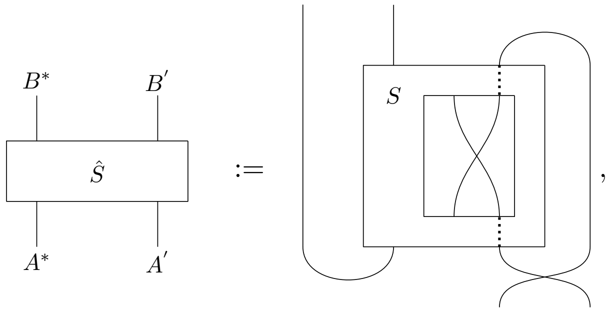
(6.24)

after which applying cups and caps gives $V = id$, a contradiction. □

This property allows us to establish a theorem, on the internalisation of polyslots.

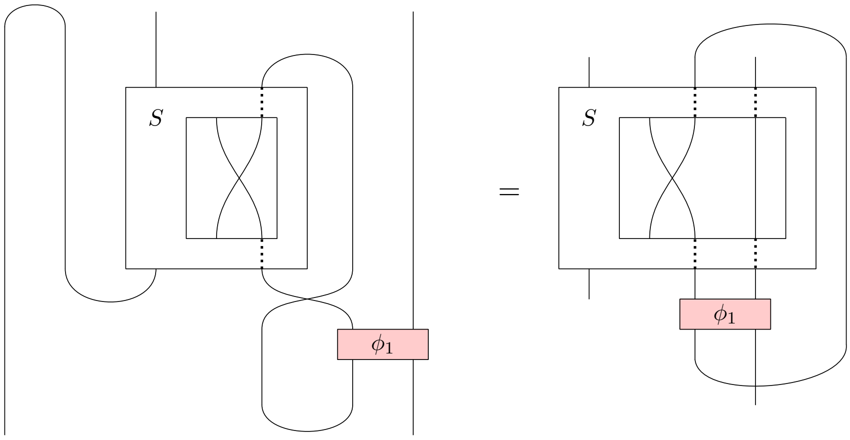
Theorem 17. *Let* **C** *be a groupoid and* **D** *a compact closed category with* $\mathbf{C} \subseteq_{\otimes} \mathbf{D}$*. Every polyslot on* **C** *is implementable by a* **D***-supermap on* **C***.*

Proof. We begin by considering single-input slots, to each slot we can assign a **D**-supermap by



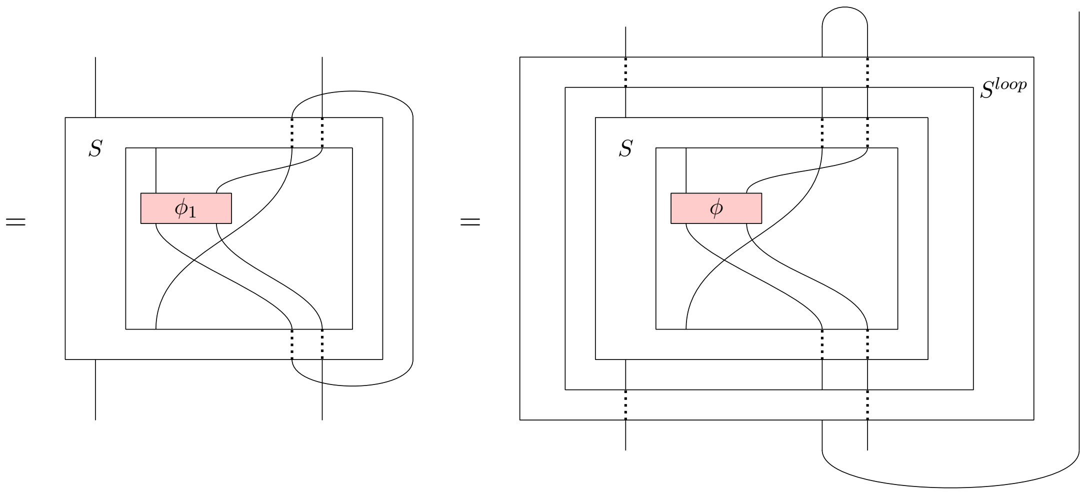
, (6.25)

and then generalize to multiple inputs by induction. Consider



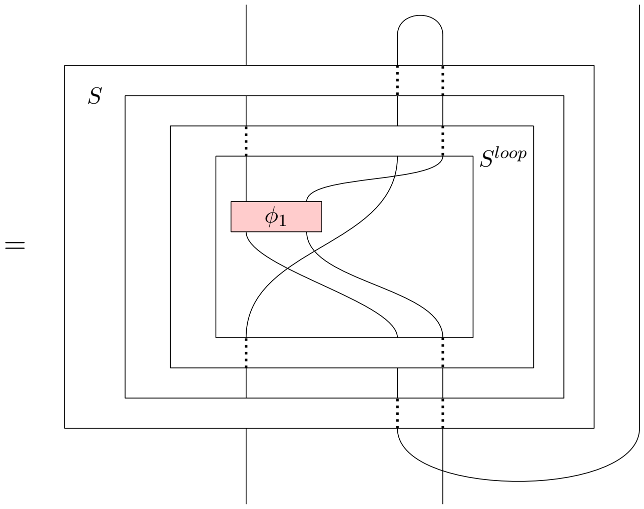
, (6.26)

and note that since S preserves no-pathing shapes, application of S^{loop} would recover the application of a cup/cap and generation of identity wire



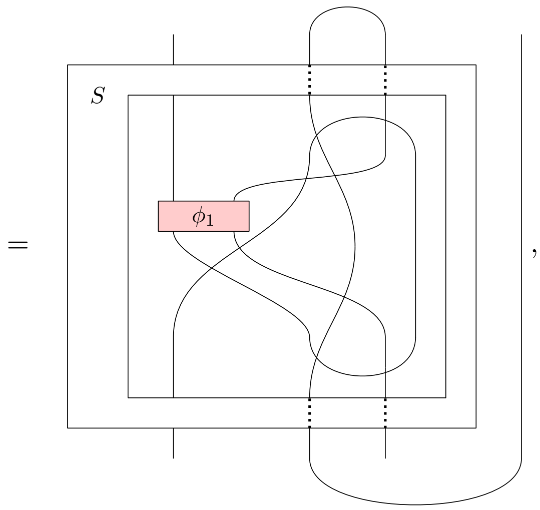
, (6.27)

since by assumption S is a slot, it commutes with S^{loop} giving:



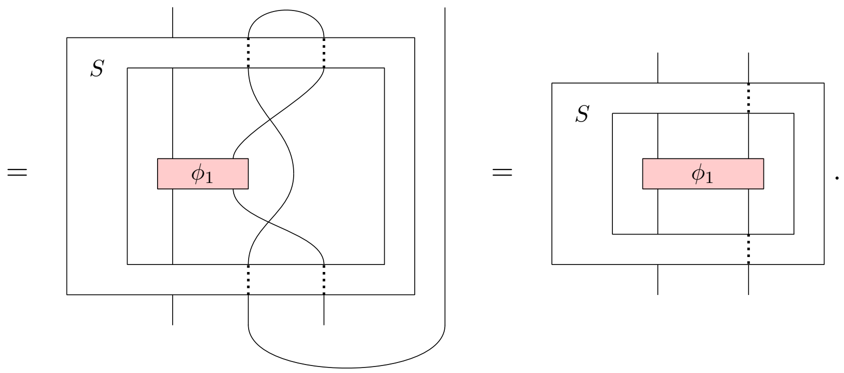
(6.28)

which after unpacking the definition of S^{loop} gives:



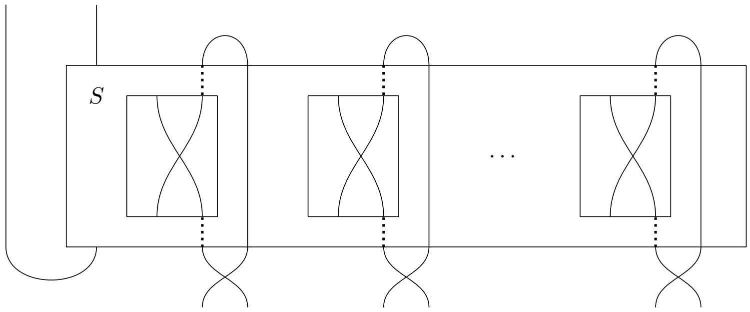
(6.29)

finally using compact closure and local-applicability gives



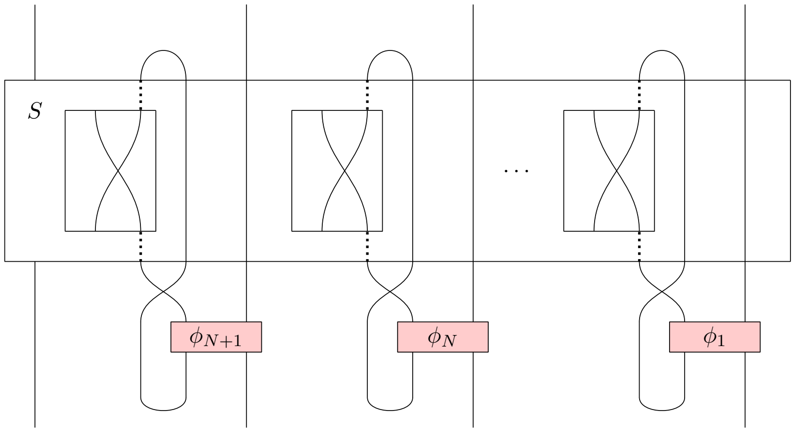
(6.30)

This argument can be extended to multiple input polyslots by induction, assuming representation by **D**-supermaps for N-parties then consider the natural candidate **D**-supermap for $N + 1$ parties



, (6.31)

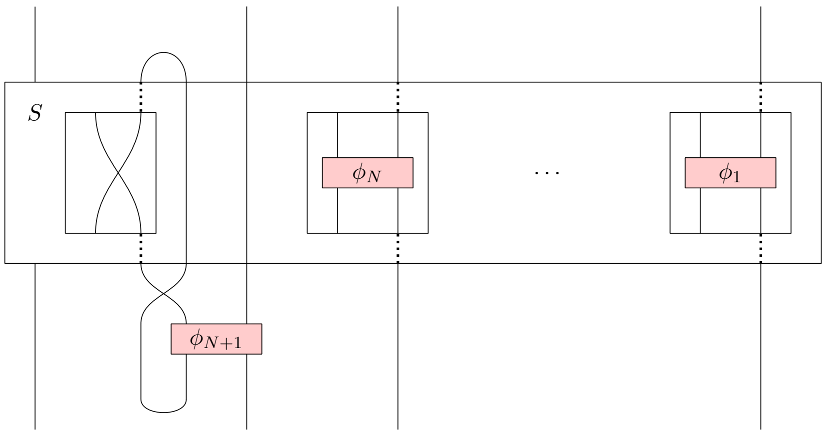
we will consider its action on some family $\phi_{(i)}$ of morphisms



. (6.32)

Filling in the first entry of a polyslot with $N + 1$ inputs returns a polyslot with N inputs, so using the induction hypothesis:

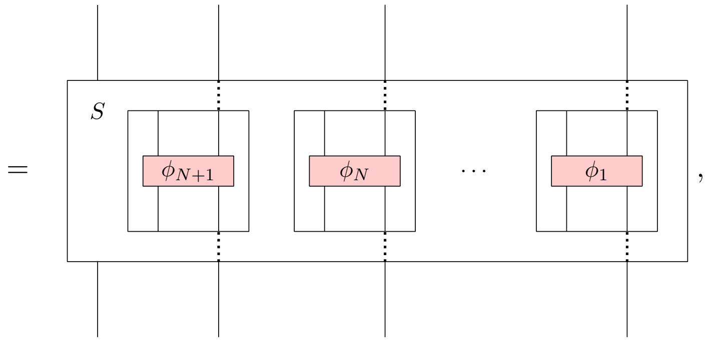
$=$



. (6.33)

Furthermore filling in entries 1 to N of an $N + 1$ input polyslot returns a slot on the final $(N + 1)^{th}$ entry, for which representation was verified in the initial stage of this proof.

Consequently we reach



(6.34)

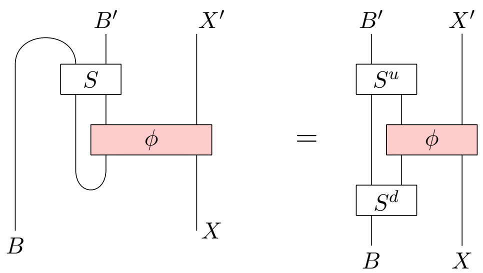
which completes the proof. □

The assignment of a **D**-supermap to each polyslot derived above, is in fact a (bijective on objects) faithful[3] functor of symmetric polycategories, meaning that it is an assignment which commutes with the composition rules of polycategories. With reference to this, we can rephrase theorem 17 as the statement that $\mathbf{pslot}[\mathbf{C}] \subseteq_{poly} \mathbf{Dsup}[\mathbf{C}]$[4]. In this thesis we are most concerned with the one-to-one correspondence, that this assignment commutes with composition is a neat bonus.

Theorem 18. *Polyslots on the unitaries are in one-to-one correspondence with superunitaries, and polyslots on the quantum channels are in one-to-one correspondence with the superchannels.*

Proof. Beginning with the equivalence between superchannels and polyslots on quantum channels, recall that there is a mapping which sends every locally applicable transformation on quantum channels to a superchannel, and consequently every slot can be mapped to a superchannel. To show that in turn every superchannel defines a slot, note that since every locally-applicable transformation on quantum channels is constructed by a superchannel as shown in chapter 5, all locally-applicable transformations on the quantum channels are slots with interchange law inherited from the interchange law for the monoidal category of completely positive maps [212, 241, 250].

For the equivalence between superunitaries and polyslots on the category of unitaries, recall that every single-input superunitary decomposes as

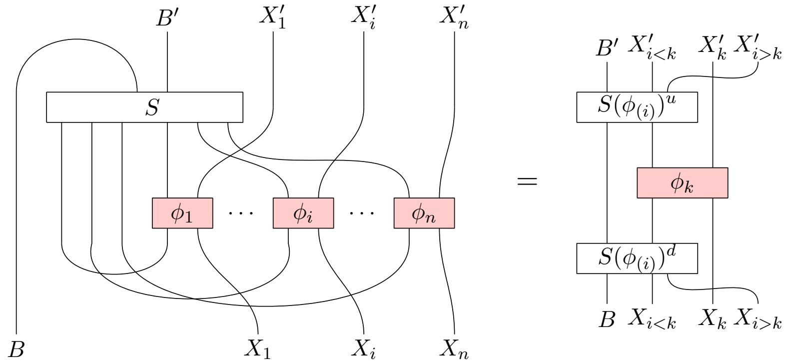


(6.35)

[3]Faithful means injective on morphisms here, and should not be confused with the notion of faithful enrichment of Chapter 3.

[4]Faithful functors give a more categorically well-behaved generalisation of the notion of sub-category.

where S^u and S^d are unitaries $\in \mathbf{U}$ [53]. A useful consequence of this result is the following one for multiparty superunitaries. Every superunitary of type $[A_1, A'_1] \dots [A_n, A'_n] \to [B, B']$ satisfies:



(6.36)

Where the $S(\phi_{(i)})^u$ and $S(\phi_{(i)})^u$ are unitaries which do not depend on ϕ_k. This can be shown by noting that fixing all but ϕ_i, the resulting map $S(\phi_1, \dots \phi_{i-1}(-)\phi_{i+1} \dots \phi_N)$ defines up to braiding a single-input superunitary, so by theorem 17 must decompose as a comb. Any family of functions which decomposes at the single party-level as a comb defines a polyslot, since combs commute with all locally-applicable transformations. $\square$

In informal terms, polyslots on unitaries are superunitaries, and polyslots on channels are superchannels. The assignments which exhibit the one-to-one correspondence are again given by bijective on objects invertible polyfunctors, meaning that the above correspondences can be phrased as an equivalence of polycategories. The polycategory $\mathbf{pslot}[\mathbf{U}]$ is equivalent to $\mathbf{Su}$ and the polycategory $\mathbf{pslot}[\mathbf{QC}]$ is equivalent to $\mathbf{QSc}$.

Note that by the comb decomposition theorem for single-input superchannels and superunitaries it is also the case that $\mathbf{srep}[\mathbf{QC}] = \mathbf{QSc}$ and $\mathbf{srep}[\mathbf{U}] = \mathbf{Su}$. However, the construction $\mathbf{pslot}[-]$ is more closely aligned to the spirit of the definition of supermaps, in which comb decomposition is treated as a theorem to be proven not an assumption to be made.

6.6 Summary

We have found that there are a variety of constructions of theories of supermaps available. Over any monoidal category we can write the following constructions

$$\mathbf{srep}[\mathbf{C}] \subseteq \mathbf{pslot}[\mathbf{C}]$$

The first assumes that when all-but-one input is filled, what remains is a comb, and the second construction assumes a strong-enough notion of locality to force the interchange law for polycategorical composition. Both constructions recover the physicists' definitions of general black-box supermaps [12,17] via our main theorems which state that $\mathbf{pslot}[\mathbf{U}] = \mathbf{Su}$ and $\mathbf{pslot}[\mathbf{QC}] = \mathbf{QSc}$.

Chapter 7

Summary, Outlook, and Conclusion

In this thesis we have tried to build a general framework for supermaps. To finish, let us summarise the concrete results of the thesis and discuss a variety of future potential research directions.

7.1 Summary of Results

The key piece of mathematics which we have used to define supermaps on any theory of processes, is the notion of a locally-applicable transformation on a symmetric monoidal category. In our main theorem, proven in chapter 5, we discovered that locally-applicable transformations on the symmetric monoidal category of quantum channels are exactly the quantum superchannels. We then gave a general abstract definition for theories of supermaps, and demonstrated that supermaps of such theories can always be used to construct locally-applicable transformations. Finally, we strengthened locally-applicable transformations to polyslots, which give a polycategory of candidate supermaps over any symmetric monoidal category, and which recover both the superchannels *and* the superunitaires when applied to the quantum channels and unitaries respectively. In short, we have developed a clear idea of what theories of supermaps are *and* how to construct them.

7.2 Outlook

Let's now consider the possible future directions which would build on the results of this thesis. A common theme amongst these future directions, is that this new compositional framework for supermaps will allow us to phrase questions that we had not previously been able to phrase, even though those question would intuitively had made sense.

Characterisation of Locally-Applicable Transformations in Infinite Dimensions
Since we now have definitions of categorical supermaps in terms of either locally-applicable

transformations or slots. In finite dimensions, single-input supermaps decompose as combs [12, 53], an immediate question then is whether our definitions of supermaps on arbitrary Hilbert spaces have the same property. First, for the case of mixed quantum theory, we would suggest the following conjecture.

Conjecture 1. *The locally applicable transformations on the monoidal category* **QO** *of quantum operations of type* $[A, A'] \to [B, B']$ *are combs*

Second, for the case of unitaries, we saw that we should in general use a stronger notion of locality, leading us to conjecture as follows.

Conjecture 2. *The slots on the the monoidal category of (infinite dimensional) unitaries of type* $[A, A'] \to [B, B']$ *are combs*

Proving these conjectures may involve use of functional analysis or may be provable using the graphical methods of categorical quantum mechanics. More generally, any question which has been asked of quantum supermaps on finite dimensional Hilbert spaces can now be asked of our definition of supermaps on arbitrary Hilbert spaces, meaning the a wide variety of new research projects can now begin from this point. On the applied side the are questions of capacity activation and advantages of quantum causal structures for computation which are specific to infinite dimensional or continuous variable settings. On the theoretical side there are questions of the characterisation of supermaps on semi-causal channels as multi-inupt combs [13, 15, 53].

Resource Theories of Higher-Order Transformations The standard approach to resource theories in quantum theory, is to take composition to be free, in the sense that two resources are equivalent if they can be reached by arbitrary iterated compositions of free transformations [48, 251]. The more composition rules that are available then, to more coarse grained the equivalences of resources should become. Whilst in standard quantum theory it is generally accepted that the key composition rules are sequential and parallel composition, as we have seen the story of the compositionally of supermaps is more subtle. By writing a general definition of theory of supermaps we have essentially defined for free what the most natural notion of composition is for supermaps, and so we will have established a notion of resource theory and sub-theory of supermaps.

Meta-Theory for Supermaps Now that we have written down some rules for theories of supermaps, we could try to construct a meta-theory **Sup** of theories of supermaps, analogous to the meta-category **Cat** with objects given by categories [21]. In **Sup** we should expect to see objects as theories of supermaps and will need to define structure preserving maps, and even higher morphisms between structure preserving maps, analogous to functors and natural transformations. If we can construct such a meta-theory, we can really begin to organise the variety of potential supermap definitions using category theory. A key question for instance, is whether one can show that combs or optics, the simplest form of supermaps are correspondingly the smallest theory. It is likely for instance that one could show that open circuits appear via some kind of universal construction in **Sup** just as the quantum channels appear as universal within the category **MonCat** of monoidal categories [198].

Type Theory for General Supermaps Other frameworks for supermaps either limited to circuits-with-holes [121] or limited to categories with additional properties such as compact closure [126–128, 132, 135], are equipped with tensor products and other more elaborate type constructors. So far our construction returns a polycategory without input tensors, one could either hope to get more general tensors by including general extension sets as objects analogously to [126], or instead by looking for promonoidal tensors as in [121].

Reconstruction of All of Higher Order Quantum Theory So far our framework has been used to define and reconstruct theories of supermaps. However, higher order quantum theory [127, 135] and the **Caus**[**C**] construction [126, 132] return super-super maps and their iterations. A key question is whether the approach of this thesis can be developed further to develop complete higher-order physical theories over any monoidal category or operational probabilistic theory. This approach is likely to come with many challenging subtleties, coming form the fact that the super-supermaps will have to be defined as locally-applicable transformations *on* symmetric polycategories rather that symmetric monoidal categories. Furthermore such a step would appear to have to work layer-by-layer and probably require some form of taking a categorical limit [21] to return a single infinitely iterated theory as an end product.

Local-Applicability in Other Contexts One of the surprises of the characterisation of superchannels by local-applicability, is that this abstract notion of local-applicability was strong enough to imply linearity. This suggests a new approach to defining standard quantum transformations is tenable, in which transformations are defined first as being locally-applicable functions on quantum states, and later proven to be the unitary or completely positive trace preserving maps. This would essentially be a flip in perspective in comparison to the standard approach of assuming linearity and then imposing complete-preservation [199, 220]. If successful, such a research direction would establish local-applicability as the key formal, and purely physically motivated, principle for defining quantum dynamics.

Post-Quantum Causal Structures The development of a framework for supermaps on arbitrary monoidal categories gives us a framework for supermaps on all operational probabilistic theories [137], including those without any assumption of finite dimensionality. Immediate questions then can be asked, on the relationship between post-quantum features of physical theories and features of their associated super theories. Concretely one may wonder whether theories with stronger than quantum correlations or interference patterns are compatible with some form of post-quantum causal structures.

Sound and Complete Diagrammatic Languages for Supermaps In chapter 4 we modelled the intuitive picture of a supermap using locally-applicable transformations, and we introduced a function-box notation to avoid writing algebra and to keep the formal mathematics that we write looking as similar as possible to the pictures we were trying to model. We played a similar game when we defined *theories of supermaps*, using an as yet-unformalised diagrammatic language for categories enriched in symmetric polycategories. In each case we so far can only claim that the diagrammatic calculations we wrote down were shorthand, an efficient way of explaining to the reader how it is that they would reconstruct an algebraic argument for themselves, if they wanted to. However, much of the power of the categorical quantum mechanics program [22, 202, 208] has come from the proof of a variety of soundness and completeness results [252–266].

Firstly, soundness results ensure that manipulations of diagrams are not just recipes but in fact are themselves the formal calculations. Consequently, two key soundness results which are needed to formalise our calculations are soundness of the string diagrams we used to represent symmetric polycategories, and soundness of function boxes for representation of locally-applicable transformations.

Whilst the soundness of the diagrams we used in this paper, while as yet unformalised, is a fairly natural expectation, there is a more speculative proposal for diagrammatic reasoning which is less clearly viable. This proposal is to flip the story of this thesis on its head, by taking the intuitive pictures which we wished to model, and seeing if they can be made into a sound diagrammatic language for whatever to most appropriate notion of theory of supermaps turns out to be. We could conjecture for instance that the intuitive pictures we have been drawing can be used to give a sound diagrammatic language for symmetric monoidal categories enriched into symmetric polycategories with cotensors compatible with enrichment. Such a result if formalisable would justify on formal grounds the regular use of similar intuitive pictures throughout the quantum literature on supermaps.

Past soundness there is the even more powerful concept of completeness of diagrammatic languages for algebraic theories, stating not only that everything proved diagrammatically is valid but furthermore that everything provable algebraically is provable diagrammatically. It seems conceivable that the intuitive pictures drawn for supermaps could even be complete for some appropriate variant of theories of supermaps as defined in chapter 4. Even further, representation of supermaps such as the quantum switch using formal diagrammatic languages is a hot topic in recent quantum foundations, one new approach could be the combination of a general diagrammatic language for theories of supermaps with specific aspects of diagrammatic languages for quantum theory such as the PBS [261] or ZX [266] calculi.

Characterisation of Routed Supermaps A key class of supermaps on constrained spaces used to construct supermaps of simpler types, are the supermaps on spaces equipped with sectorial constraints. An immediate open question is whether local-applicaiblity is a strong enough requirement to recover such supermaps, crucially they escape our characterisation theorem since whilst they are causal they are not in general *normal*. A characterisation here would probably require more subtle use of the controlled swap gate introduced in [228].

Consistent Circuits for Indefinite Causal Structures in Infinite Dimensions Now that we have a stable generalisation of supermaps to infinite dimensional quantum theory, a key question is whether the construction of all explicitly known unitary supermaps from routed supermaps [211] could now be generalised to infinite dimensions. If successful such an approach would establish categorical methods as not only useful for defining supermaps but also useful as a tool for constructing interesting examples of supermaps such as those which break causal inequalities [18, 213].

7.3 Final words

In this thesis, we developed a process-theoretic approach to the study of supermaps. Whilst the language of category theory was used here, the tools of classical category theory generally were not, we used category theory as a map or filing system rather than using it for solving open problems in the study of supermaps.

Whilst some success was found in characterising supermaps using minimal mathematical assumptions, it appears as described in the outlook that we have only really scratched the surface. Indeed, we don't have tensor products, we don't know how the framework actually behaves in infinite dimensions, we don't know how to cope with super-supermaps. Furthermore, we don't know how to characterise our definitions in terms of more familiar mathematical languages in all but the cases in which supermaps have already been defined. In short, we are still just at the beginning of the development of a compositional framework for general higher-order transformations.

Given the variety of applications that have been found in the field of applied category theory, from putting other intuitive flowchart-like pictures into formal mathematics, one may wonder what other applications could come from our formalisation. We don't know where else to find researchers who would like to use the idea of a black-box supermap yet, in whichever monoidal categories their field cares about the most. Hopefully we will find them soon, and that they will find that this formalisation, and whatever comes from the next chapter of this story, works in some way for them too.

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Appendix A

Proofs of Properties of Closed Monoidal Categories

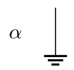
In this section we prove some theorems of chapter 3, in which causality is studied in the context of general closed symmetric monoidal categories.

A.1 HOCCs have no correlations with single-state objects

The notations and terminologies used here are taken from [126].

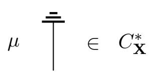
Theorem 19. *Every HOCC is a closed symmetric monoidal category which has no correlations with single-state objects*

Proof. A general state on $\mathbf{X} \otimes \mathbf{Y}$ is a member of the set $(C_X \times C_Y)^{**}$ where C_X is the set of states on $\mathbf{X}$, C_Y is the set of states on C_Y and C^* is the set of effects which normalise elements on C, i.e. $\forall \rho \in c : \pi \circ \rho = 1$. Let $\mathbf{Y}$ be a single-state object, since $\mathbf{Y}$ is flat its unique state must be a scalar multiple of the maximally mixed state.



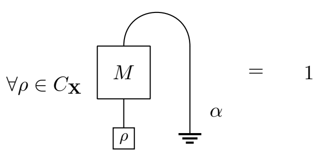
(A.1)

Since C_X is flat it follows that a scalar multiple of the discard process exists inside C_X^*.



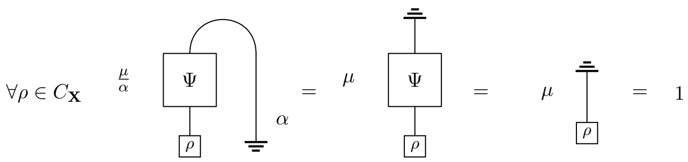
(A.2)

The elements of the set $(C_X \times C_Y)^*$ are up to process-state duality the processes M in the underlying category such that,



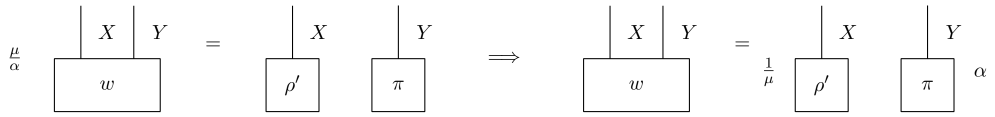
(A.3)

Note that any first-order causal process Ψ



(A.4)

which entails that $\frac{\mu}{\alpha}\Psi \in (C_X \times C_Y)^*$. In turn since $\{\frac{\mu}{\alpha}\Psi \mid \Psi \text{ causal }\} \subseteq (C_X \times C_Y)^*$ then it follows that $(C_X \times C_Y)^{**} \subseteq \{\frac{\mu}{\alpha}\Psi \mid \Psi \text{ causal }\}^*$. For any $w \in \{\frac{\mu}{\alpha}\Psi \mid \Psi \text{ causal }\}^*$ it is immediate that $\frac{\mu}{\alpha}w \in \{\Psi \mid \Psi \text{ causal }\}^*$ which in turn implies the following decompositions,



(A.5)

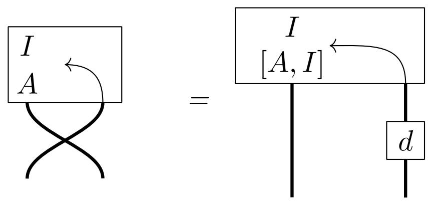
By assumption the usage of an effect of the form $\mathbf{Y} \to \mathbf{I}$ (which will be normalised by the right hand side of the composition) on w produces a state on $\mathbf{X}$. This in turn confirms that the left hand side of the decomposition is indeed a state of $\mathbf{X}$, and so any $w \in (C_X \times C_Y)^{**} \subseteq \{\frac{\mu}{\alpha}\Psi \mid \Psi \text{ causal }\}^*$ must decompose as the unique state of $\mathbf{Y}$ in parallel with a state of $\mathbf{X}$. ☐

A.2 The existence of canonical processes of closed symmetric monoidal categories

In this section we prove the existence of two key structural morphisms in any closed symmetric monoidal category.

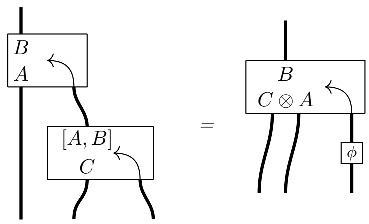
Theorem 20. *The following hold in any closed symmetric monoidal category $\mathcal{C}$:*

- *For each object A there exists a unique dualiser d_A satisfying*



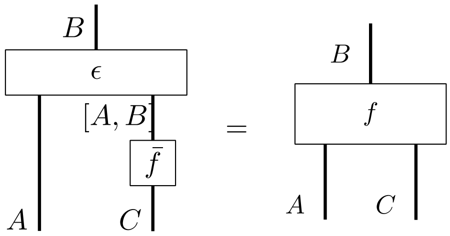
(A.6)

- *For each triple A, B, C there exists a unique static currying ϕ_{ABC} satisfying*



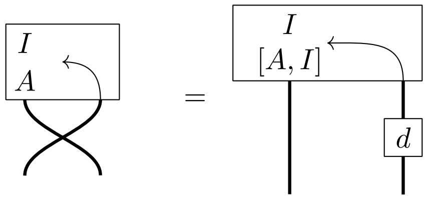
(A.7)

Proof. Each proof follows by one or more applications of the existence of the curried version of *any* process, guaranteed by the closed monoidal structure of $\mathcal{C}$. Since $\mathcal{C}$ is closed monoidal we know that for every morphism $f : (A \otimes C) \to B$ there exists a unique morphism $\bar{f} : C \to [A, B]$ such that



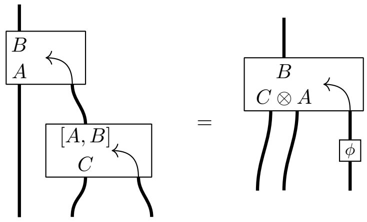
(A.8)

taking f the right hand side of the condition we wish for d_A to satisfy:



(A.9)

we see that d_a can be taken to be the currying of the right hand side, the existence and uniqueness of such a d_A are guaranteed by the defining condition of a closed monoidal category. Finally the defining condition for ϕ:



(A.10)

is again precisely the condition that ϕ be the currying of the morphism on the right-hand side of the condition. That such a ϕ exists and is unique is then again immediately implied by the closed monoidal structure of $\mathcal{C}$. □

A.3 Lifting isomorphism with double dual

In this section we will use the notation $[f, g]$ to mean the supermap which pre-composes with f and post-composes with g, the formal definition of $[f, g]$ is given in Appendix A. We will furthermore regularly use the notations $[A, g]$ and $[f, B]$ as shorthand for $[id_A, g]$ and $[f, id_B]$ respectively.

Theorem 21 (Lifted double duals)**.** *Let $\mathcal{C}$ be any closed symmetric monoidal category, if $\otimes$ preserves equivalence with double duals then $[-,=]$ preserves equivalence with double duals.*

Proof. We first give a sketch proof, outlining the sequence of internal isomorphisms used to show that $[[A, B], I]I] \cong [A, B]$, we then expand on this demonstrating that the above isomorphism is actually witnessed by $d_{A \Rightarrow B}$. Firstly assuming d_A and d_B are isomorphisms then $d_{[B,I]}$ is an isomorphism since the contravariant functor $[-, I]$ preserves isomorphisms. Furthermore since $\otimes$ preserves equivalence with double duals

$$A \otimes [B, I] \cong [[A \otimes [B, I], I]I] \tag{A.11}$$

Again using that d_B is an isomorphism gives

$$[A, B] \cong [A, [[B, I], I]] \cong [A \otimes [B, I], I] \tag{A.12}$$

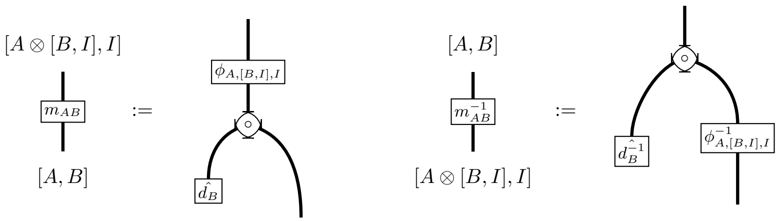
again since the contravariant functor $[-, I]$ preserves isomorphisms this implies,

$$[[A, B], I] \Rightarrow I \cong [[[A \otimes (B, I]), I], I], I] \tag{A.13}$$

the right hand side can be simplified using the first point on $\otimes$.

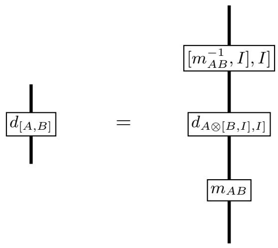
$$[[A, B], I], I] \cong [A \otimes [B, I], I] \cong [A, B] \tag{A.14}$$

So there indeed exists an isomorphism of the form required, to move beyond a sketch proof it must be shown that this isomorphism is in fact $d_{[A,B]}$. Using ϕ and the invertible (by assumption) canonical morphism $d : B \to [[B, I], I]$ in its static form $\hat{d}_B : I \to [B, [[B, I], I]]$ an invertible morphism m can be built.



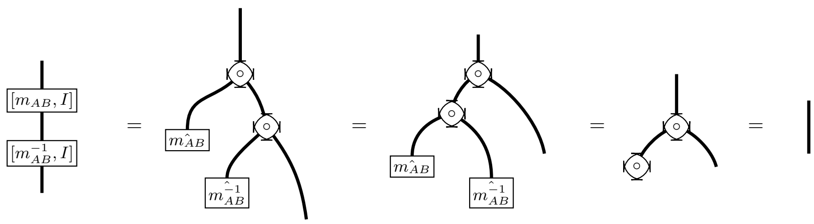
(A.15)

$d_{[A,B]}$ can be expressed in terms of m and $d_{[A\otimes[B,I],I]}$ in the following way,



(A.16)

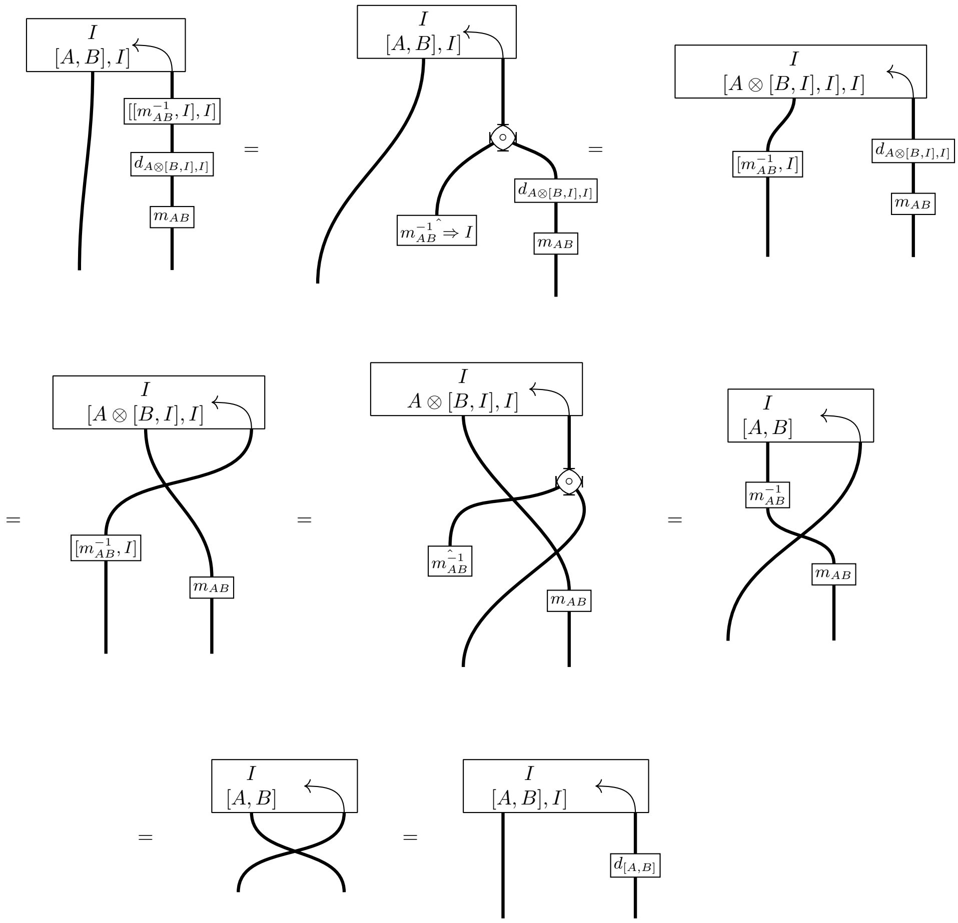
Where since m is an isomorphism $[m, I]$ and $[[m, I], I]$ are isomorphisms too.



(A.17)

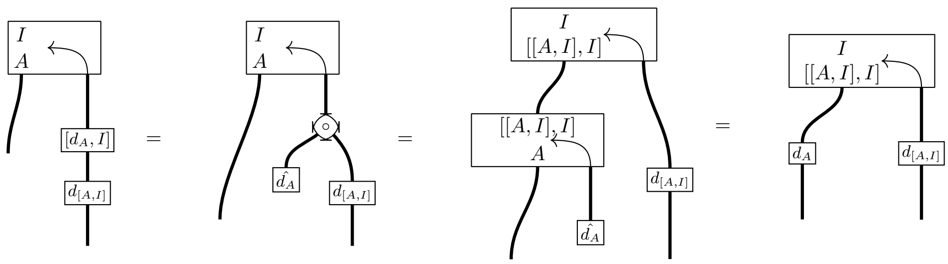
The proofs of the identities used above can be found in Appendix A. The proof that $d_{[A,B]}$

decomposes as above is then given as follows.



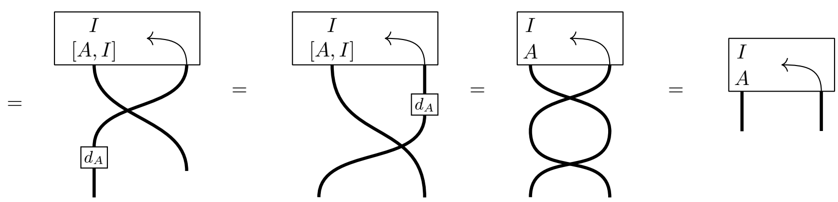
(A.18)

By assumptions d_A and d_B are isomorphisms, so $[d_B, id]$ is an isomorphism. It can be shown that $[d_B, id]$ is always the the right inverse of $d_{[B,I]}$ since first by expanding the definition of $[d_B, I]$



(A.19)

and then using the definition of any canonical morphism d_X twice.



(A.20)

Since $[d_B, id]$ is an isomorphism and $[d_B, id]$ is a right inverse for $d_{[B,I]}$, it follows that $d_{[B,I]}$ must be an isomorphism. Since $\otimes$ preserves isomorphism with double dual $d_{A\otimes[B,I]}$

must be an isomorphism and by the same reasoning as for B it follows that $d_{[A\otimes[B,I],I]}$ is an isomorphism. This completes the proof that every part of the given decomposition of $d_{[A,B]}$ is then an isomorphism, entailing that $d_{[A,B]}$ itself must also be an isomorphism. □

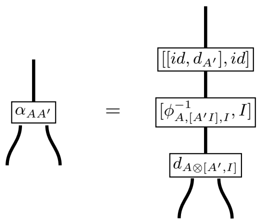
A.4 Wires with no-signalling states

Theorem 22. *Let $\mathcal{C}$ be a deterministic closed symmetric monoidal category with no correlations with single-state objects, then if*

- *$\otimes$ preserves equivalence with double duals*
- *A and A' each have enough states and are canonically equivalent to their double duals*

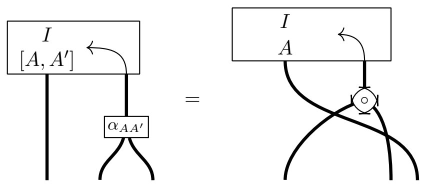
then the object $[A, A']$ has no-signalling states.

Proof. We first show that every effect $\Pi : [A, A'] \to I$ can be written as an application of a discard effect and an insertion of a state. This is a consequence of the isomorphism $A \otimes [A', I] \cong [A, A'] \Rightarrow I$ constructed by the following morphisms.



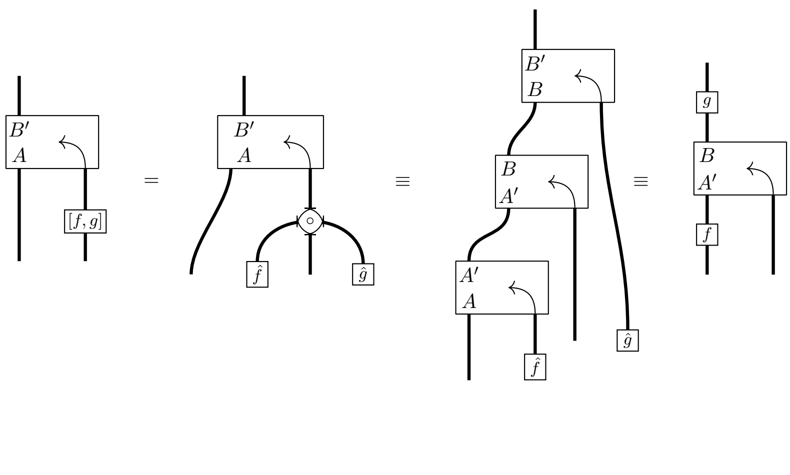
(A.21)

Indeed one can show the following identity



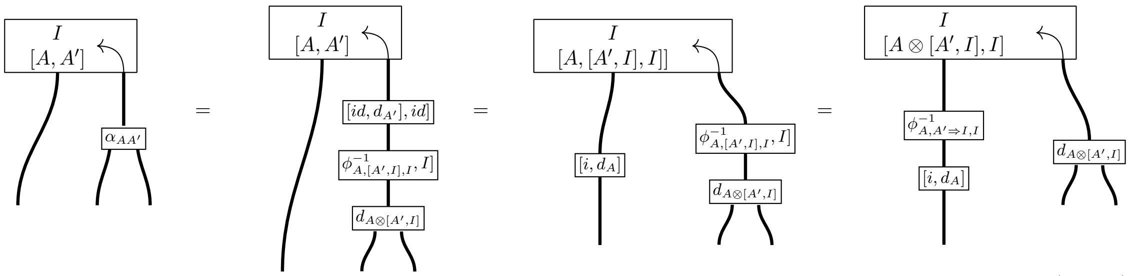
(A.22)

Using the general formula



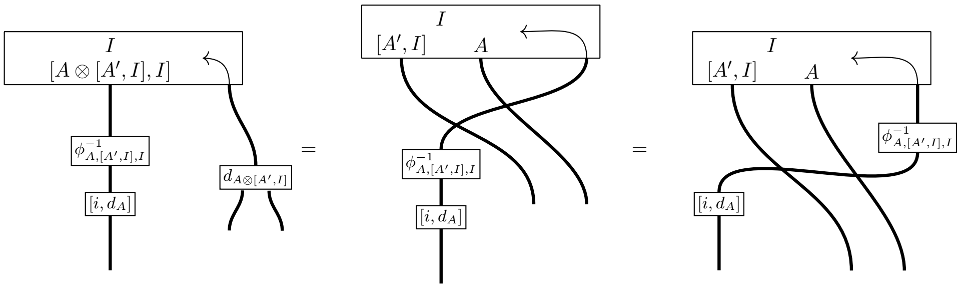
(A.23)

twice.



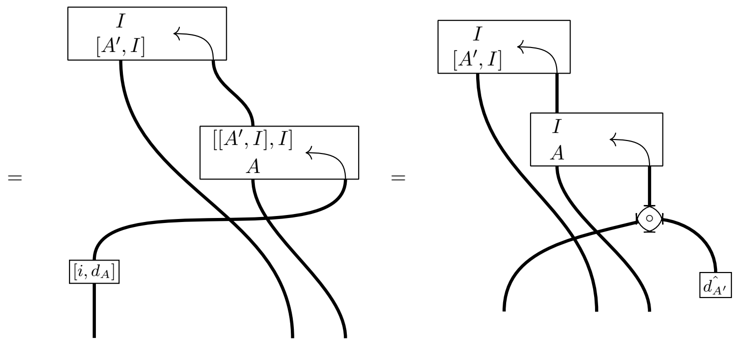
(A.24)

Then using the defining property of d,



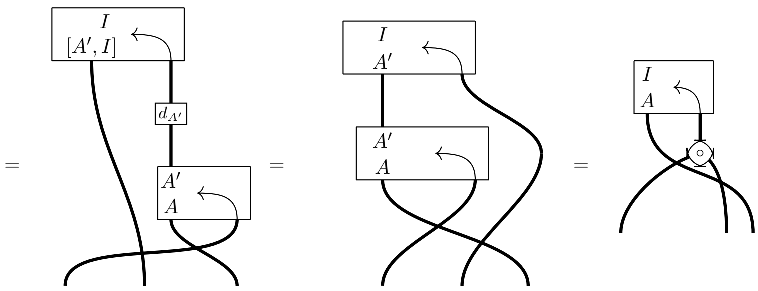
(A.25)

and the natural isomorphism ϕ,



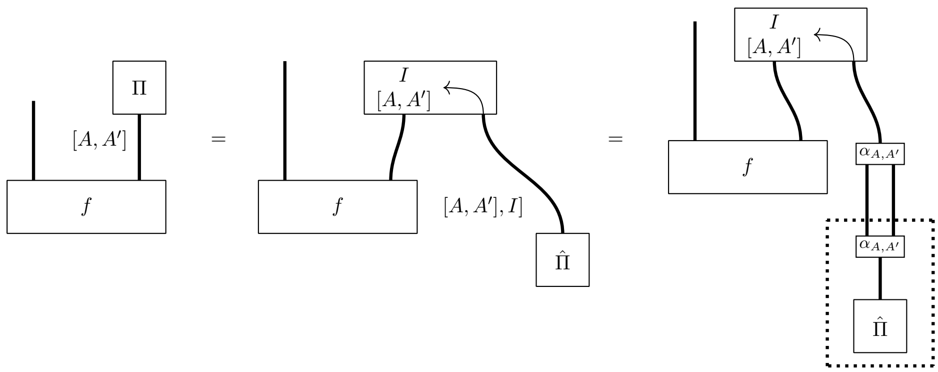
(A.26)

and the defining identity of the sequential composition supermap twice we reach



(A.27)

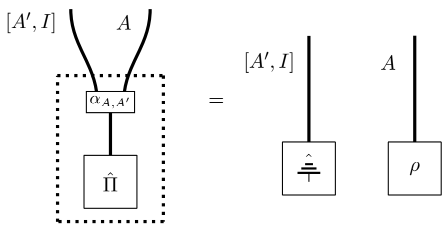
With this identity in mind we note that for *every* effect $\Pi : [A, A'] \to I$



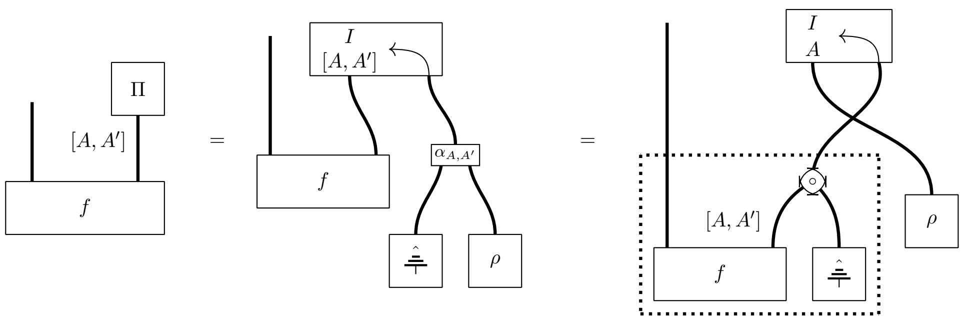
(A.28)

we then use the property of no correlations with single-state objects on the state highlighted on the bottom left,

(A.29)

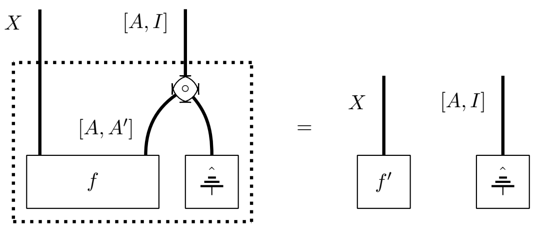


to reach



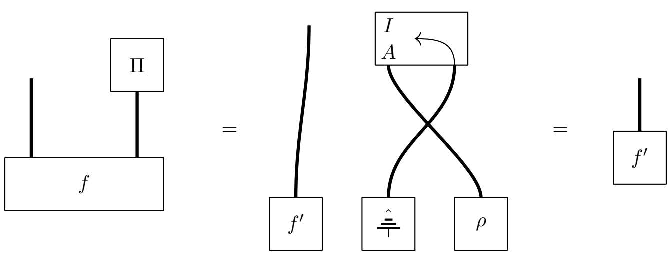
(A.30)

This time we use no correlations with single-state objects on the bipartite state highlighted on the bottom right,



(A.31)

this finally entails that there exists some state f' such that for every effect Π.



(A.32)

which is precisely the statement that $[A, A']$ has no-signalling states.

$\square$

Appendix B

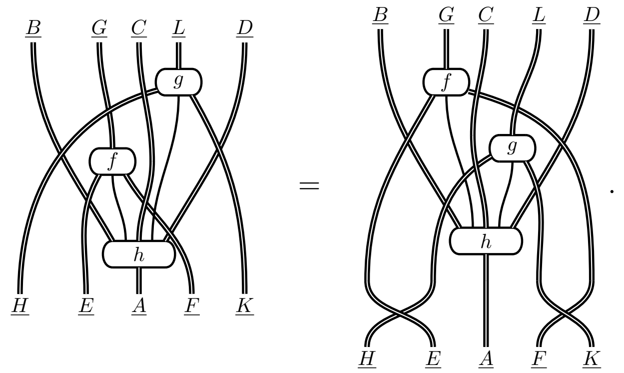
Polycategories of Supermaps

In this appendix, we show that supermap definitions using compact closed categories, and supermap definitions using polyslots, both return symmetric polycategories [194,197]. Within the main text we have also shown that both theories enrich the symmetric monoidal categories they act one, and each theory could furthermore be seen as a model of theories of supermaps as defined in chapter 6 by noting that in both cases underlying identity morphisms and famillies of identity functions appear as polymorphisms of type $[A_1 \otimes \dots \otimes A_n, A_1' \otimes \dots \otimes A_n'] \to [A_1, A_1'] \dots [A_n, A_n']$. Let us here be more explicit about symmetric polycategories, following [216]. A symmetric polycategory $\mathbf{P}$ is a collection of objects $A, B, \dots$ and for each pair of lists $\underline{A}$ and $\underline{B}$ a collection $\mathbf{P}(\underline{A}, \underline{B})$ of morphisms with:

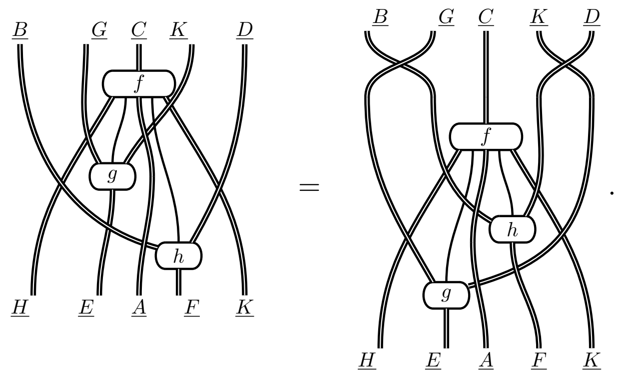
- A functorial action by permutations, meaning for morphism $f : \underline{A} \to \underline{B}$ and each pair of permutations $\sigma : [n] \to [n]$ and $\rho : [m] \to [m]$ a new morphism denoted $\rho(f)\sigma$ such that $\rho'(\rho(f)\sigma)\sigma' = (\rho \circ \rho')(f)(\sigma' \circ \sigma)$.
- For each pair $f : \underline{A} \to \underline{B}X\underline{C}, g : \underline{D}X\underline{E} \to \underline{F}$ of morphisms a new composed morphism $g \circ_X f : \underline{DAE} \to \underline{BFC}$.
- For each object A and identity morphism $i_A : A \to A$.

Composition is subject to associativity and identity laws alongside:

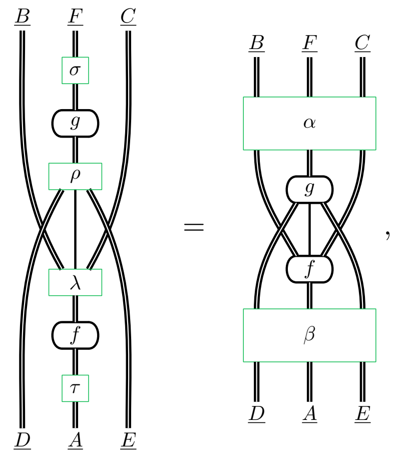
- Interchange 1:



- Interchange 2:



- Equivariance with respect to permutations:

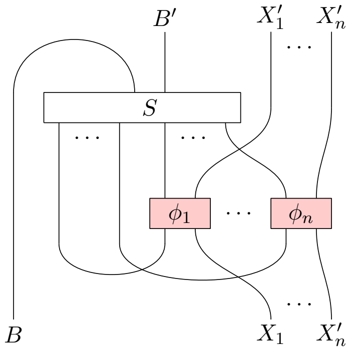


It will help us to be more precise about the last equivariance requirement, which can be written explicitly as $(\sigma g \rho) \circ_X (\lambda f \tau) = \lambda_{X\to\underline{A}}\sigma_{i_B(-)i_C}(g \circ_X f)\tau_{i_D(-)i_E}\rho_{X\to\sigma(\underline{A})}$ where for instance $\sigma_{i_B(-)i_C}$ means $i_{\underline{B}\otimes\sigma\otimes i_{\underline{C}}}$ and $\rho_{X\to\underline{A}}$ represents ρ in which the role of X is replaced by the entire list $\underline{A}$. By functorality, when equivariance with respect to τ and σ for ρ and λ set to the identity is known, what remains to be checked is the equivariance law with τ and σ set to the identity. Note that equivariance with respect to swaps can be used to deduce equivariance for general λ and ρ by decomposing λ and ρ into swaps and using functorality.

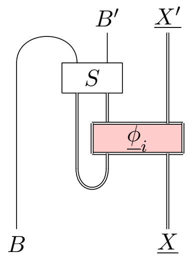
We will now prove that the minor diagrammatic generalisation of supermaps built from compact closed categories and the general definition of polyslots, both return symmetric polycategories. For the former case, equivariance and interchange are not the main issue, proving that the composed morphism is a supermap presents the main difficulty. In the latter case, proving that the composition of two polyslots is a polyslot becomes easy, whereas equivariance and interchange become less clear.

B.1 Polycategory of D-supermaps

We will find that when dealing with listed data naive diagrammatic representations become cumbersome, so for readability, we adopt a convention analogous to the convention used for genuine lists in multi/polycategories, choosing for instance to represent the diagram



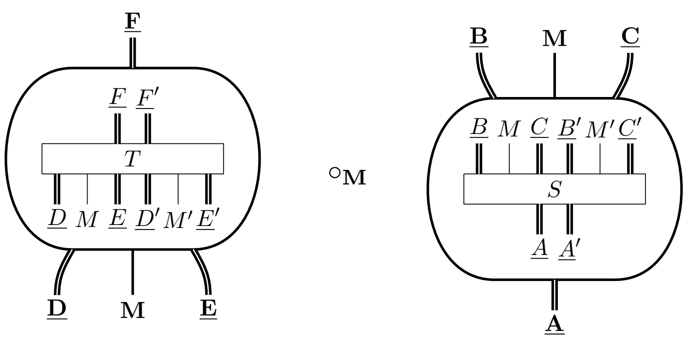
with the following compacted notation:



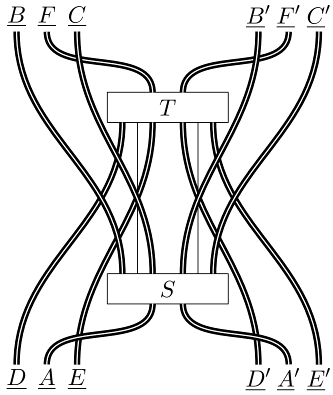
Such a language is not formalised but is used to convey the essence of proofs, with the unpacking of details left to the interested reader with access to larger pieces of paper.

Lemma 16. *A symmetric polycategory* $\mathbf{Psup}[\mathbf{C}]$ *can be defined with objects given by pairs* $[A, A']$ *of objects of* $\mathbf{C}$ *and morphisms of type* $S : \Gamma \to \Delta$ *given by the* $\mathbf{D}$*-supermaps of type* $S : \Gamma \to \Delta$ *with:*

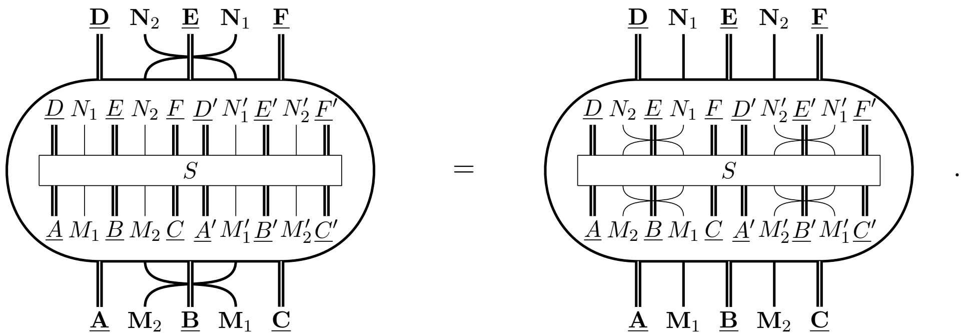
- *Composition rule given by taking:*



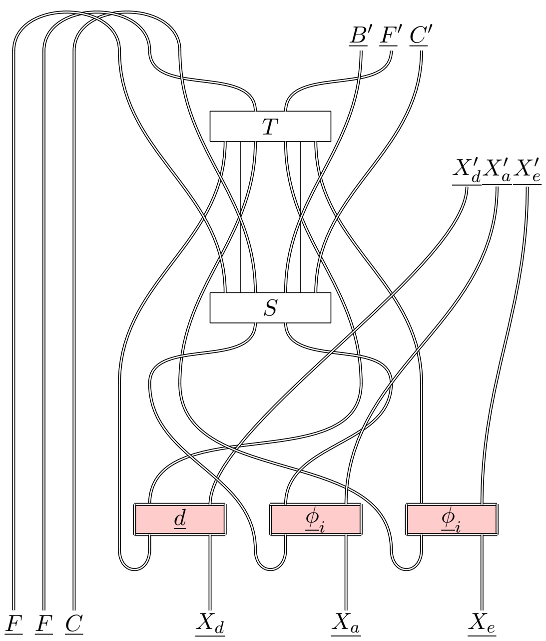
to be



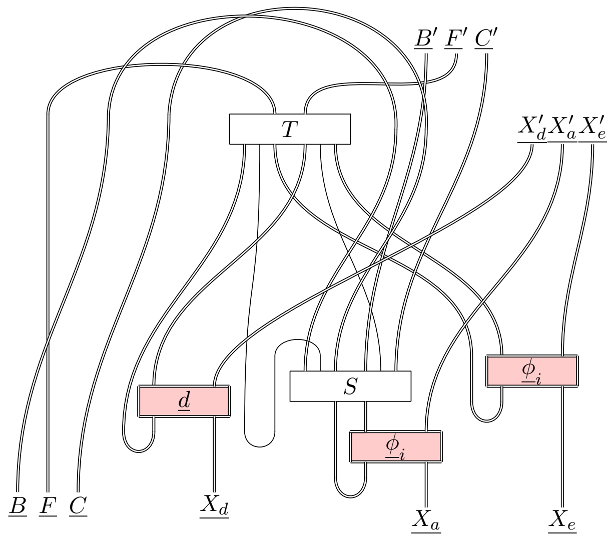
- *Action by permutations generated by:*



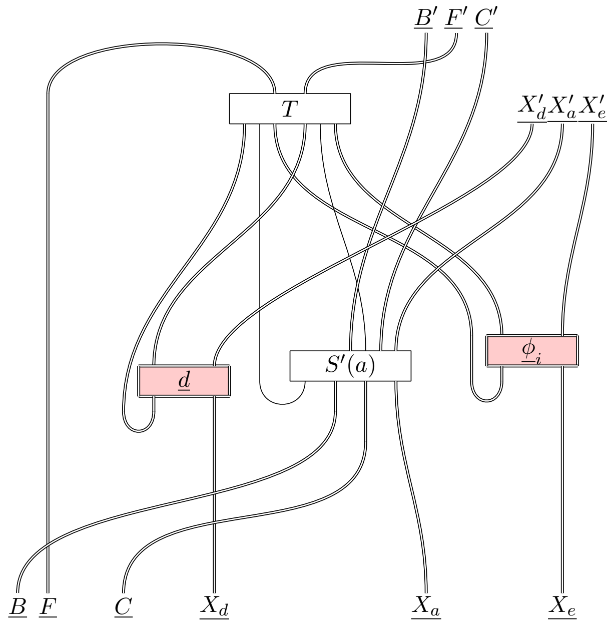
Proof. The action $\sigma(-)\rho$ by some general permutation σ on inputs or outputs is given by finding a decomposition of σ in terms of swaps and then generating the action for σ by repeated application of the actions of the associated swaps as defined above. Such a decomposition always exists and is well formed by the defining equations for the symmetry of **D**. Functorality is inherited from the symmetry of **D**. This composition rule returns a new **D**-supermap since the application of $T \circ_M S$ can be written



which by the interchange law for symmetric monoidal categories can be converted to

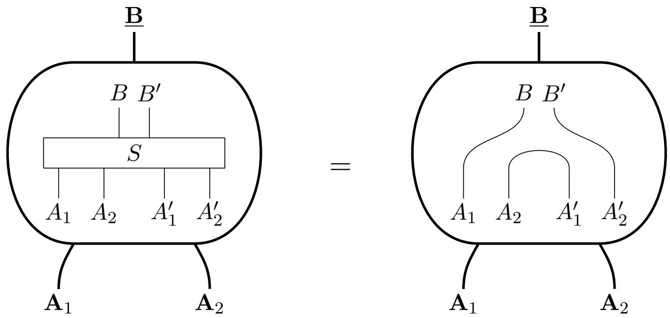


where since S is a **D**-supermap we can replace the action of S by a new morphism $S'(a)$ of **C** to give

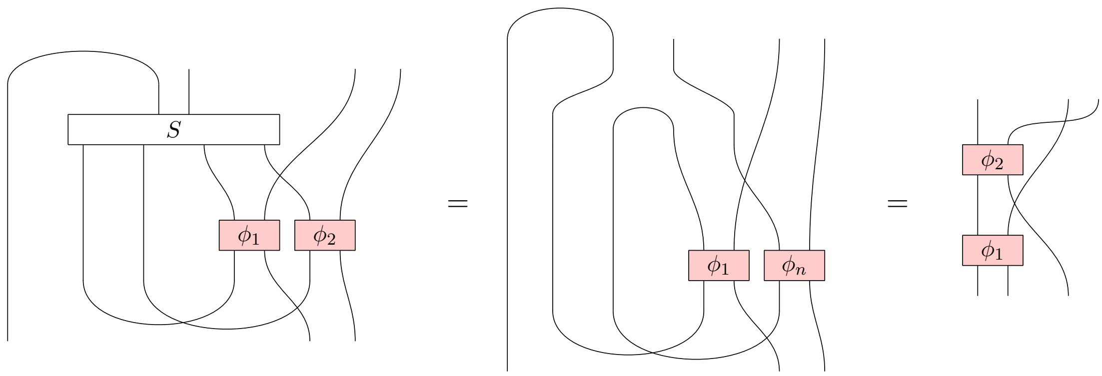


what remains is the actions of T on a series of channels with B, C considered as extensions of the morphism $S'(a)$, consequently, the entire global diagram is a morphism of **C**. The required associativity, interchange, and equivalence laws are inherited directly from the interchange laws and symmetry of the symmetric monoidal structure of **D**. □

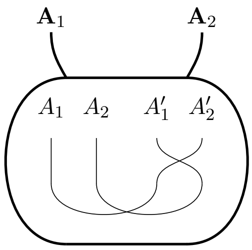
It is noted in the main text that composition along multiple wires ought not to be allowed, so as to avoid the creation of time-loops, this point can be made at a more technical level now an explicit definition of supermap has been given. A simple example demonstrates why two-wire composition rules are in general forbidden. Since **C** is a symmetric monoidal category, for any $\mathbf{C} \subseteq \mathbf{D}$ with $\mathbf{D}$ compact closed then there exists a $\mathbf{D}$-supermap of type $S : [A, A][A, A] \to [A, A]$ which performs sequential composition:



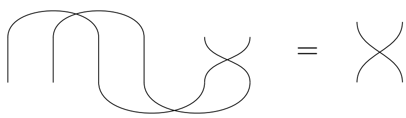
This is indeed a supermap since for all ϕ_1, ϕ_2 then:



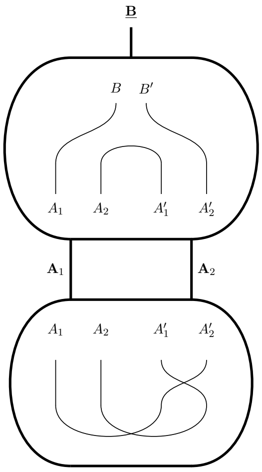
which since $\mathbf{C}$ is a symmetric monoidal category must be in $\mathbf{C}$. Next note that there exists a $\mathbf{D}$-supermap of type $\phi : \emptyset \to [A, A][A, A]$ given by:



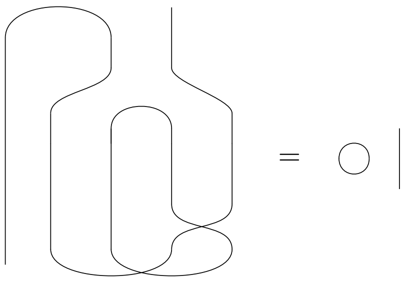
Indeed note that it is a supermap since the following



is a member of $\mathbf{C}$ given that $\mathbf{C}$ is symmetric monoidal. However, if we were to try to compose ϕ and S along both of their output/input wires, to give meaning to the following diagram



then a loop would be formed:



There is no guarantee that this re-normalisation by a scalar preserves membership of **C**, indeed in the study of quantum causal structure such loops are often interpreted as time-loops, and in the category **U** we find that such a re-normalisation does not preserve membership of **U**. In the above sense we can see that the natural emergence of a poly-categorical semantics can be understood as a compositional semantics which prevents the forming of time-loops.

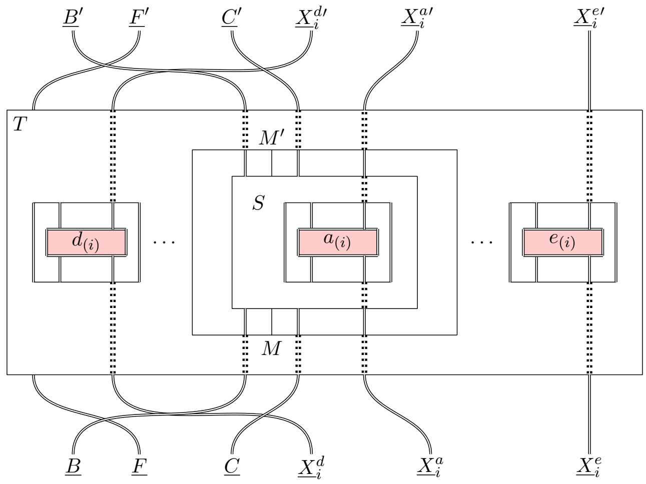
B.2 Polycategory of polyslots

To prove the following results algebraically is possible but extremely unreadable due to the need to keep track of symmetries, for readability we prefer to present our proofs in diagrammatic shorthand.

Theorem 23. *The polyslots on* **C** *define a polycategory* **pslot**[**C**] *with:*

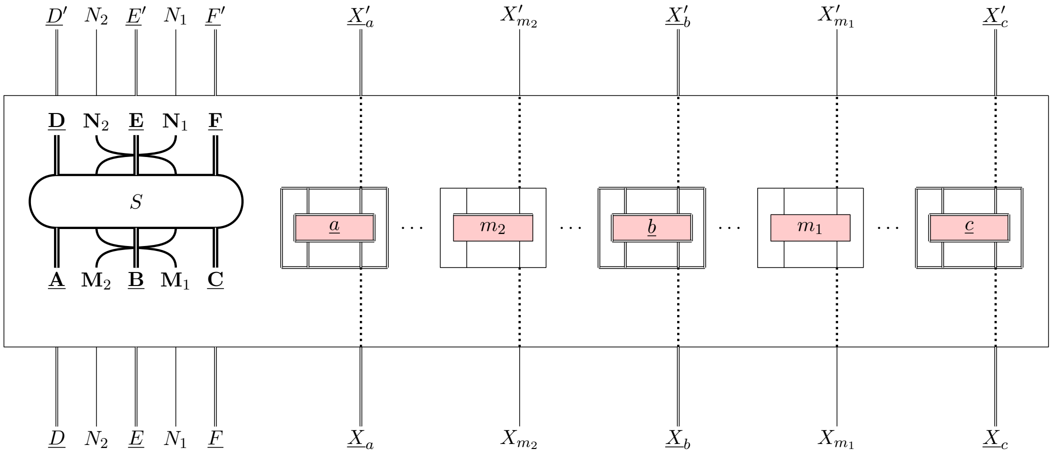
- *Objects given by pairs* $[A, A']$ *with* A, A' *objects of* **C**.
- *Poly-morphisms of type* $S : [A_1, A'_1] \dots [A_n, A'_n] \to [B_1, B'_1] \dots [B_m, B'_m]$ *given by polyslots of type* $S : [A_1, A'_1] \dots [A_n, A'_n] \to [B_1 \otimes \dots \otimes B_m, B'_1 \otimes \dots \otimes B'_m]$.

- *Composition* $T \circ_M S$ *of* $S : \underline{\mathbf{A}} \to \underline{\mathbf{B}}\mathbf{M}\underline{\mathbf{C}}$ *and* $S : \underline{\mathbf{D}}\mathbf{M}\underline{\mathbf{E}} \to \underline{\mathbf{F}}$ *given by taking* $T \circ_M S(d_{(i)}, a_{(j)}, e_{(k)})$ *to be*

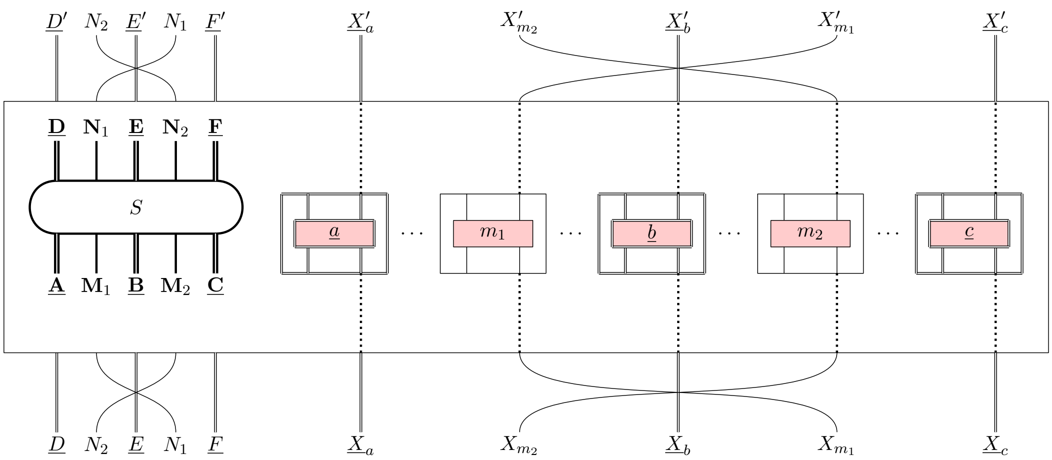


.

- *Action by permutations generated by taking*



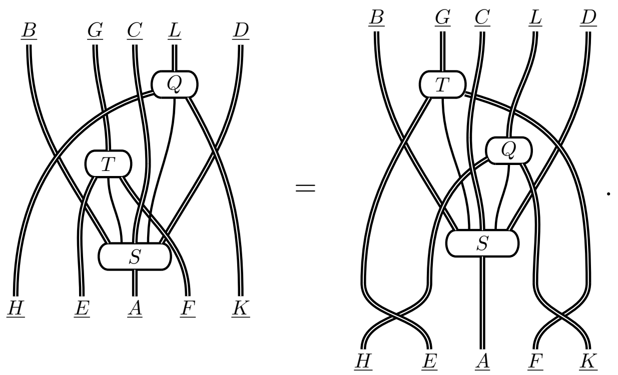
to be



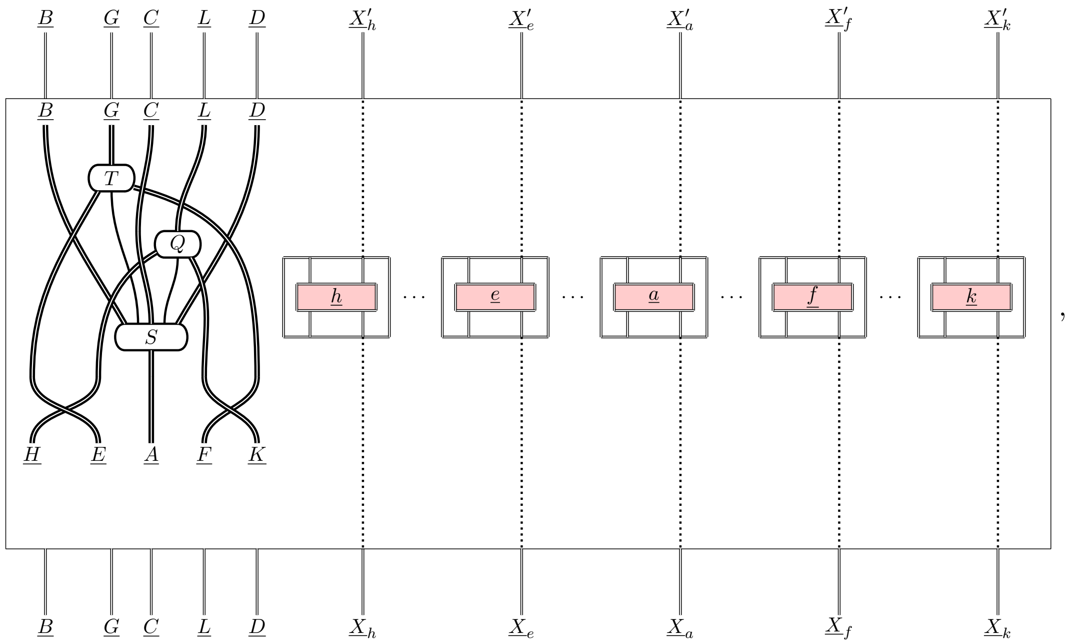
.

Proof. As was the case for **Dsup**[**C**] the composed action for a general permutation can be given by composing the actions for any decomposition of σ into swaps. That this assignment to general permutations is well-formed and functorial for outputs follows from the symmetry of **C**, for inputs instead this follows from both the symmetry of **C** and the symmetry of **Set** which was used the flip the order of intputs to S. Associativity is inherited from **Set**. The locally-applicable transformation given by choosing the identity function for each extension is indeed a slot, and acts as the identity in **pslot**[**C**]. The composition of two polyslots returns a polyslot, filling in all but one of the $d_{(i)}$ returns a slot up-to braids since $S(a_{(j)})$ is a morphism of **C**, filling in all but one of the $e_{(i)}$ returns a slot up-to braids for the same reason. Filling in all but one of the $a_{(i)}$, S acts as a slot as does T, and the sequential composition of two slots is a slot.

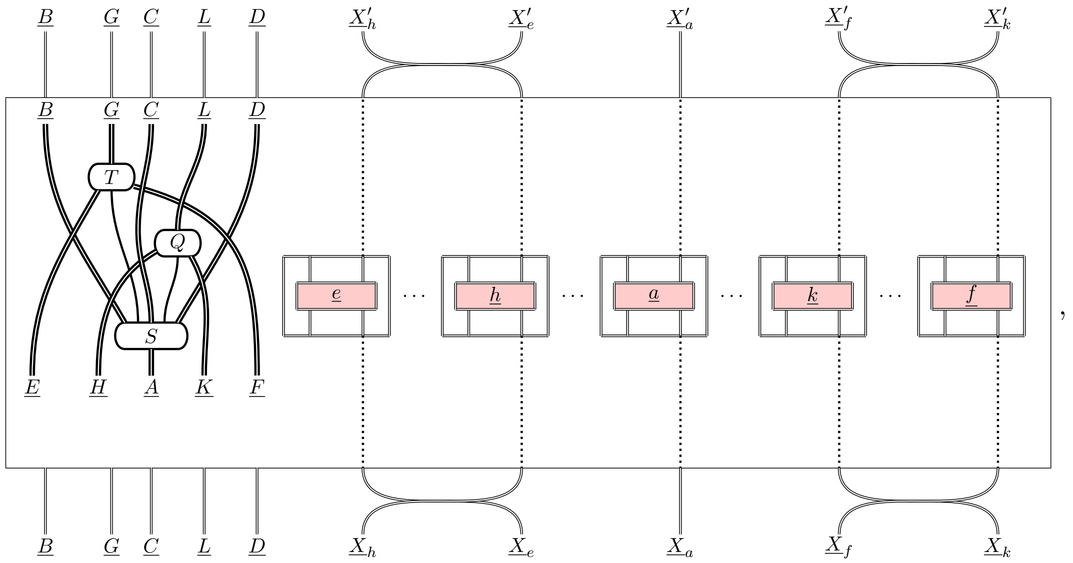
We now check the interchange laws for composition, the most involved one to prove being the following:



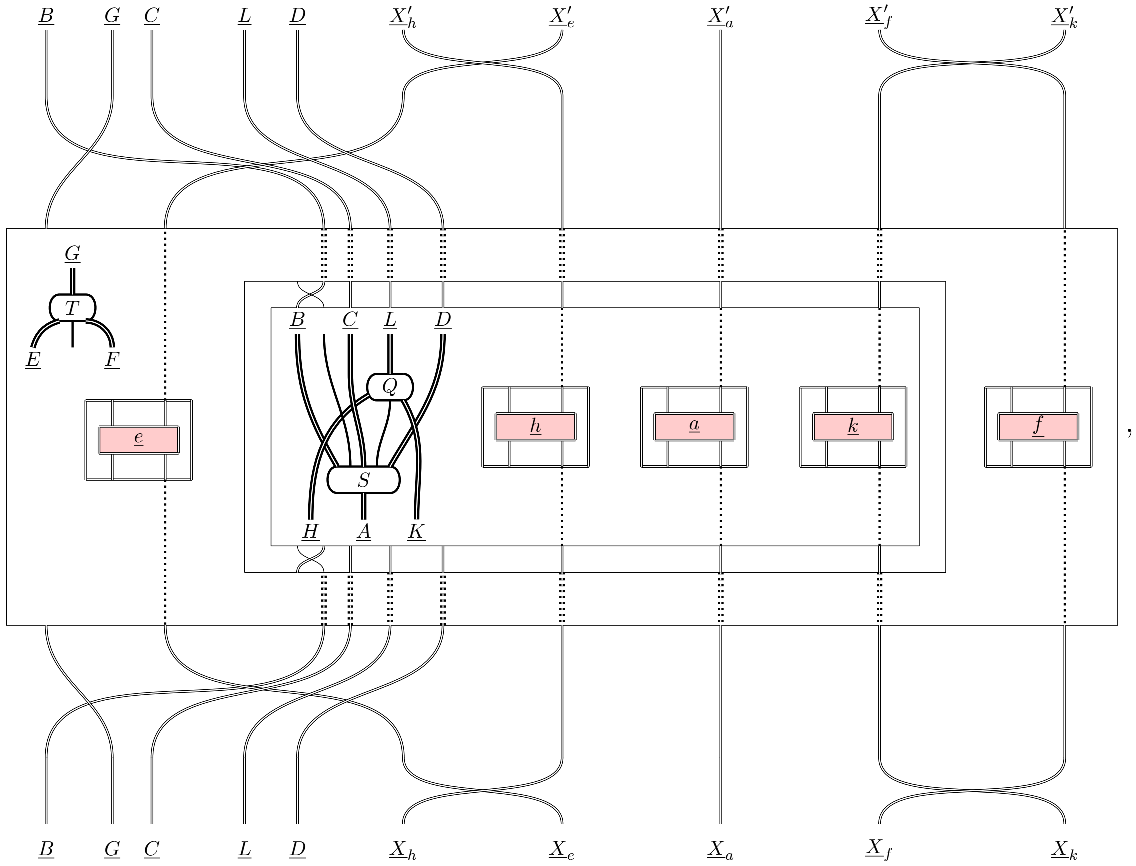
Consider



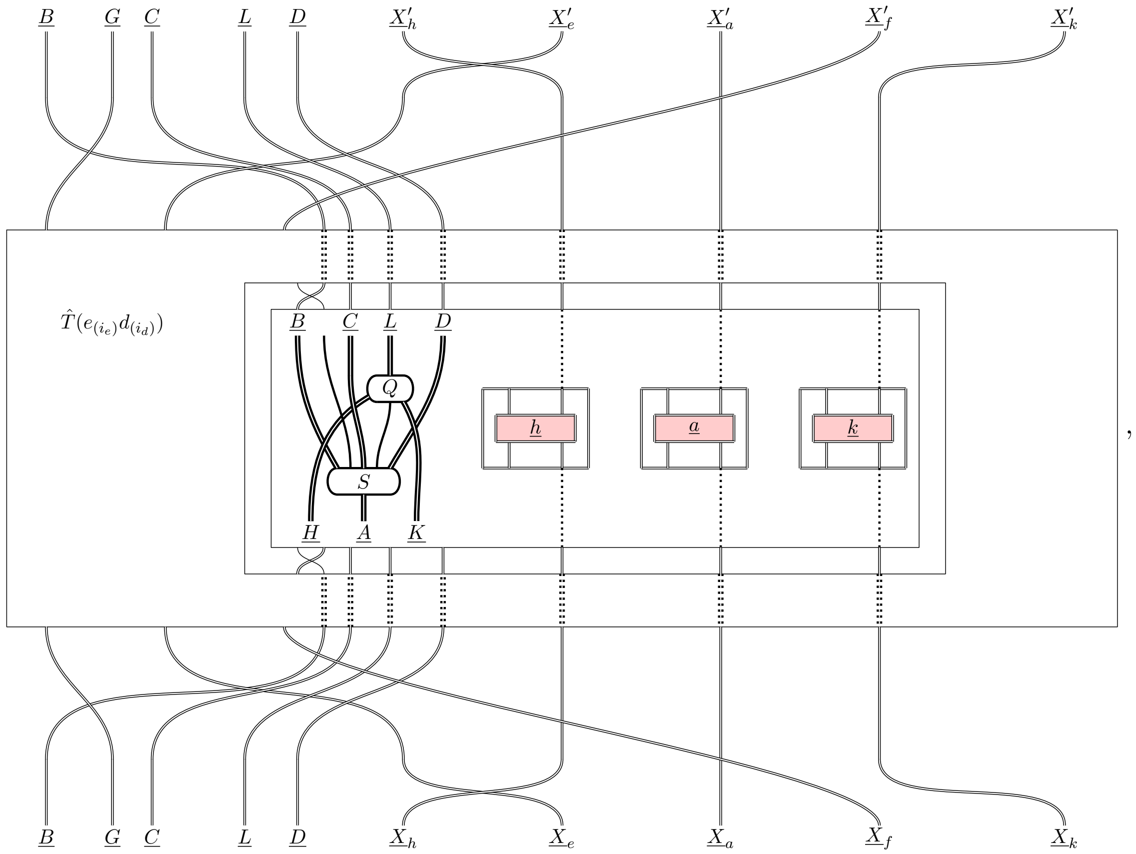
applying the symmetric action gives:



using the composition rule gives:

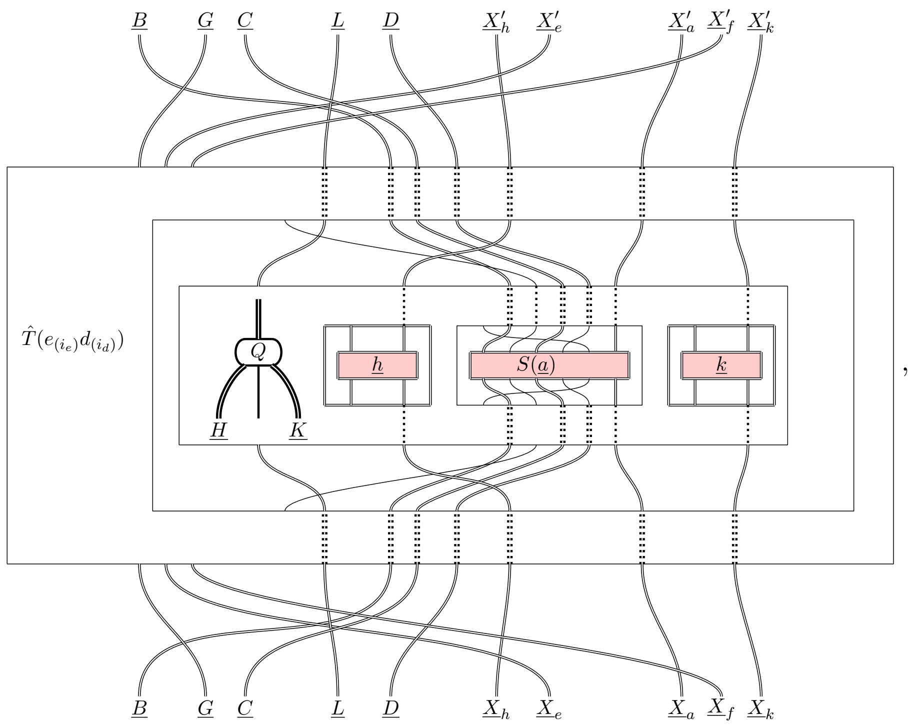


or equivalently using the definition of slot induced by a polyslot :



,

where $Z = \underline{B} \otimes \underline{C} \otimes \underline{L} \otimes \underline{D} \otimes \underline{X}_h \otimes \underline{X}_a \otimes \underline{X}_k$ and similarly for Z'. We then use the composition rule again to give:



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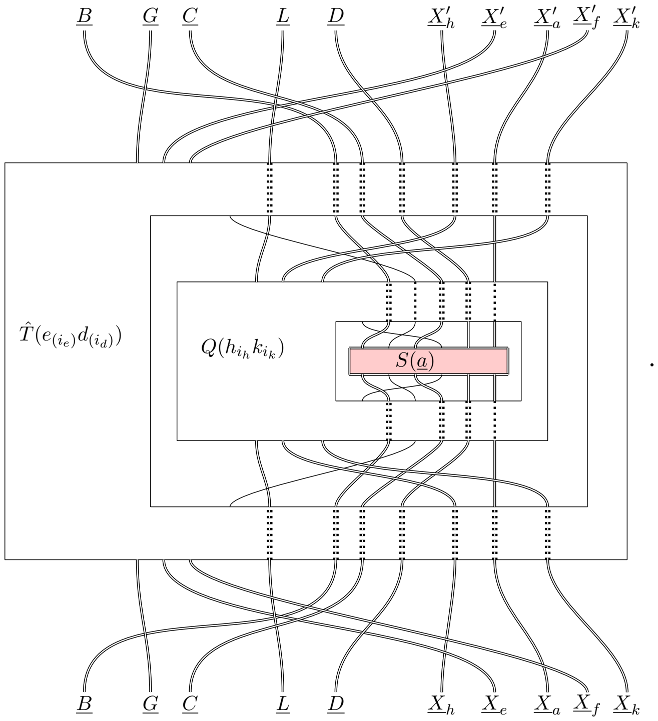
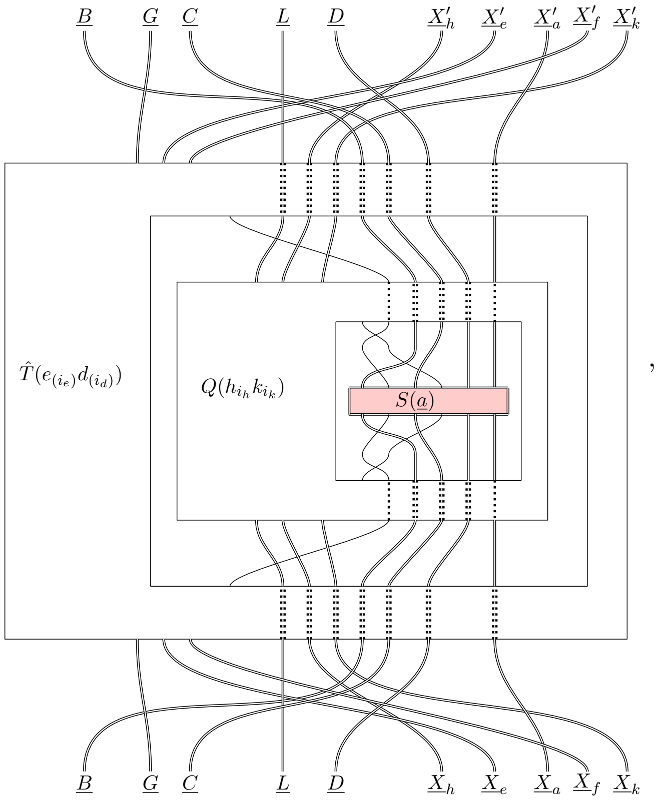
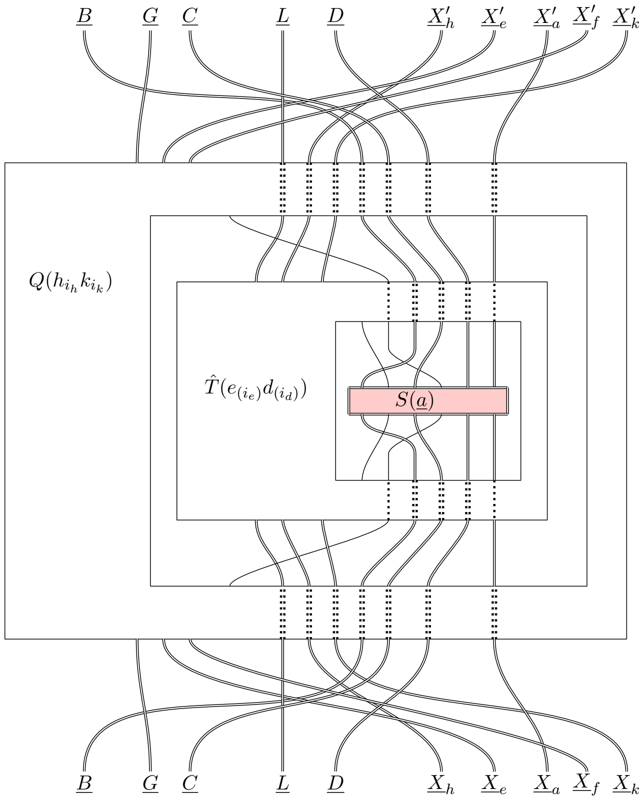
which after converting into slot form gives:



Using a series of swaps to set up the defining condition for slots gives:

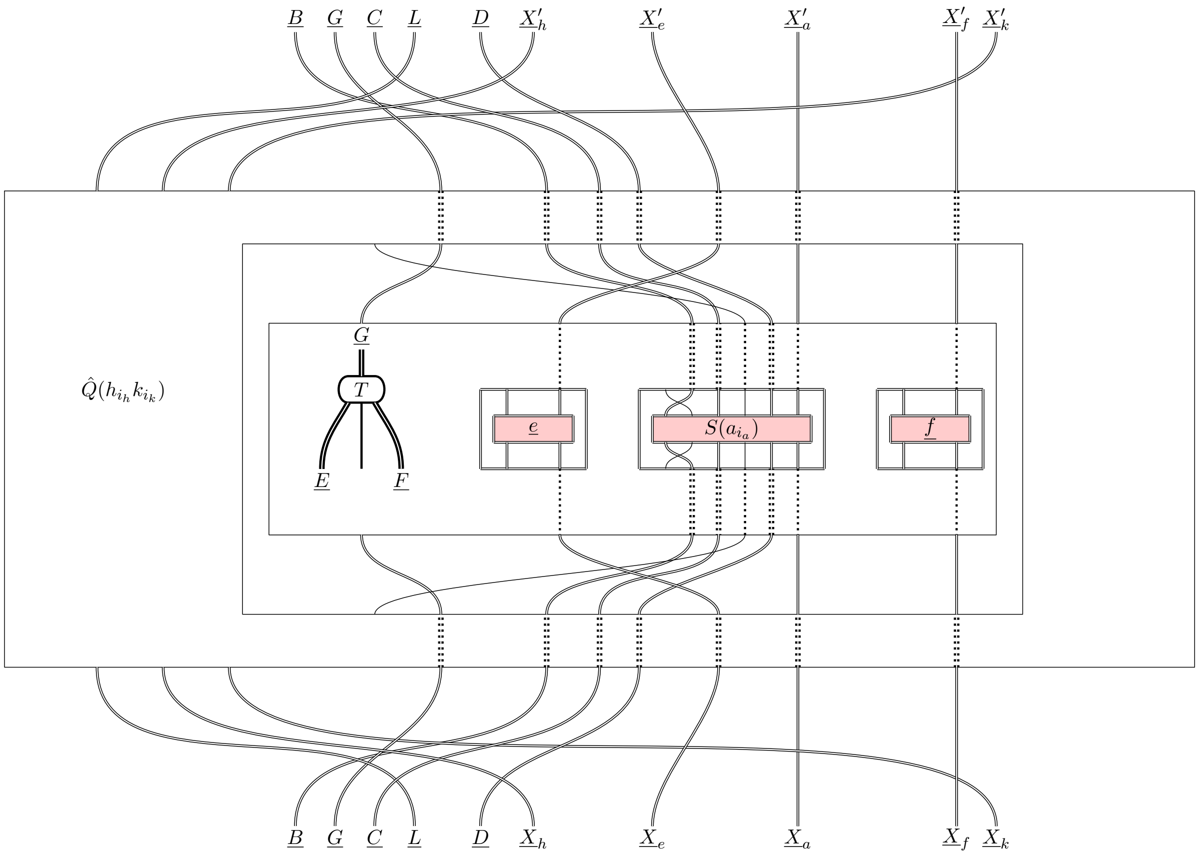


after-which the slot equation can finally be used to return:



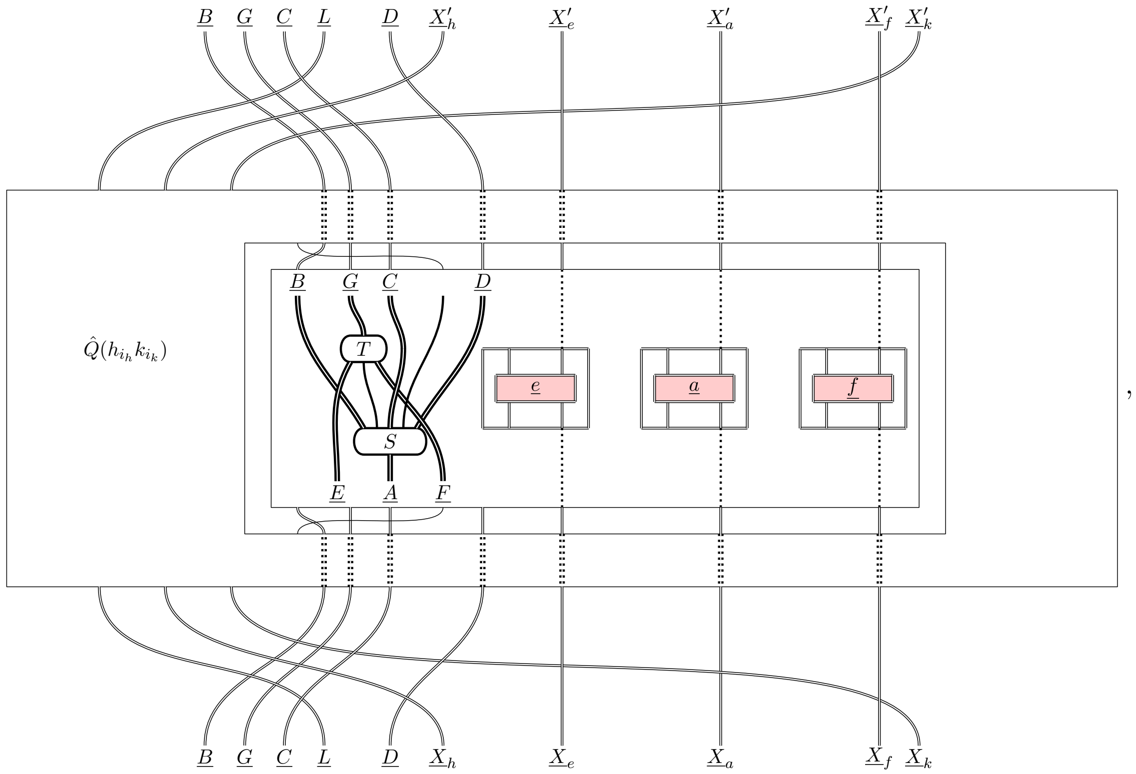
.

Unpacking the definition of $\hat{T}$ gives:

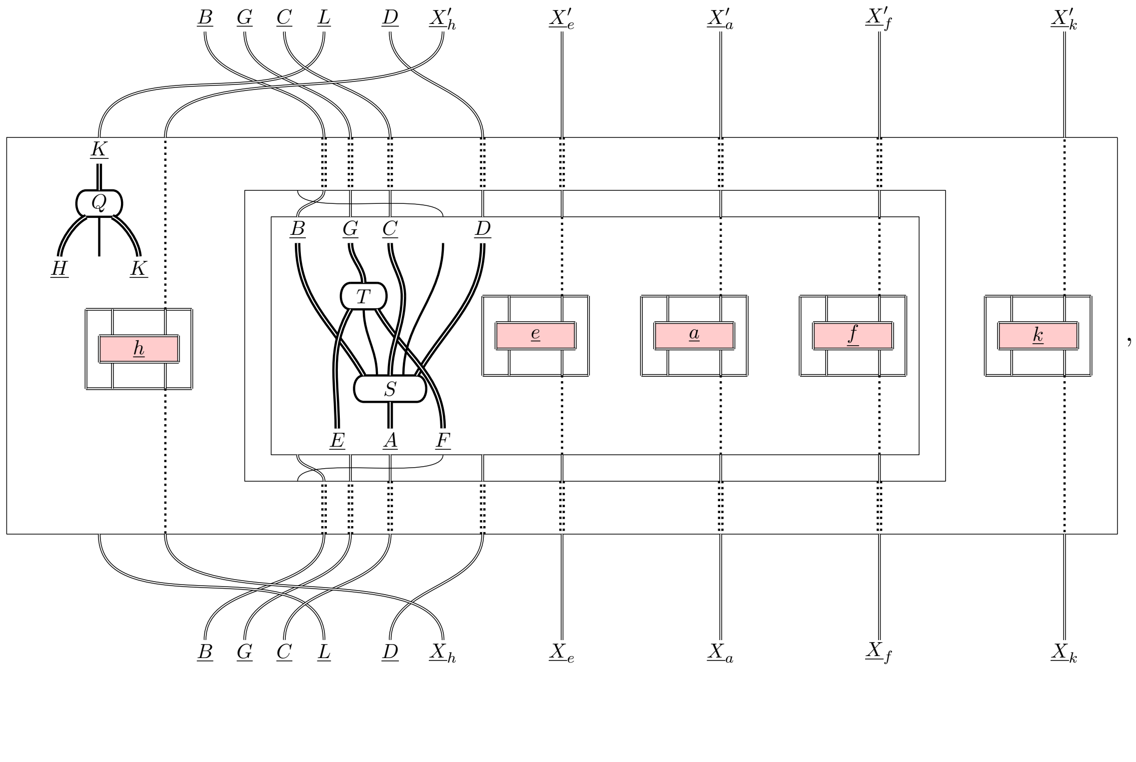


,

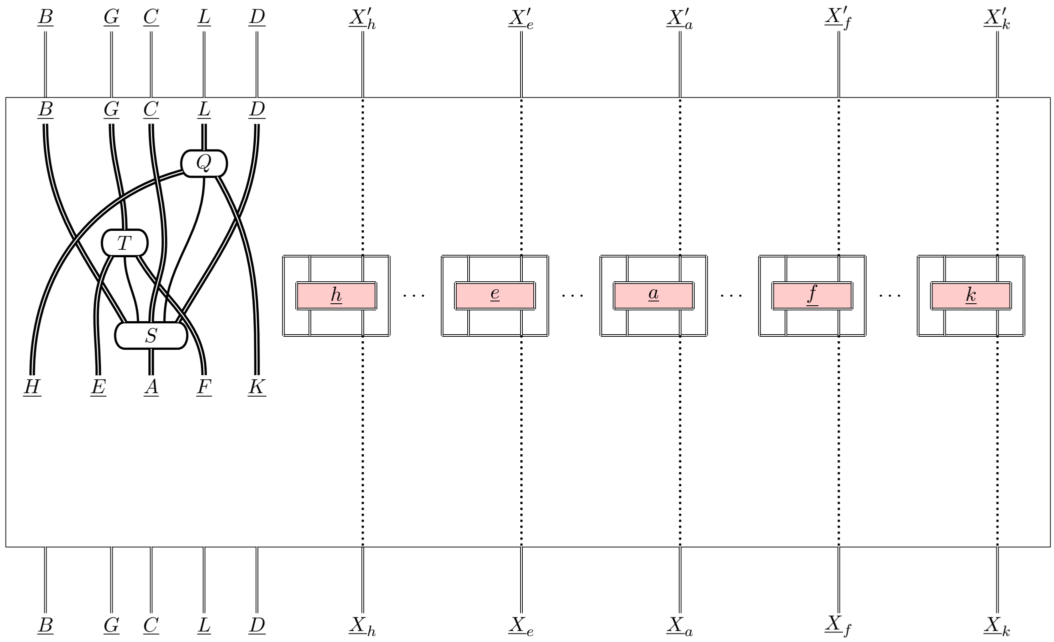
re-packaging the composition between T and S gives



after-which unpacking the definition of $\hat{Q}$ gives



and finally repackaging the composition rule completes the proof:



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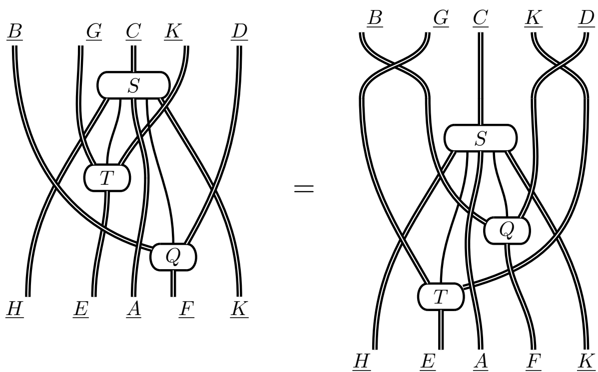
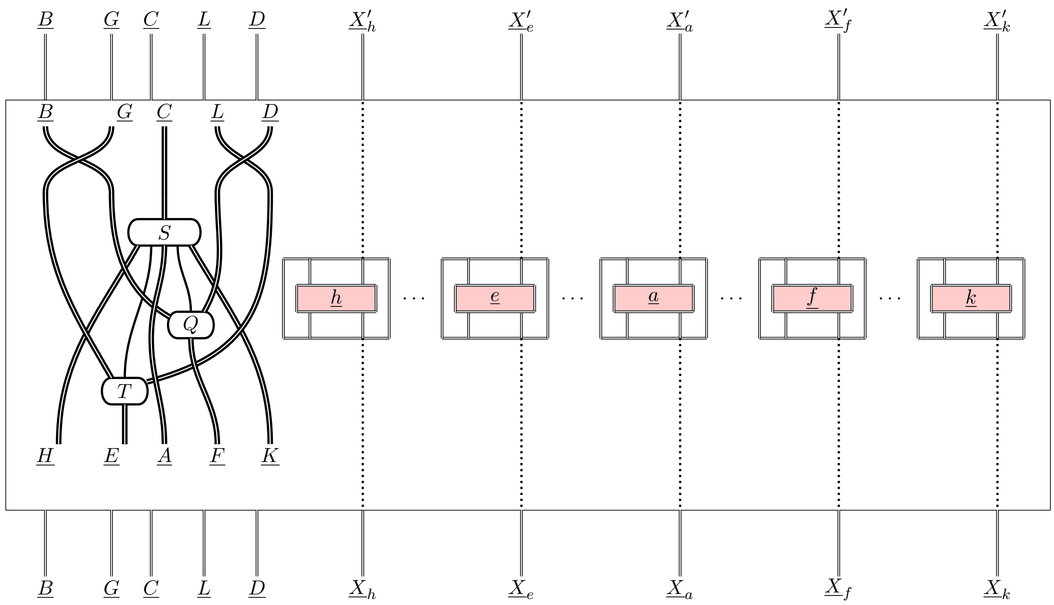
The other interchange law which needs to be checked:



,

is more straightforward. We begin by considering the latter term:



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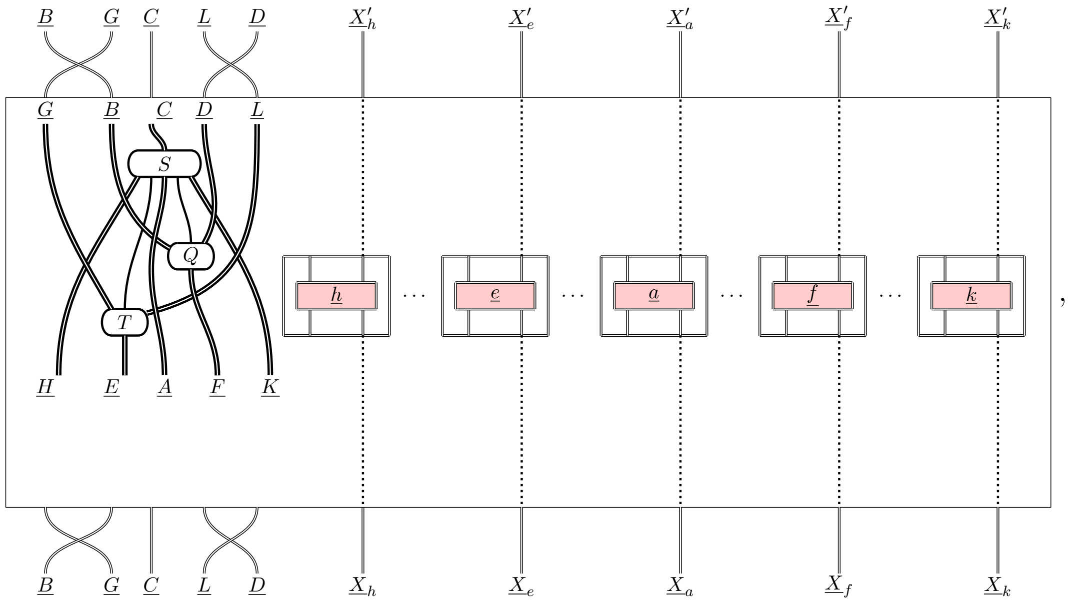
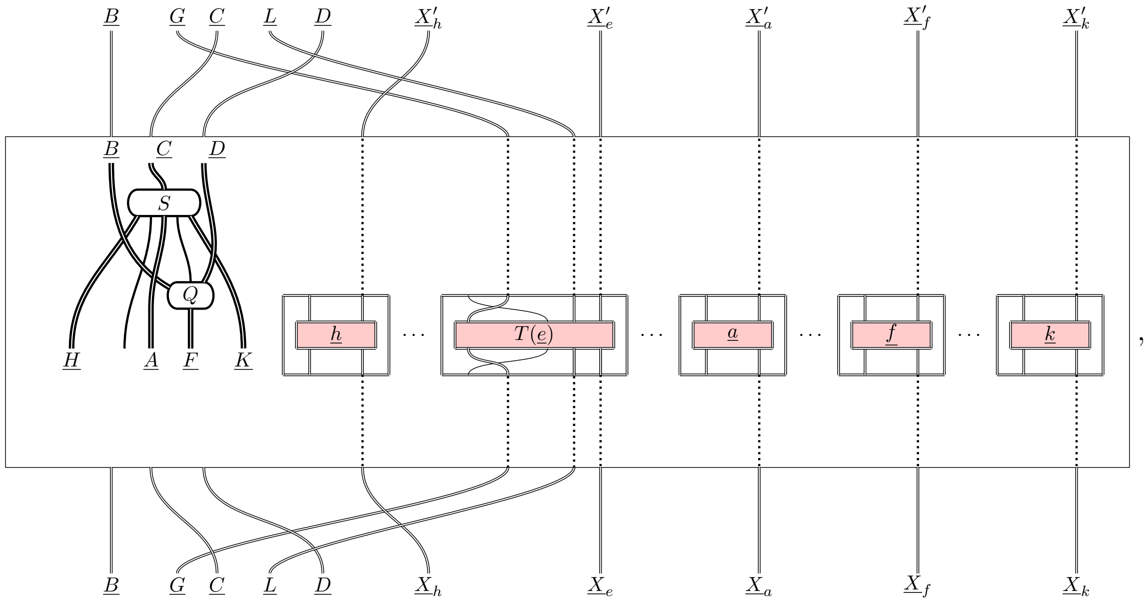
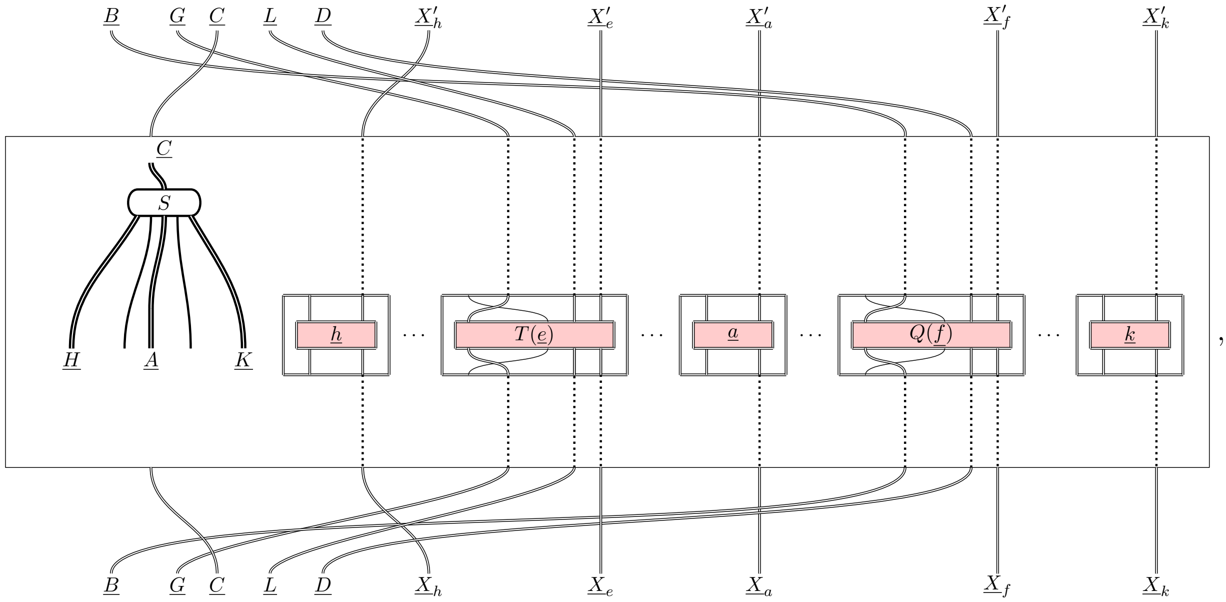
and then use the definition of the symmetric action:



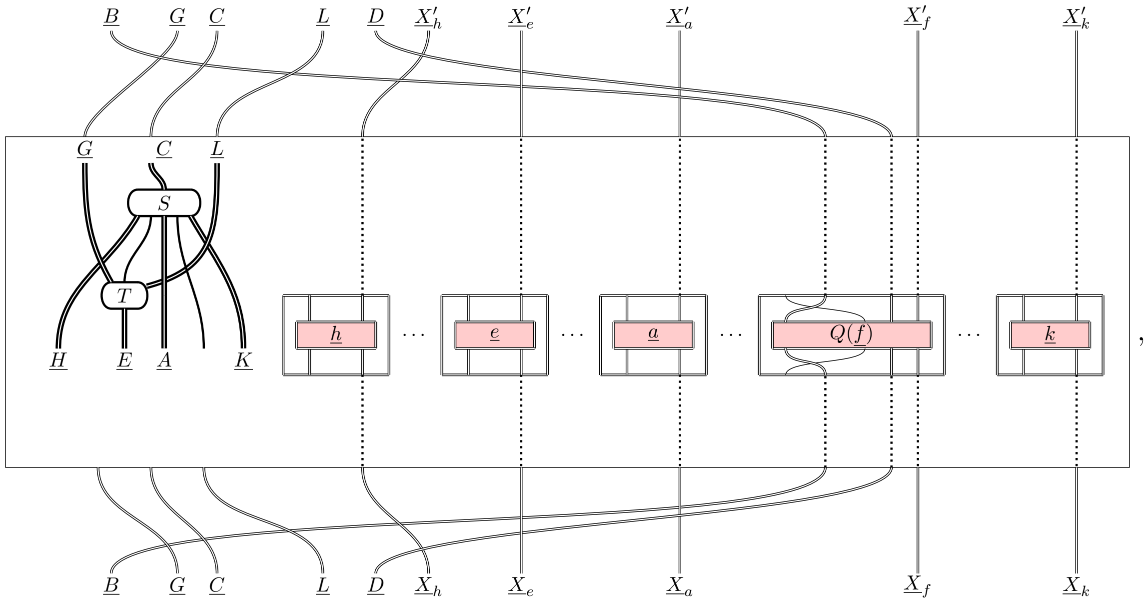
then we use the definition of composition along T:



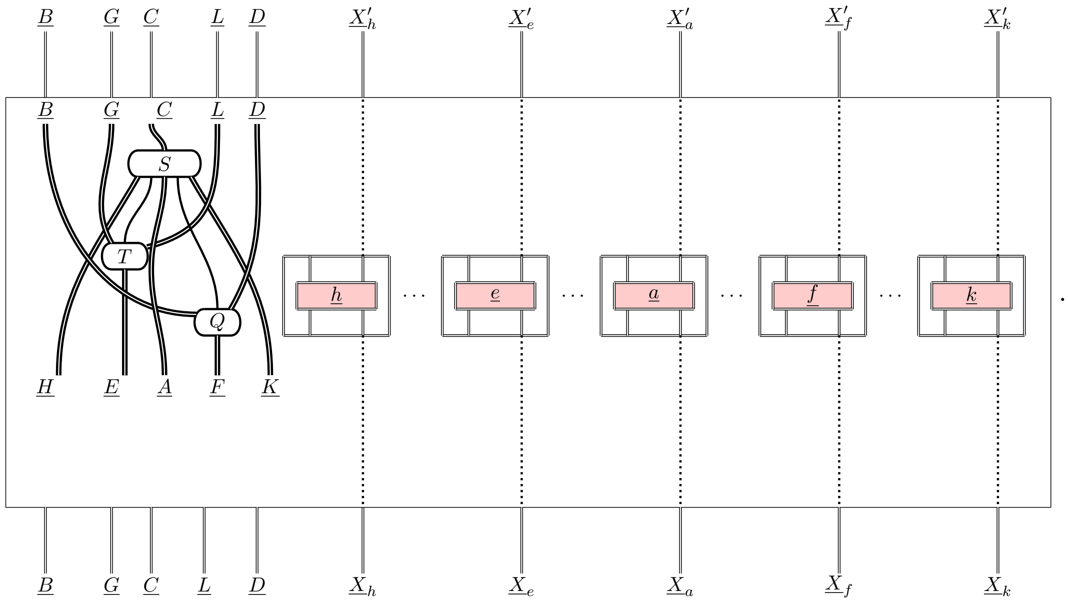
and then use the definition of composition along Q:



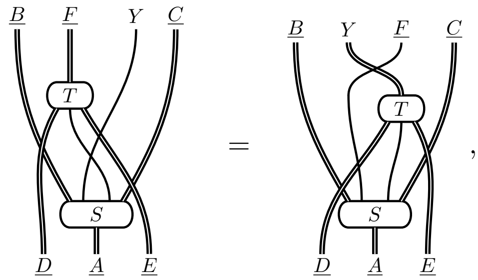
then using the definition of composition along T:



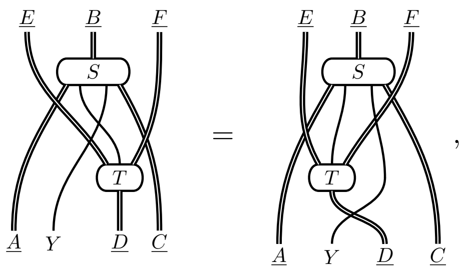
and finally the using the definition of composition along Q gives the result:



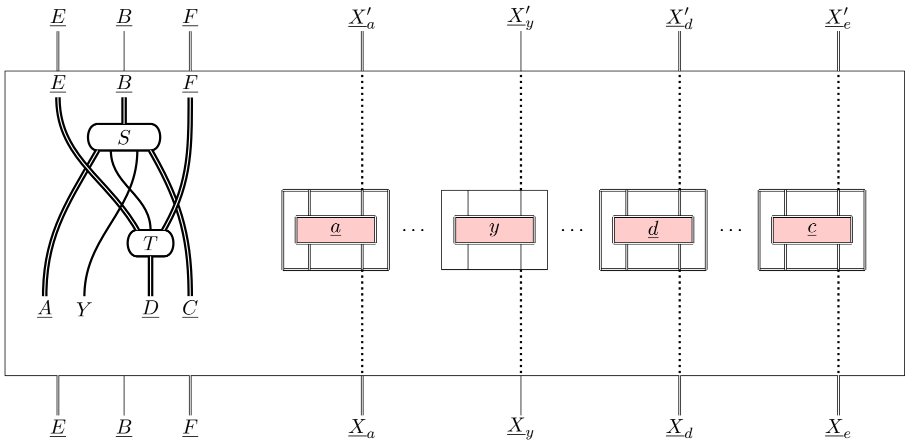
Finally, we check equivariance with respect to permutations. Given the functorial action by permutations, the checking of equivariance can be performed piece-wise by checking on swaps and then concluding general equivariance by functorality. This means that to check equivariance we will just need to establish that:



and further that:



along with the horizontal reflection of each of these laws. Let us show how to prove the first, with the proof of the rest given by the same methods. To begin, consider:



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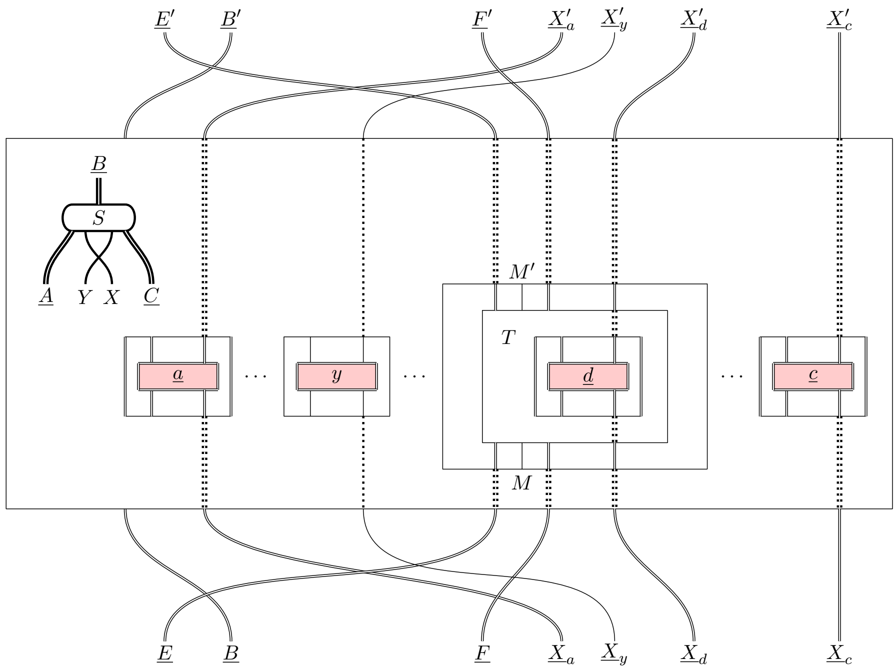
using the polsylot composition rule gives:



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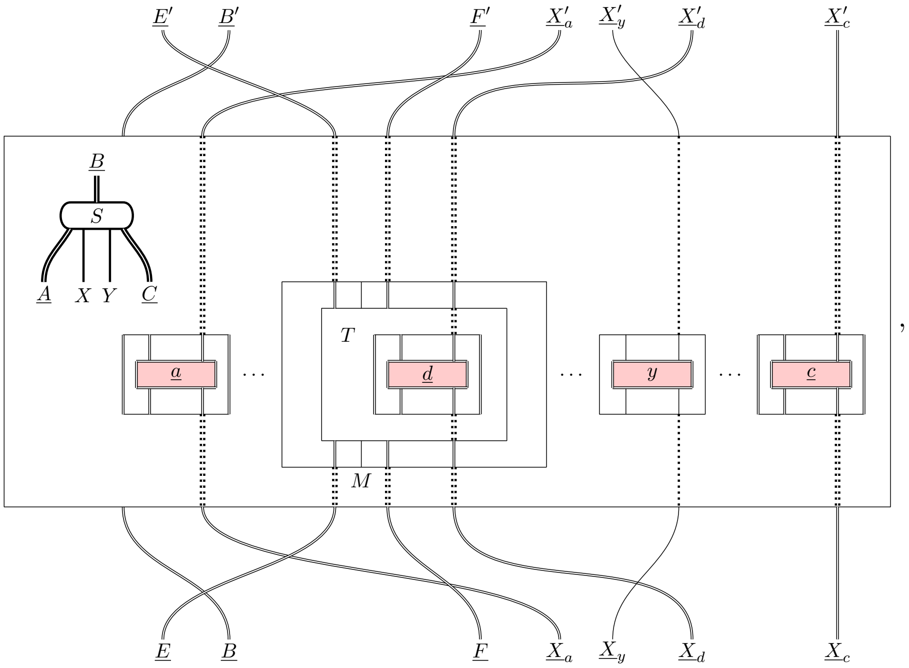
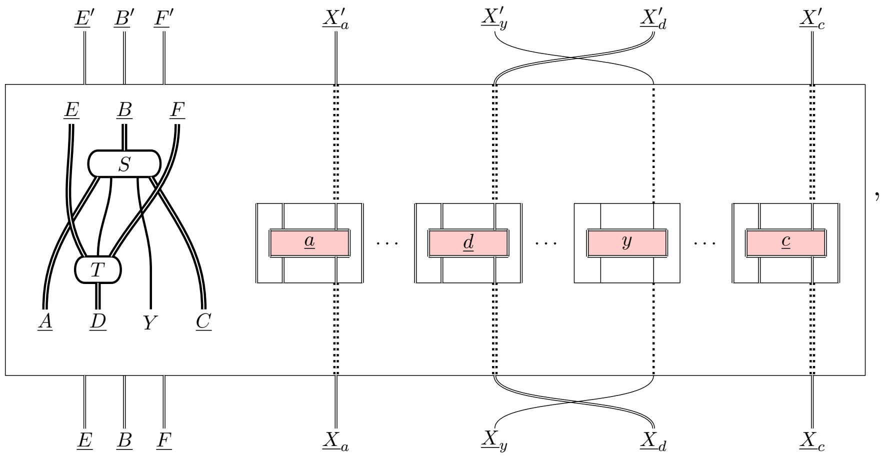
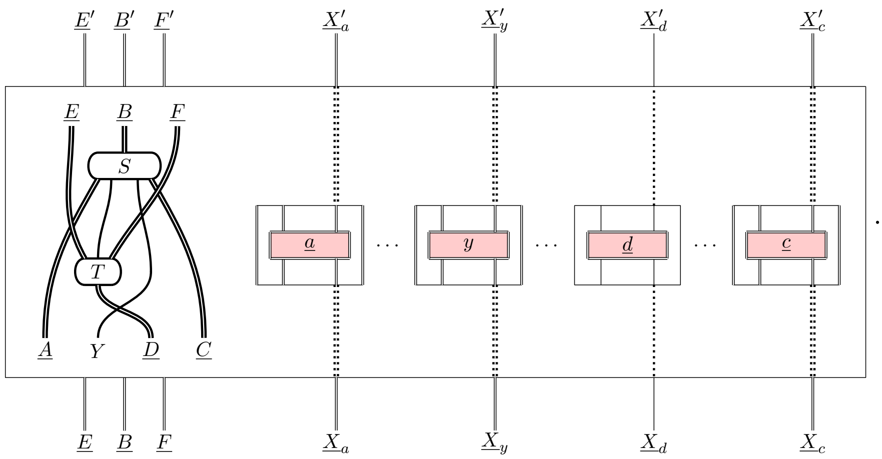
then using the action under a swap for S gives:



untangling the polyslot composition rule then gives:



which using the action under a swap completes the proof:



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