SIMULATING QUANTUM PROCESSES USING CLASSICAL STRUCTURES

Contents

Acknowledgments

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Abstract. In earlier work, it has been shown that Frobenius algebras with certain properties can be used to simulate classical data. They are especially useful because in their definition itself they incorporate the most major difference between classical and quantum data- which is the absence of copying and deleting operations for arbitrary quantum states. In later work, it has been shown that a special, commutative, dagger-Frobenius algebra (classical structure) corresponds to an orthonormal basis for a finite-dimensional Hilbert Space. In this paper, we show how classical structures can be used to axiomatize maximally entangled states for two qubits. We then go on explain how this new representation of quantum informatic protocols using classical structures can be used to encode the flow of quantum data. We explain how this process encodes all the possible observational branches simultaneously, and thus has the ability to tell us the end result that can be achieved in any particular protocol. We end with a summary of the results obtained, and suggestions for future work.

1. Introduction

1.1. Need for a Category-theoretic approach.

The mathematical foundations for quantum mechanics were laid over 70 years ago by the German mathematician John Von Neumann [8]. Since then, the mathematical roots of Quantum mechanics have been placed firmly in the framework of Hilbert Spaces (Mostly finite-dimensional, but also infinite-dimensional). Although the descriptions of various phenomena and properties of the quantum world is quite cumbersome due to the extensive use of matrices, there has never been any need to question this framework, as it is entirely accurate and complete. However, in recent times some researchers have argued that this approach to quantum mechanics is very "low-level", and that we require a "high-level" approach in order to make significant breakthroughs in the field of quantum computing and quantum information protocols. The "high-level" approach that they have chosen is through Category Theory. Let us first refresh our memory and look at the definition of a category.

Definition 1. A category $\mathbf{C}$ consists of:

1. A family $|\mathbf{C}|$ of objects.
2. For each A, B $\in |C|$, a set of morphism (or arrows) going from A to B, which is denoted by $\mathbf{C}$(A,B)
3. In addition to the above, we also have a binary operation (composition) defined on the set of all morphisms. This operation must satisfy the following three conditions:
   (a) For any $f \in \mathbf{C}(A, B)$ and $g \in \mathbf{C}(B, C)$, we must have that $g \circ f \in \mathbf{C}(A, C)$. This essentially means that the set of morphisms must be closed under composition.
   (b) For any three morphism *f*,*g* and *h* for which types match, we have the following associative law:

$$h \circ (g \circ f) = (h \circ g) \circ f \tag{1.1}$$

(c) For each $A \in |C|$, there exist an identity morphism 1_A, which is such that for any $f \in \mathbf{C}(A, B)$, we have:

$$f \circ 1_A = f = 1_B \circ f \tag{1.2}$$

The three conditions given above basically guarantee that the set of morphisms under the binary operation of composition forms a semi-group with identity. Here, note that the term 'monoid' is used instead of a semi-group with identity, and the two signify exactly the same thing. Further, a monoid can actually be viewed as a category with a single object. We can think of each element of the monoid as a morphism in our one-object category, and then the monoid multiplication is nothing but composition of morphisms in the category.

Example 2. A few examples of simple categories are:

1. **Set**- which consists of sets as objects, and functions between them as morphisms.
2. **Rel**- which consists of sets as objects, but the morphisms in this case are relations between the sets, and not functions.
3. $\mathbf{FdVec_K}$- which consists of finite-dimensional vector spaces over the field K as objects, and linear maps between them as morphisms.
4. **FdHilb**- this is the category that we are most interested in, and it consists of finite-dimensional Hilbert spaces (i.e. Vector spaces over the field of complex numbers which come with an inner-product defined on them) in place of objects, and linear maps between them as morphisms.

A lot of interesting structures can be defined to make the categories more interesting and complex, but we will only go into those ones which are necessary for the purpose of this paper.

Now, there are many reasons why Category theory would be used as a tool to explore any mathematical structure. Here are a few:

- It allows you to study the structure at a sufficiently abstract level in order to be able to make connections between the structure at hand, and other seemingly disconnected areas in Mathematics.
- Category theory allows you to express properties and features of the structure in a diagrammatic language (under certain assumptions). Thus it provides a universal language for talking about finite dimensional Hilbert spaces and other related structures.
- Category theory is especially useful while exploring properties of a physical system, since it provides a natural setting for the physical systems (which become objects in the category), and processes between them (morphisms).

Of course, 'Category Theory' is a very broad area of study indeed, and in order to describe something as complex as finite-dimensional Hilbert spaces accurately, a lot more structure needs to be introduced into our simple categories. There is a need for additional modalities which will capture the vector-space structure, as well as the inner-product space structures which are present in a finite-dimensional Hilbert space. This has been done in various texts such as [14, 3, 5]. These texts (along with many others) explain not just the category theoretic framework, but also the diagrammatic representation of different protocols. However, as we would expect, the introduction of complicated structures which will capture different aspects of Hilbert spaces pulls us away from our initial objective- which was to keep our description as simple and "high-level" as possible.

1.2. **Advantages of our approach.**

Other authors have probably put it more succinctly, but I would give the analogy of building a house. Now, when you are building a house, it is important to pay attention to the details- i.e, where each brick should go, and how two bricks should be alligned with each other and so on. These details are very important for ensuring that the foundations of the house are sound, and that there is no chance of it collapsing in the future! But while building the house, we also need to look at the larger picture- such as the architectural blueprint, or an aerial view of the house, in order to check the aesthetics and overall functionality of the house. This is essentially what we are trying to do with the new category-theoretic approach to quantum mechanics.

The status of Hilbert spaces as the mathematical foundation of Quantum mechanics remains unchallenged, but the attempt is to examine the entire framework and formalism from a higher level and thus make connections with other areas, and also explore quantum informatic protocols and phenomena. We hope that our approach will help to design new protocols, and to check the correctness of existing ones. As we will see later, the approach highlighted in this paper allows you to encode the flow of information in all observational branches simultaneously, and thus allows for a compact and streamlined approach.

One of the primary reasons for the growing popularity of Category theory as a tool for exploring mathematical structures, especially in the context of physical theories, is the fact that most properties or statements lend themselves to a very convenient diagrammatic representation. The convention that we shall follow to represent most properties and protocols is the same as the one followed in [4] and [5].

To begin with, we shall represent objects as arrows, and morphisms as boxes. So, a morphism $f : A \to B$ will be expressed as:

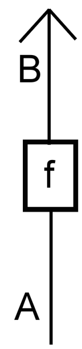


Fig (1.1)

As we go further and become acquainted with monoidal categories, will will see how this diagrammatic representation carries over to the newer concepts such as strictness and compactness. Finally, we are going to look at dagger compact categories, and then the concept of internal monoids and comonoids, which will lead us to the concept of Frobenius algebras. Throughout this journey, the diagrammatic representation which has been begun above will play an integral role in helping us appreciate the material.

1.3. **Why Frobenius algebras.**

In the past few years, a lot of work has been done on how quantum mechanical properties and processes can be described in the more abstract settings of dagger-compact categories (for e.g, see [5]), or categories with other structures. It was established long back that even in the bizarre world of quantum mechanics, faster-than-light communication is not allowed, even though it may seem that way due to the way entangled states behave. In other words, even though there may be perfect correlation between two spatially separated systems (normally qubits), and performing an operation on one may instantaneously change the other, this change will be meaningless without some form of classical communication. Due to this simple fact, classical communication and manipulation of classical data actually plays a very important role in any *useful* quantum informatic protocol.

Various kinds of paradigms have been explored to describe the manipulation of classical data within the setting of symmetric monoidal categories and dagger-compact categories. Some of the objects that have been explored are biproducts, Frobenius algebras, and branching trees, etc. However, a completely satisfactory structure which incorporates all the salient features of both classical and quantum data has not yet emerged. Frobenius algebras are relatively simple objects which incorporate copying and deleting into their very definition, and hence have been used to model classical data in some recent work [1, 9]. A recent, and very exciting result about them is the fact that special kinds of Frobenius algebras actually correspond exactly to what we call quantum observables, i.e. orthonormal bases for finite dimensional Hilbert spaces (see [2]) .

The reason why this result is so important is that in quantum mechanical protocols, measurements can only be made with respect to a particular orthonormal basis (hence the term 'quantum observable'). As we already know, the result of these measurements can be predicted only in a probabilistic manner, and it is normally the results of such measurements which need to be communicated by way of a classical channel during quantum protocols in order to make the whole protocol 'meaningful'. As we will see later, the act of making a measurement and sending a copy of the measurement to a different location can be simulated very nicely using the copying operation in a classical structure. This is why we feel that classical structures are more intricately connected with quantum mechanical protocols than has perhaps been thought previously.

1.4. **Layout of this paper.**

We have already explained our basic motivation for using category theory and classical structures (i.e. special, commutative dagger-Frobenius) in particular to study quantum informatic protocols. In the Section 2, we will go through the basics of quantum mechanics to ensure that our theoretical approach in the rest of the paper does not diverge too much from the real picture. In the third section, we will look at the mathematical framework that has been used to describe the flow of quantum information in existing texts such as [14, 6, 5]. We will build up the mathematical framework and finally define 'classical structures', which are the mathematical objects of greatest interest to us.

In Section 4 we will prove the main result of this paper, which is the axiomatization of bases (for two qubits) which consist of maximally entangled states. We will axiomatize these bases by using a diagrammatic condition, but will leave the explanation of the theoretical foundations for coming up with this particular condition till Section 5. In this section, we will examine our representation of quantum protocols using classical structures more closely, and also apply the concepts to the entanglement swapping protocol. Finally, in Section 6 we will analyse the results and observations made in this paper, and try and relate it with other ongoing work in this area.

2. The basics of quantum mechanics revisited

Before we begin describing the abstract setting in which we will examine quantum mechanics, it would be useful to remind ourselves of the postulates of quantum mechanics and the basic rules and properties which govern the evolution of physical systems in the quantum world. We will first simply state the postulates as given in widely recognised texts on quantum mechanics such as [10]. Once we have stated them, we will then give brief explanations for each of them. Here are the postulates:

(1) Associated to any isolated physical system is a complex vector space with inner product (i.e. a Hilbert Space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space. The inner product of two vectors $|\psi\rangle$ and $|\phi\rangle$ is denoted as : $\langle\psi|\phi\rangle$. This inner product is linear in the second variable, and congugate linear in the first variable.

(2) The evolution of a *closed* quantum system is described by a Unitary transformation. In other words, the state $|\psi\rangle$ of a closed system at time t_1 is related to the state $|\psi'\rangle$ of the same system at time t_2 by a Unitary operator U which depends only on the times $t_1$and t_2.

$$|\psi'\rangle = U\,|\psi\rangle \tag{2.1}$$

(3) Quantum measurements are described by a collection $\{M_i\}$ of *measurement operators.* These are operators acting on the state space of the system being measured. The index i refers to all the possible measurement outcomes that may occur in the experiment. If the state of the system is $|\psi\rangle$ immediately before performing the experiment, then the probability that result i occurs is given by:

$$p(i) = \langle\psi|M_i^\dagger M_i|\psi\rangle \tag{2.2}$$

The condition that the measurement operators must satisfy is:

$$\sum_i M_i^\dagger M_i = I \tag{2.3}$$

(where I is the identity operator)

These measurements are sometimes called non-destructive or projective measurements because after the measurement outcome i is obtained, the system is left in the state:

$$|\psi'\rangle = \frac{M_i|\psi\rangle}{\sqrt{\langle\psi|M_i^\dagger M_i|\psi\rangle}} \tag{2.4}$$

(4) The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have n systems and system number l is prepared in the state $|\psi_l\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes \ldots\ldots \otimes |\psi_n\rangle$.

The first postulate says nothing but what we had already said in the first section- i.e. each quantum mechanical system can be described mathematically as a vector in a Hilbert space. It would be important to note here that in this paper, we are dealing only with finitary quantum mechanics- which means that we are assuming that the underlying Hilbert space is finite-dimensional. Physically, this translates into looking at only those physical properties which have a finite number of possible states, such as the spin of an electron, or other such things.

The second postulate talks about which kinds of transformations are allowed in a *closed* physical system. Now, by a closed system, we basically mean a system which is not being influenced by any external factor. So without being too precise, the second postulate states that if a system undergoes a natural transformation, without being influenced by any external factors, then the transformation can be described a Unitary operation. We will look at the definition of a unitary transformation more closely in the third section, but for now, recall that in the matrix calculus of **FdHilb**, the condition for a matrix B to be a unitary matrix is that the inverse of the matrix should be equal to its adjoint (and the adjoint, in **FdHilb** is the cojugate transpose of the matrix). So in mathematical terms, we have:

$$[B]^{-1} = [B]^{\dagger} = \overline{[B]}^{T} \tag{2.5}$$

Note that due to the nature of unitary transformations, they are all necessarily reversible. This fact makes it much easier to deal with them during quantum informatic protocols, as we will see a little later on. In fact, if we look a little closely at the conditions on a square matrix which make it unitary, we will find that a unitary operation on a Hilbert space actually signifies a change in basis (where the old and new bases are both orthonormal). Equation (2.5) actually says that if we look at each column of the matrix $[B]$ as a vector, then the inner product of any vector with itself is one, and the inner product of any two distinct vectors is zero. Thus, if our Hilbert space is n-dimensional, having an orthonormal basis $\{e_1, e_2, \ldots\ldots, e_n\}$ and $[B]$ is an $n \times n$ matrix such that

$$[e_1, e_2, \ldots\ldots, e_n]\,[B] = [f(e_1), f(e_2), \ldots\ldots, f(e_n)] \tag{2.6}$$

Then, $[B]$ is a unitary matrix if and only if $\{f(e_1), f(e_2), \ldots\ldots, f(e_n)\}$ also forms an orthonormal basis for the Hilbert space.

The third postulate describes the nature of measurements in quantum mechanics. It is worth noting here that for any quantum system, there could be more than one possible set of measurement operators $\{M_i\}$ that we could use in order to make our measurement. The set of measurement operators actually corresponds to a basis of the underlying Hilbert space, and as we already know, a particular Hilbert space can have more than one (actually, infinitely many) basis. One of the most important features of quantum mechnics, and perhaps the most striking difference between the quantum and classical world, is that we cannot predict the measurement outcome while measuring an arbitrary state, and that the outcomes can only be assigned certain probablilities, which are governed by eqn (2.2). This equation is also known as the Born rule.

In the course of this paper, we will frequently come across the term ‘tensor product’, which is used to describe composite systems constructed from two or more smaller systems. The fourth postulate simply tells us that the composite system of n smaller systems will be the tensor product of all n systems. This may not seem very important at first sight, but it is actually the most essential differentiating feature between the quantum and classical worlds. In the classical world, the composite system of two or more sub-systems is usually described by the direct sum (or the cartesian product), in which all the properties of the composite system can be traced back to one of its constituents. However, in a tensor product, we can have composite systems which are truly entangled, i.e. the system cannot be decomposed into two or more smaller parts.

Example 3. To make the situation a little clearer mathematically, let us consider two vector spaces X and Y over some underlying field $\mathbf{F}$. Let X be an m-dimensional vector space, and let $\{a_1, a_2, \ldots\ldots, a_m\}$ be a basis for X. Let Y be an n-dimensional vector space having a basis $\{b_1, b_2, \ldots\ldots, b_n\}$. Now, the Direct sum of X and Y, say $(X \oplus Y)$ will be an $(m+n)$-dimensional space, whereas the tensor product, say $(X \otimes Y)$ will be an (mn)-dimensional space. An arbitrary vector ϕ in the space $(X + Y)$ can be written as:

$$\phi = c_1 a_1 + \ldots + c_m a_m + c_{m+1} b_1 + \ldots + c_{m+n} b_n \, (\forall i,\, c_i \in \mathbf{F}) \tag{2.7}$$

So clearly, any such vector ϕ can be expressed as the sum of two vectors $\alpha \in X$ and $\beta \in Y$. However, an arbitrary vector in the tensor product space $(X \otimes Y)$ is of the form:

$$\psi = \sum_{i,j} c_{ij} (a_i \otimes b_j) \tag{2.8}$$

Here the sum is taken over all possible values of i and j. Thus, the basis for $(X \otimes Y)$ consists of $m \times n$ vectors. Now, if a vector is the tensor product of two vectors, $\alpha \in X$ and $\beta \in Y$, then it must be of the form:

$$\alpha \otimes \beta = (c_1 a_1 + \ldots + c_m a_m) \otimes (c_{m+1} b_1 + \ldots + c_{m+n} b_n) \tag{2.9}$$

It is clear that all vectors of the form given in eqn (2.7) cannot be expressed in the form of a tensor product of two vectors belonging to smaller subspaces (i.e. they cannot all be expressed in the form of eqn (2.9)). The vectors belonging to $(X \otimes Y)$, which cannot be expressed in the form (2.9) for any value of m and n, are said to be entangled states, or entangled vectors.

Having reminded ourselves of the rudiments of quantum mechanics, and the properties of finite-dimensional Hilbert spaces and the tensor product, we can now look at the more abstract mathematical structures which we will be using to explore quantum informatic protocols and properties.

3. The algebraic framework

3.1. **Strict Symmetric Monoidal Categories.**

As we discussed in the previous section, a major difference between quantum and classical information is the nature of composite systems. In the classical world, systems are coupled with each other in the form of cartesian products. Thus, all the properties of the composite system can be traced back to one of the components. However, in the quantum world, we have a purely tensor product- which enables the composite system to have properties which cannot be traced back linearly to any of the components. It also enables an entire range of *entangled states*, i.e. quantum states which simply cannot be decomposed into two or more smaller components. Now, without further ado, let us define the various structures which will lead us to the specific setting in which we shall examine quantum protocols. Most of the definitions given here follow the same notation and conventions as the ones followed in [4].

Definition 4. A Monoidal Category is a category which comes with a *unit* object I, and a binary operation (the 'tensor product') defined on all possible pairs of objects in the following manner:

$$- \otimes - : C \times C \to C \tag{3.1}$$

The monoidal category must also come with the following three isomorphisms:

$$\lambda_A : A \simeq I \otimes A \tag{3.2}$$

$$\rho_A : A \simeq A \otimes I \tag{3.3}$$

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C \tag{3.4}$$

Furthermore, these three isomorphisms must be *natural,* which essentially means that these isomorphisms must commute with any other morphisms which could be applied to the objects, and that they must also commute with each other. This is actually expressed as a set of coherence conditions in Maclane [7].

If we denote the set of all morphisms from A to B by $\mathbf{C}(A,B)$, then the tensor product is also defined for all pairs of morphisms in the following manner:

$$-\otimes-:\ \mathbf{C}(A,B)\times\mathbf{C}(C,D)\rightarrow\mathbf{C}(A\otimes C,\ B\otimes D)\ ::\ (f,g)\longmapsto f\otimes g \tag{3.5}$$

Again, when we define the $-\otimes-$product on morphisms, this operation must also satisfy conditions similar to equations (3.2), (3.3) and (3.4), which stipulate that the operation is associative, and that each morphism f has 1_I (i.e. the Identity morphism on the unit object I) as its unit object. These conditions translate into the following:

$$\alpha_{f,g,h}:\ f\otimes(g\otimes h)\simeq(f\otimes g)\otimes h \tag{3.6}$$

$$1_I\otimes f\simeq f\simeq f\otimes 1_I \tag{3.7}$$

And as above, the isomorphisms which specify these conditions must satisfy certain coherence conditions to make sure that they are *natural.*

We now come to, what is in my view, the most important condition for a monoidal category. This condition also demontrates the power of the diagrammatic calculus which is the greatest advantage afforded by this entire approach to quantum mechanics. The condition simply states that for four morphisms: $f\in\mathbf{C}(A,B)$, $g\in\mathbf{C}(B,C)$, $h\in\mathbf{C}(D,E)$ and $k\in\mathbf{C}(E,F)$, we have that:

$$(g\otimes k)\circ(f\otimes h)=(g\circ f)\otimes(k\circ h) \tag{3.8}$$

We can see below the diagrammatic representation of the LHS of eqn (3.8). In the LHS of the equation, the morphisms g and k are tensored with each other and put in a bracket. This is physically interpreted to mean that g and k are spatially placed next to each other. They are shown to be bracketed in the diagram. Similarly, f and h are placed next to each other and put inside the same bracket. Since f and h are placed temporally before g and k, they appear below g and k in the diagram. Thus, in equations which contain the composition sign, the actions on the right of the '$\circ$' sign happen first, and then the actions on the left of the '$\circ$' sign. Similarly, in our graphical representation, time flows from bottom to top, so that the actions at the bottom happen first, and then the ones on the top. Thus, the LHS of eqn (3.8) can be represented as follows:

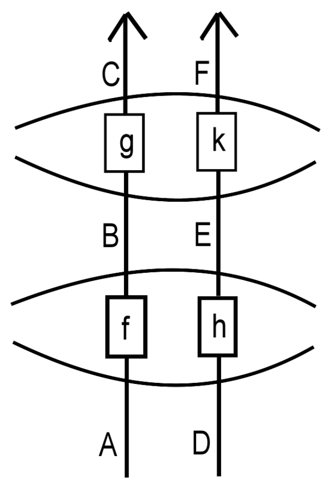


Fig (3.1)

On the other hand, in the RHS of the equation, we have g and f, which are temporally next to each other (i.e, g follows f), being placed inside the same bracket, and similarly k and h being placed inside another bracket to emphasise that they are temporally next to each other. Since the two parts are then coupled with a tensor product, in the diagram we place them spatially next to each other. This is depicted in the following manner:

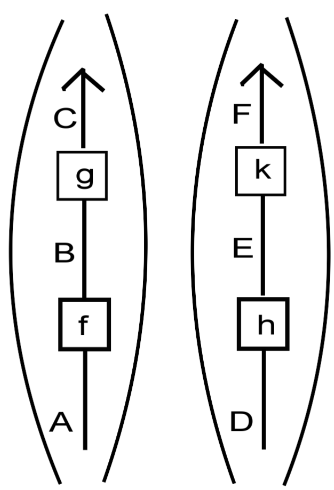


Fig (3.2)

Now, we can see that eqn (3.8) must hold true, since we can simply change the position of the brackets in Fig(3.1) to get Fig(3.2). In fact, if we remove the brackets from both the figures, then they are identical.

Definition 5. A Symmetric Monoidal Category is a Monoidal Category which comes with a fourth natural isomorphism in addition to the three given as equations (3.2), (3.3) and (3.4). This isomorphism is called the 'symmetry isomorphism':

$$\sigma_{A,B} : A \otimes B \simeq B \otimes A \tag{3.9}$$

In addition to eqn (3.9), the tensor product for morphisms must also satisfy one additional condition, namely:

$$1_A \otimes 1_B = 1_{A \otimes B} \tag{3.10}$$

Now, A *strict* symmetric monoidal category is one in which all the natural isomorphisms mentioned above are simply the identity morphisms. In other words, to obtain the conditions for a strict symmetric monoidal category, we just need to replace the $\simeq$ sign by the $=$ sign in all the equations above.

We have already described how objects and morphisms can be represented in a diagrammatic manner (fig. (1)). This representation can be extended to the realm of strict symmetric monoidal categories in a very convenient way. In the setting of a *strict* monoidal category, the unit object I is represented simply as empty space, so that the identity isomorphisms given in equations (3.2) and (3.3) are totally self-evident. Also, in a strict monoidal category, morphisms of the type $\psi : I \to A$ ('states' or 'elements'), and morphisms of the type $\phi : A \to I$ ('effects' or 'co-elements') have a special importance which will become more clear later on. Apart from sates and effects, another very important concept in quantum mechanics is that of scalars. At this abstract level of monoidal categories, scalars are simply defined to be morphisms from the unit object I to itself. Thus, the set of scalars is formally written as:

$$\mathbf{S}_C = \mathbf{C}(I, I) \tag{3.11}$$

It turns out that this set is always a monoid with categorical composition as monoid multiplication. This should not surprise us, because as we discussed earlier, a monoid is nothing but a category with only one object (in this case the unit object I). The non-trivial and truly beautiful fact about scalar monoids is the observation, first made by Kelly and Laplaza in [13], that even for non-symmetric monoidal categories, this monoid $\mathbf{S}_C$ will always be commutative. We will discuss the properties and the role of this scalar monoid in greater detail after we introduce the concept of an adjoint to a morphism is Section 3.2.

3.2. **Dagger-compact Categories and FdHilb.**

Before we move on to Dagger-compact categories, we must first define what a functor is, and what $\mathbf{C}^{op}$ signifies.

Definition 6. A functor is simply a structure preserving map from one category to another.

Since a category consists of both objects and morphisms, it is clear that the functor must map the objects from one category onto the objects of another category. In addition to this, it must also map the morphisms from one category onto the morphisms of the other category. To put it explicitly,

If F, is our functor from category $\boldsymbol{C}$ to category $\boldsymbol{D}$, then:

$$F : |\boldsymbol{C}| \longmapsto |\boldsymbol{D}| :: A \to F(A) \tag{3.12}$$

$$F : \boldsymbol{C}(A, B) \longmapsto \boldsymbol{D}(F(A), F(B)) :: f \to F(f) \tag{3.13}$$

When we say that this functor must be a 'structure preserving' map, we mean that it must preserve identities and the composition of morphisms, i.e.

For any object $A \in |\boldsymbol{C}|$, we must have

$$F(1_A) = 1_{F(A)} \tag{3.14}$$

And for any two morphisms f and g for which types match, we must have

$$F(g \circ f) = F(g) \circ F(f) \tag{3.15}$$

In this sense, a functor performs a similar function for categories to the one that a Group homomorphism performs for groups.

Now, $\mathbf{C}^{op}$is another categry, which contains exactly the same objects as the category $\mathbf{C}$, but in which all morphisms are reversed. This means that we have the following two relations:

$$A \in \boldsymbol{C}^{op} \Longleftrightarrow A \in \boldsymbol{C} \tag{3.16}$$

$$f \in \boldsymbol{C}^{op}(B, A) \Longleftrightarrow f \in \boldsymbol{C}(A, B) \tag{3.17}$$

Obviously identities are preserved in going from $\boldsymbol{C}$ to $\boldsymbol{C}^{op}$, since the reverse of an identity is nothing but the identity itself.

Definition 7. A strict *dagger* monoidal category $\boldsymbol{C}$ is a strict monoidal category which comes with a functor: $\dagger : \boldsymbol{C}^{op} \to \boldsymbol{C}$, which takes each object to itself, and each morphism f to its adjoint $f^{\dagger}$. On morphisms, this functor is involutive (applying it twice is the same as not applying it), and contravariant (applying the dagger to the composition of two morphisms is the same as applying the dagger to each morphism separately, and then reversing their order). To put it in mathematical terms, we have the following three conditions:

$$\forall A \in |\mathbf{C}|,\ A^{\dagger} = A \tag{3.18}$$

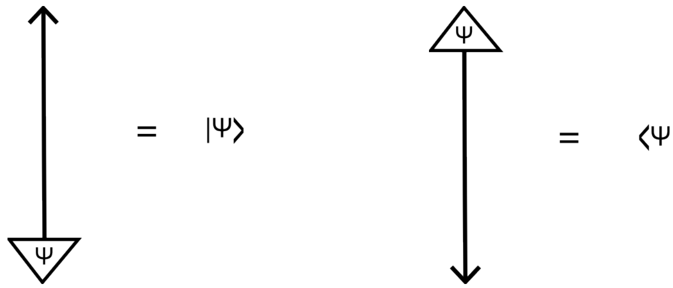
$$\forall f \in \mathbf{C}(A, B),\ (f^{\dagger})^{\dagger} = f \tag{3.19}$$

$$For\, any\, f \in \mathbf{C}(A, B),\ g \in \mathbf{C}(B, C) :\ (g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger} \tag{3.20}$$

In addition to these three conditions, we also require that the adjoint of a tensor product of two morphisms should be the same as the tensor product of the adjoints of each morphism taken separately. In other words,

$$(f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger} \tag{3.21}$$

Note that if $f : A \to B$, then $f^{\dagger} : B \to A$. With the adjoint in hand, we can relate our diagrammatic calculus to the well-known Dirac notation which is used so extensively in various texts. In standard Dirac notation (also known as the bra-ket notation), the *states* or *elements* (i.e. mappings $\psi : I \to A$) are exactly the ‘kets’ and the adoints of such mappings (i.e $\psi^{\dagger} : A \to I$), which are technically *effects* or *co-elements* are exactly the ’bras’. This is depicted graphically below. Remember that since we are in the realm of strict monoidal categories, the unit object I is shown simply as empty space:



Fig(3.3)

If you observe the diagrams, you will notice that if you rotate the bra (or ket) by 90 degrees in a clockwise direction, you obtain the direction of the arrow in the corresponding diagram.

We can now introduce the concept of a unitary morphism, which is very important in the context of quantum mechanical protocols.

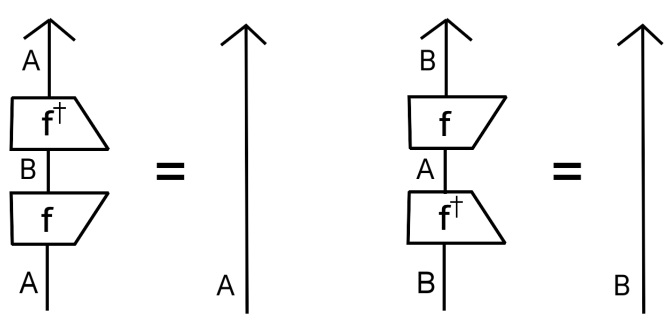
Definition 8. A morphism in a strict dagger monoidal category is said to be *unitary* if its inverse and adjoint are the same, i.e. $f \in \mathbf{C}(A, B)$ is said to be unitary if

$$f^{-1} = f^{\dagger} \Rightarrow (f^{\dagger} \circ f) = 1_A, \, and \, (f \circ f^{\dagger}) = 1_B \tag{3.22}$$

We will find that the graphical calculus that we had introduced earlier can be extended in a very nice way to incorporate the concept of the adjoint [4, 5]. In order to introduce this concept, we just need to modify our representation of a morphism and introduce some asymmetry. Thus, instead of representing a morphism $f : A \rightarrow B$ as simply a rectangular box, we will now represent it as an asymmetric trapezium. Now, we can represent the adjoint $f^{\dagger} : B \rightarrow A$ by simply turning the whole picture upside down, or by flipping it along a horizontal axis. Thus, f and $f^{\dagger}$ can be represented as the following:

Fig(3.4)

We can now express eqn (3.22) in a very simple diagrammatic manner. Both parts of the equation are shown below. As before, time flows from bottom to top. The interpretation of this diagrammatic equation is that when a morphism is composed with it's mirror image (with respect to a horizintal plane), then the two cancel out. This can be seen in the figure below:



Fig(3.5)

Now that we have introduced the concept of an adjoint and the definitions of states and effects and their relation to Dirac's 'Bra-ket' notation, we can now examine the all-important inner-product of two vectors in a Hilbert space. First, it would be useful to note that since a state $\psi : I \rightarrow A$ is a linear mapping from the set of scalars$\mathbf{S}_C$ to the Hilbert space $\mathcal{H}$, hence it is completely defined by the image of $1 \in \mathcal{C}$. Thus, we can also refer to a state as a vector or an element of $\mathcal{H}$, since the vector (which will be the image of 1) completely defines the mapping.

We are now equipped to define the *inner product* of two vectors (or states) $\phi, \psi \in \mathcal{H}$ as:

$$\langle\phi|\psi\rangle = \phi^\dagger \circ \psi : I \rightarrow I \tag{3.23}$$

As we had said earlier, a mapping from $I \rightarrow I$ is nothing but a scalar in our underlying field. Thus, the inner product given by eqn (3.23) is nothing but a scalar. Even though our definitions of adjoints and inner-products, etc are still quite abstract, many familiar things can be derived even from the definitions that we have already given, as shown in [4]. For example, we can recover the defining property of an adjoint of a functor, as given in standard texts such as [10]. If $\phi : I \rightarrow A$ and $\psi : I \rightarrow A$ are two states, then the defining property of the adjoint of a morphism $f : A \rightarrow A$ is usually given to be:

$$\langle f^\dagger \circ \psi|\phi\rangle = \langle\psi|f \circ \phi\rangle \tag{3.24}$$

We can easily prove this relation with the tools already at hand. From eqn (3.23), we have,

$$\langle f^\dagger \circ \psi|\phi\rangle = (f^\dagger \circ \psi)^\dagger \circ \phi \tag{3.25}$$

Since the adjoint is a contravariant and involutive operator, and since the associative law applies to composition of morphisms, we get

$$(f^\dagger \circ \psi)^\dagger \circ \phi = (\psi^\dagger \circ f) \circ \phi = \psi^\dagger \circ (f \circ \phi) \tag{3.26}$$

Finally, using eqn (3.23) again, we arrive at the RHS of eqn (3.24) as follows:

$$\psi^\dagger \circ (f \circ \phi) = \langle\psi|f \circ \phi\rangle \tag{3.27}$$

Using this property of of the adjoint functor, we can also prove the defining property of unitary morphisms- which is that they must preserve the inner product. In other words, if U is a unitary morphism from $A \to A$, then using eqn (3.23) we have:

$$\langle U \circ \psi | U \circ \phi \rangle \quad = \quad (U \circ \psi)^{\dagger} \circ (U \circ \phi) \tag{3.28}$$

Now, by the contravariance of the †-functor and the associativity of the '∘' operation, we get:

$$(U \circ \psi)^{\dagger} \circ (U \circ \phi) = (\psi^{\dagger} \circ U^{\dagger}) \circ (U \circ \phi) = \psi^{\dagger} \circ (U^{\dagger} \circ U) \circ \phi \tag{3.29}$$

Finally, since U is a unitary operation, and using eqn (3.23) yet again, we have:

$$\psi^{\dagger} \circ (U^{\dagger} \circ U) \circ \phi = \psi^{\dagger} \circ \phi = \langle \phi | \psi \rangle \tag{3.30}$$

Definition 9. A strict dagger *symmetric* monoidal category is both a strict dagger monoidal category, AND a strict symmetric monoidal category for which, given any two objects $A, B \in |\mathbf{C}|$, we have that

$$\sigma^{\dagger}_{A,B} = \sigma^{-1}_{A,B} = \sigma_{B,A} \tag{3.31}$$

This simply means that if we regard $\sigma_{A,B}$ to be a morphism from $A \otimes B$ to $B \otimes A$, then it is a unitary operation.

Now in order to represent the measurements and preperation of entangled states, the structures that we have introduced so far do not suffice. To represent these things faithfully and accurately, we have to move to a slightly different setting.

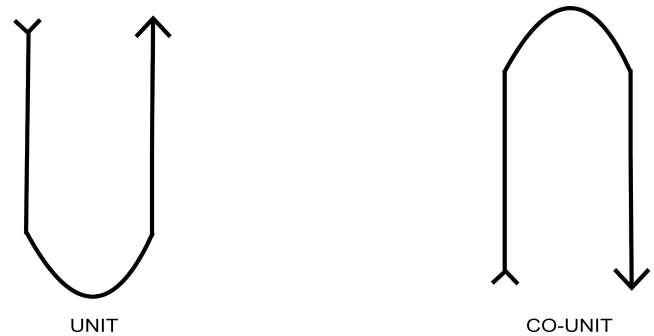
Definition 10. A *compact closed category* is a symmetric monoidal category in which to each object A, we assign a dual object $A^{\star}$and two linear mappings. The first is called the unit:

$$\eta_A : I \to A^{\star} \otimes A \tag{3.32}$$

And the second one the co-unit, defined as:

$$\varepsilon_A : A \otimes A^{\star} \to I \tag{3.33}$$

In our graphical calculus described earlier, we had represented an object as an upward arrow. Now we will represent the dual to an object A, i.e. $A^{\star}$, as a downward arrow labelled A. Thus, units and co-units are represented as cups and caps, as shown below:

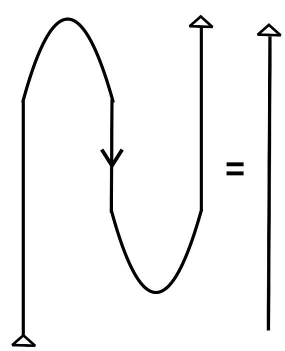


Fig(3.6)

Now, if we recall what λ_A, ρ_A and $\alpha_{A,B,C}$ stood for (equations (3.2), (3.3) and (3.4)), then the condition on the unit and co-unit is that the following diagram, and the dual one for $A^\star$ should commute:

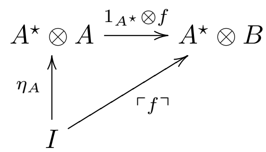
$$\begin{array}{ccccc} A & \xrightarrow{\rho_A} & A \otimes I & \xrightarrow{1_A \otimes \eta_A} & A \otimes (A^\star \otimes A) \\ \downarrow{\scriptstyle 1_A} & & & & \downarrow{\scriptstyle \alpha_{A,A^\star,A}} \\ A & \xleftarrow{\lambda_A^{-1}} & I \otimes A & \xleftarrow{\varepsilon_A \otimes 1_A} & (A \otimes A^\star) \otimes A \end{array}$$

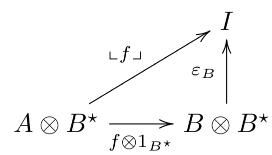
Following the same diagrammatic rules described above, we can represent the above condition as the following picture:



Fig(3.7)

Definition 11. For any morphism $f : A \to B$, the name, $\ulcorner f \urcorner$, and the coname $\llcorner f \lrcorner$ of the morphism in a compact closed category are defined by the following diagrams [5]:





Note that by these definitions, each morphism $g' : I \to A^\star \otimes B$ is actually the name of some morphism $g : A \to B$. The same goes for every morphism of the type $g'' : A \otimes B^\star \to I$, which is the coname of the same morphism g.

As far as our diagrammatic representation was concerned, we will represent the preperation of an entangled state as a name, and an observation branch as a coname. Thus, the standard bell-state is nothing but the name of the identity morphism in the category **FdHilb**.

3.3. **Internal classical structures and Frobenius algebras.** We have already discussed what a monoidal category consists of. So, if $(\mathbf{C}, \otimes, I)$ is a monoidal category then,

Definition 12. An *internal monoid* in the category $\mathbf{C}$ consists of an object $M \in |\mathbf{C}|$, together with a pair of morphisms,

'μ', known as multiplication

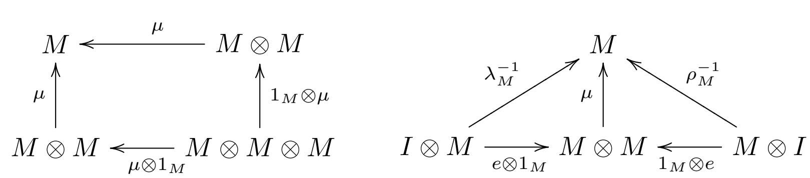
$$\mu : M \otimes M \to M \tag{3.34}$$

and 'e', known as the multiplicative unit

$$e : I \to M \tag{3.35}$$

These morphisms are defined in such a manner so that the following diagrams commute:

(3.36)



The origin of the term 'internal monoid' is due to the fact that monoids can be looked at equivalently as internal monoids in the category **Set**, i.e. the category which consists of sets as objects, and functions between the sets as morphisms.

The dual concept of the monoid is the *comonoid*, which is defined in a very similar way, just by reversing the arrows.

Definition 13. An *internal comonoid* is an object , $C \in |\mathbf{C}|$, together with a pair of morphisms, 'δ', known as comultiplication

$$\delta : C \to C \otimes C \tag{3.37}$$

and 'ϵ', known as the comultiplicative unit

$$\epsilon : C \to I \tag{3.38}$$

As before, these morphisms are defined in such a manner so that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\delta} & C \otimes C \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle 1_C \otimes \delta} \\ C \otimes C & \xrightarrow[\delta \otimes 1_C]{} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccccc} & & C & & \\ & {\scriptstyle \lambda_C}\swarrow & \downarrow{\scriptstyle \delta} & \searrow{\scriptstyle \rho_C} & \\ I \otimes C & \xleftarrow[\epsilon \otimes 1_C]{} & C \otimes C & \xrightarrow[1_C \otimes \epsilon]{} & C \otimes I \end{array} \tag{3.39}$$

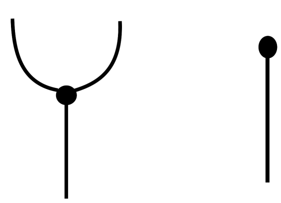
Now, the above structures- the internal monoid and comonoid- can be defined for any arbitrary monoidal category. However, if we have the luxury of defining adjoints for operators, i.e. if we are working in a *dagger* monoidal category, then internal comonoid automatically defines an internal monoid, and vice-versa. Thus, if we have an internal comonoid given by

$$(X,\ X \xrightarrow{\delta} X \otimes X\ ,\ X \xrightarrow{\epsilon} I\ ) \tag{3.40}$$

then, the internal monoid is defined in the following way by making use of $\delta^\dagger$ and $\epsilon^\dagger$:

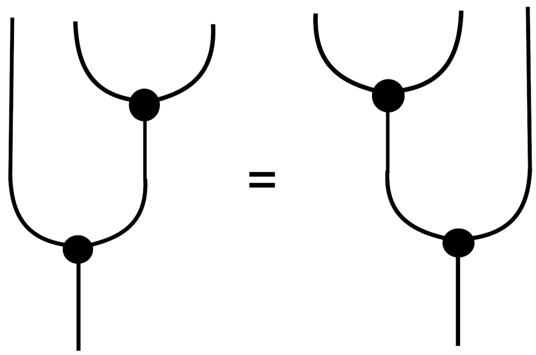
$$(X,\ X \otimes X \xrightarrow{\delta^\dagger} X\ ,\ I \xrightarrow{\epsilon^\dagger} X\ ) \tag{3.41}$$

There is again, a very convenient way to express all the above equations in a diagrammatic way. If we express the comultiplication δ, and the comultiplicative unit (henceforth called just the unit) ϵ, as the following:



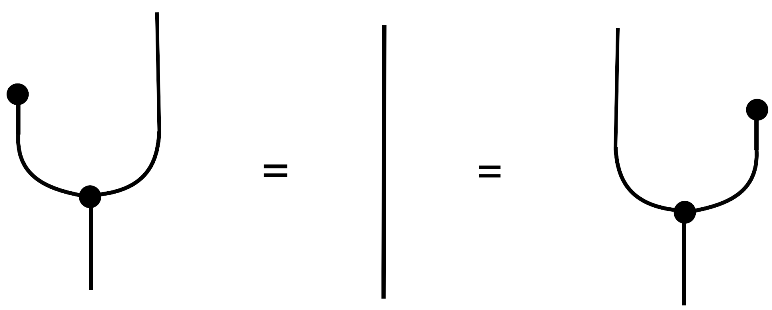
Fig(3.8)

Using this same depiction, we can easily express the conditions given in the form of the the commuting of the diagrams in eqn (3.39). The diagram on the left in this equation says nothing but:



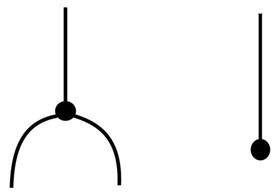
Fig(3.9)

Similarly, the conditions as the expressed as the commutation of the second diagram in eqn (3.39) can be expressed as:



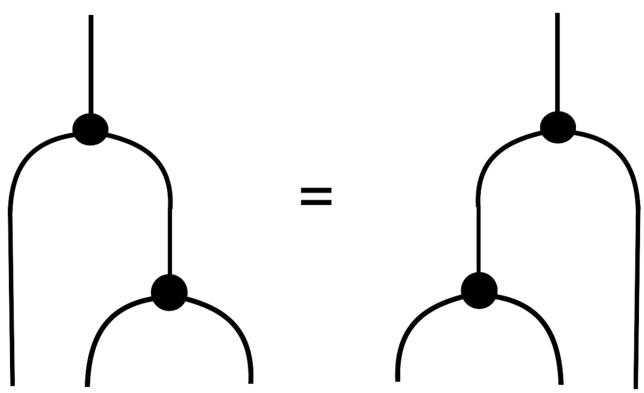
Fig(3.10)

In a dagger monoidal category, since the multiplication and counit are just defined as the adjoints to the comultiplication (i.e.$\eta = \delta^{\dagger}$), and the adjoint to the unit (i.e $e = \epsilon^{\dagger}$), therefore, their diagrammatic representations can just be obtained by turning the diagrams for comultiplication and the unit upside down. Thus, the multiplication and the counit are represented respectively as:



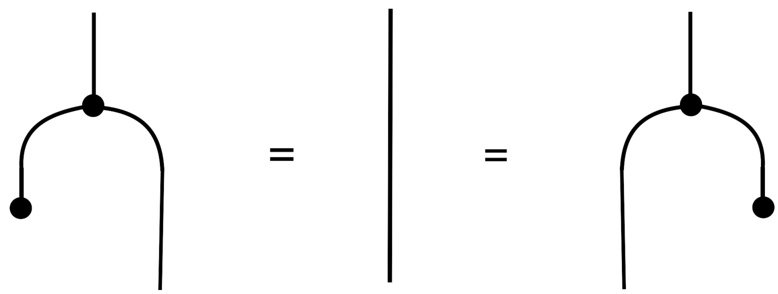
Fig(3.11)

Again, the conditions expressed as the two diagrams in eqn (3.36) can be expressed in a very similar manner to the diagrams in eqn (3.39). The first diagram reads very much like the associative law- as it should, since we have already discussed how a monoid can be seen as an internal monoid in the category **Set**, and the associative law for a monoid corresponds to the first diagram in eqn (3.36), which can be depicted as:



Fig(3.12)

Similarly, the condition expressed as the commutation of the second diagram in eqn (3.36) can be shown in the form of the following diagram:

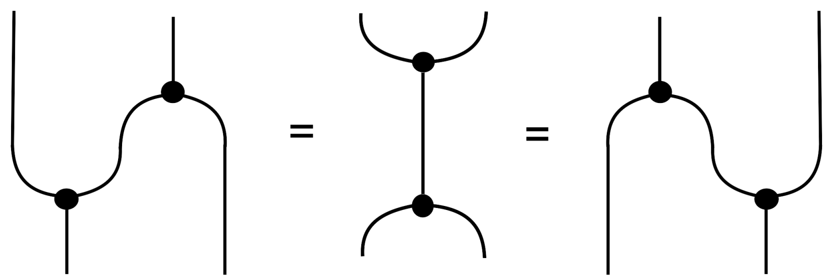


Fig(3.13)

Definition 14. A *Frobenius algebra* in a dagger symmetric monoidal category consists of an internal comonoid and monoid (which is defined by the adjoints to the comonoid operations, as shown above), which in addition, satisfy the following condition:

(3.42)
$$\begin{array}{ccc} X \otimes X & \xrightarrow{1_X \otimes \delta} & X \otimes X \otimes X \\ & \searrow^{\delta^\dagger} & \\ {\scriptstyle \delta \otimes 1_X}\big\downarrow & X & \big\downarrow{\scriptstyle \delta^\dagger \otimes 1_X} \\ & \searrow^{\delta} & \\ X \otimes X \otimes X & \xrightarrow[1_X \otimes \delta^\dagger]{} & X \otimes X \end{array}$$

This condition is known as the *Frobenius condition*, and it is the defining property of a frobenius algebra. This condition can once again be represented diagrammatically in a very convenient way, as shown below:



Fig(3.14)

Definition 15. A *Classical structure* is a special, commutative dagger-Frobenius algebra in a dagger symmetric monoidal category **C**. By 'special', we mean that

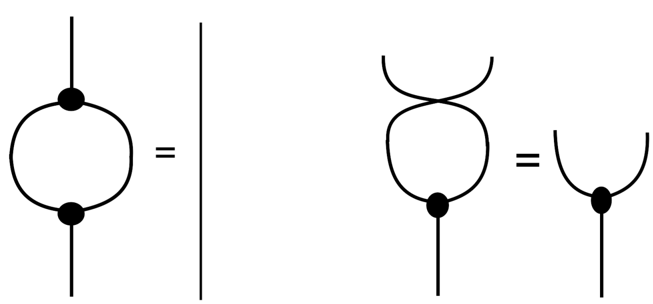
(3.43) $$\delta^{\dagger} \circ \delta = 1_X$$

This condition is shown diagrammatically in Fig(3.15), on the left side.

The second condition placed is that the Frobenius algebra should be commutative, i.e.

(3.44) $$\sigma_{X,X} \circ \delta = \delta$$

This can be shown as the picture on the right in Fig (3.15).



Fig(3.15)

The above-mentioned properties lead to a very elegant result given in [1], and in more detail in [17]. This result is known as the *spider theorem*, which essentially states that if we have a morphism which can be depicted as a connected network generated from a classical structure in a symmetric dagger monoidal category, then the morphism is completely charecterised by the number of 'input' wires (the domain) going into the network from the bottom, and the number of final 'output' wires coming out of it (the codomain). This is a very important result which helps us simplify more complicated network into a representation as a spider.

Due to the result in [2], we know that in the category **FdHilb**, there is a bijective correspondence between classical structures, and orthonormal bases. The way to understand this better is to say that in **FdHilb**, every orthonormal basis defines a comultiplicative operation (henceforth called a 'copying' operation) δ, and a comultiplicative unit ϵ (henceforth called a 'deleting' operation). If $\{|b_1\rangle$, $|b_2\rangle, \ldots\ldots, |b_n\rangle\}$ is an orthonormal basis for the Hilbert space, then the copying operation is given by,

$$\delta \equiv X \longmapsto X \otimes X \;::\; |b_i\rangle \rightarrow |b_i\rangle \otimes |b_i\rangle \tag{3.45}$$

And the deleting operation, the comultiplicative unit is defined as,

$$\epsilon \equiv X \longmapsto I \;::\; |b_i\rangle \rightarrow 1 \tag{3.46}$$

SInce we are working in a dagger symmetric monoidal category, the multiplication and the multiplicative unit are defined as the adjoints to the two linear operations given above. So we have:

$$\delta^\dagger \equiv X \otimes X \longmapsto X \;::\; |b_i\rangle \otimes |b_j\rangle \rightarrow \delta_{ij}|b_i\rangle \tag{3.47}$$

(Here δ_{ij} is the Kronecker delta symbol, which is $= 1$ when $i = j$, and 0 otherwise)

$$\varepsilon^\dagger \equiv I \longmapsto X \;::\; 1 \rightarrow \sum_{\mathrm{i}=1}^{\mathrm{n}} |b_i\rangle \tag{3.48}$$

4. Axiomatizing bases of maximally entangled states

4.1. **The Teleportation protocol.** Before we get to the actual result, it is important to briefly describe the teleportation protocol which, as the reader will soon realise, is the main motivation behind the result. Apart from this, teleportation occupies a very significant place in the dialogue on Quantum mechanics, and therefore merits a closer inspection and discussion in its own right. To give a very brief and naive account for why the teleportation protocol is so important, consider the following facts.

For the last part of his life, Einstein struggled to make sense of the seemingly illogical and counter-intuitive body of theoretical physics known as quantum mechanics. In 1935, in their famous paper which later became known as the EPR paradox, Einstein, along with Rosen and Podolsky argued that if quantum mechanics was indeed an accurate and complete picture of the world, then there would be "paradoxes" like the one described in their paper [11]. The first big blow to the EPR paradox came by way of a theoretical breakthrough by Bell [12], in his seminal paper published in 1964. This paper basically described how the predictions of Quantum mechanics could not be explained by any local hidden-variable theory. The only thing lacking at this time was the experimental confirmation of the bizarre predictions made by the theory of quantum mechanics. The teleportation protocol is the physical realisation of the of the predictions made by the theory of quantum mechanics. In other words, we can now perform exactly the same kinds of experiments which were thought to be impossible, and a paradox in 1935 by some of the greatest minds of our times.

What we are trying to achieve in the teleportation protocol is the transfer of a qubit in an unknown state, say $|\psi\rangle = a|0\rangle + b|1\rangle$ from one party (Alice), to another party (Bob) by sending only two bits of classical information. We allow Alice and Bob to share two qubits in a maximally entangled state, and it is this pair of qubits which will act as a communication channel for our protocol. This maximally entangled state is usually taken to be the bell-state: $B_1 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Formally, the teleportation protocol proceeds as follows:

Alice makes a 2-qubit measurement simultaneously on the unknown qubit $|\psi\rangle$ and her half of the shared bell-state. She makes this measurement in the bell-basis, in which each member is obtained by performing a unitary operation (actually one of the Pauli matrices) on the first qubit of the bell-state B_1. If we recall, the Pauli matrices are the following:

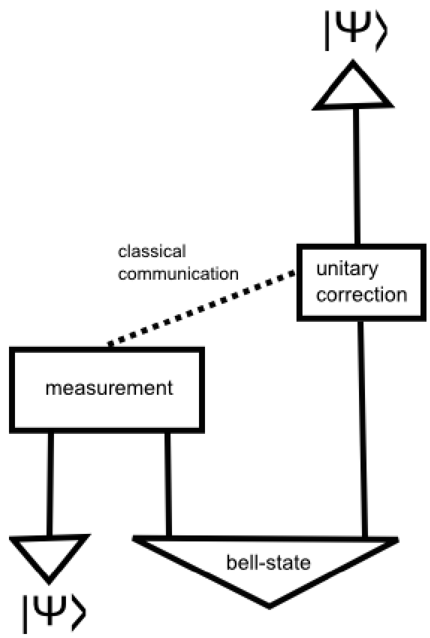
$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, P_4 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Thus, the four elements of the bell-basis are obtained by performing each of the four Pauli matrices successively on the first qubit of the bell-state. The bell-basis can be written (upto scalar multiples) as:

$$\{\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\ \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\} = \{|B_1\rangle, |B_2\rangle, |B_3\rangle, |B_4\rangle\}.$$

After making the measurement, she communicates the result of the measurement to Bob via some kind of classical channel. Note that she requires at least two bits of classical information (assuming a binary language) to send the result to Bob, since there are four possible outcomes of the measurement. Once Bob receives the measurement outcome from Alice, he then performs a unitary operation (sometimes called a unitary 'correction') on the qubit in his possession. The unitary correction that he performs is actually the same Pauli matrix that was performed on the bell-state to obtain the measurement outcome. In other words, if the measurement outcome was B_i, then Bob must perform the operation $P_i^{-1} = P_i$ on his qubit.

It can now easily be shown that the qubit in Bob's possession is nothing but the initial unknown state $|\psi\rangle = a|0\rangle + b|1\rangle$. This protocol can be verified very easily, and this has been done in various texts like [10], so we will not go into it. A simple diagrammatic representation of the above protocol is given below. In this representation, time flows from bottom to top.



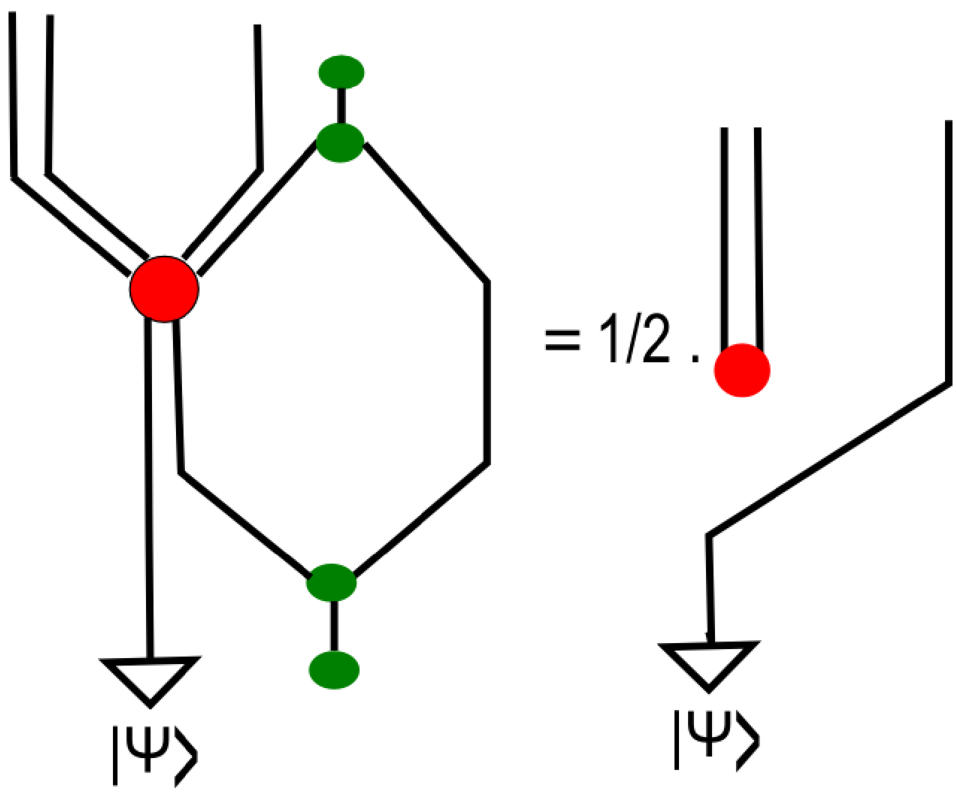
Fig(4.1)

The flow of information during the teleportation protocol has been explained in detail in texts such as [5, 6]. From these explanations it is clear that we can transfer quantum information by measuring not just the bell basis, but in fact in any basis which consists of maximally entangled states (see definition below). In fact, if we use some other basis, and our measurement outcome is some maximally entangled state $(U \otimes I) \circ (|00\rangle + |11\rangle)/\sqrt{2}$, then the unitary correction that Bob will need to perform is $U^{-1} = U^{\dagger}$(since U is a unitary operation). It just so happens that for the Pauli matrices, the inverse is the matrix itself, and hence we said that Bob will need to perform the operation P_i itself.

Definition 16. By a *maximally entangled* state for 2 qubits, we mean a standard bell-state, $(|00\rangle + |11\rangle)/\sqrt{2}$ in which one qubit has been acted on by any unitary. In other words, a maximally entangled state corresponds to: $(1_Q \otimes U) \circ (|00\rangle + |11\rangle)/\sqrt{2}$ $(= (U^T \otimes 1_Q) \circ (|00\rangle + |11\rangle)/\sqrt{2})$ for some 1-qubit unitary U.

4.2. **The actual result.** We will now show that we can axiomatize a maximally entangled state (for two qubits) in a diagrammatic manner using classical structures corresponding to bases for one and two qubits, and making them interact with each other. The idea is to determine the set of conditions that a 4×4 unitary matrix must satisfy in order to transform the computational basis for 2 qubits: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ into another basis for two qubits: $\{|b_1\rangle,|b_2\rangle,|b_3\rangle,|b_4\rangle\}$, such that the classical structure corresponding to the new basis satisfies a certain diagrammatic condition. To be more precise, we will be axiomatizing bases consisting of four maximally entangled states.

The diagrammatic condition that we will use is given below. In it, the red circle represents the classical structure corresponding to the basis $\{|b_1\rangle,|b_2\rangle,|b_3\rangle,|b_4\rangle\}$ for two qubits, whereas the green circle represents the classical structure corresponding to the computational basis $\{|0\rangle, |1\rangle\}$for one qubit.



Fig(4.2)

Now if we look at the above diagram closely, it resembles the teleportation protocol, since we have an arbitrary input tensored with the green co-unit $\epsilon^{\dagger}$(eqn 3.48) followed by the co-multiplication $\delta^{\dagger}$. As explained below, this is actually like the bell-state (though not normalised). Then we perform the copying operation with respect to the red classical structure corresponding to our new basis $B = \{|b_1\rangle,|b_2\rangle,|b_3\rangle,|b_4\rangle\}$. As we will see, this is very much like taking a measurement with respect to the basis B and then copying the result. After this, we send one copy of the result to the other side (to Bob in the teleportation protocol), and he uses it to perform some kind of operation on the second qubit of the bell-state that he shared with Alice in the beginning. As we will see later in Section 5, this operation can actually be shown to be a unitary transformation.

It is worth noting here that in the diagram, we are taking the trace when we first perform the co-unit and comultiplication, followed by multiplication and the unit operation with respect to the green classical structure corresponding to the computational basis $\{|0\rangle, |1\rangle\}$ for one qubit. As we know from various texts like [10], if we take the trace with respect to some other observable, it should not affect the overall outcome of the protocol. Indeed, this is the case, because we can easily replace the green circle with some other classical structure, say blue (corresponding to some other basis $\{|+\rangle, |-\rangle\}$for one qubit, and we will still obtain the same result.

Theorem 17. *A special, commutative, dagger-Frobenius algebra (classical structure) satisfies the above diagrammatic condition iff it corresponds to a basis* $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ *for 2 qubits in which* $|b_i\rangle = (I\otimes U_i)\,\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$*, for each i, where* U_i *represent a unitary operation on one qubit (or, equivalently, a* 2×2 *Unitary matrix). Further, the four unitaries* $U_1, U_2, U_3 \, and \, U_4$ *must satisfy the condition*

$$Trace(U_i^{-1} \circ U_j) = 2 \times \delta_{ij} \tag{4.1}$$

Proof. We will first show the forward implication, which is that if a classical structure satisfies the diagrammatic condition, then it must correspond to a basis of maximally entangled states. The backward implication is easier to prove, and will be shown later. Now, let us suppose that the *classical structure* shown as the red circle in Fig(A) corresponds to an orthonormal basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ for two qubits. Then, let U be the unitary matrix which transforms the computational basis into the new basis. We can write U as the following.

$$U= \frac{1}{\sqrt{2}}\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix}$$

In other words, if our new basis is $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$. Then,

$$(|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle) = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)\,[U] \tag{4.2}$$

It may not be clear right now as to why we are using this factor of $\frac{1}{\sqrt{2}}$ outside our matrix when we specify U. However, there is a good reason for this which will emerge as we proceed with the main result in this paper. Now we want to explicitly lay down the conditions that this matrix U must satisfy so that the classical structure corresponding to the new basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ satisfies the diagrammatic condition.

Suppose that the arbitrary input in the diagram is $|\psi\rangle = a|0\rangle + b|1\rangle$.

In the diagram, the green circle represents the classical structure corresponding to the computational basis for one qubit, i.e. $\{|0\rangle,|1\rangle\}$. We are attempting to classify all the bases which correspond to classical structures which behave like the red circle in the diagram. We are following the same diagrammatic rules in representing the copying operation, the deleting operation and their adjoints as we followed in Section 3 (the only difference is that in Fig(4.2), we are using straight lines instead of curved ones). Thus, the green co-unit followed by the green copying operation translates into:

$$I \longrightarrow \sum_{i=1}^{2} |b_i\rangle \longrightarrow \sum_{i=1}^{2} |b_i b_i\rangle :: \; 1 \longrightarrow (|0\rangle + |1\rangle) \longrightarrow (|00\rangle + |11\rangle) \tag{4.3}$$

Recall that in the category **FdHilb**, the unit object I is nothing but the underlying field, which we take to be the field of complex numbers, $\mathcal{C}$. Also, since we are working within a *strict* monoidal category, the unit object I is shown diagrammatically as just empty space.

Now, if we move upwards from the bottom (in the diagram), then on the left we only have the arbitrary input $|\psi\rangle$, and on the right we have the green co-unit followed by the comultiplication. Thus, in mathematical terms we have:

$$(a|0\rangle + b|1\rangle) \otimes (|00\rangle + |11\rangle) = a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle \tag{4.4}$$

$$= (a|00\rangle + b|10\rangle) \otimes |0\rangle + (a|01\rangle + b|11\rangle) \otimes |1\rangle \tag{4.5}$$

From eqn (4.2), we also have that

$$(|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle)[U^{-1}] = (|00\rangle, |01\rangle, |10\rangle, |11\rangle) \tag{4.6}$$

And since U is a unitary matrix, therefore $U^{-1} = \overline{U}^{\dagger}$. In other words, if we look at each column on the LHS separately, we obtain the following 4 equations:

$$|00\rangle = \frac{1}{\sqrt{2}}(\overline{u_{11}}|b_1\rangle + \overline{u_{12}}|b_2\rangle + \overline{u_{13}}|b_3\rangle + \overline{u_{14}}|b_4\rangle) \tag{4.7}$$

$$|01\rangle = \frac{1}{\sqrt{2}}(\overline{u_{21}}|b_1\rangle + \overline{u_{22}}|b_2\rangle + \overline{u_{23}}|b_3\rangle + \overline{u_{24}}|b_4\rangle) \tag{4.8}$$

$$|10\rangle = \frac{1}{\sqrt{2}}(\overline{u_{31}}|b_1\rangle + \overline{u_{32}}|b_2\rangle + \overline{u_{33}}|b_3\rangle + \overline{u_{34}}|b_4\rangle) \tag{4.9}$$

$$|11\rangle = \frac{1}{\sqrt{2}}(\overline{u_{41}}|b_1\rangle + \overline{u_{42}}|b_2\rangle + \overline{u_{43}}|b_3\rangle + \overline{u_{44}}|b_4\rangle) \tag{4.10}$$

Now, if you look at the diagram, you will see that we first have the 3-qubit state represented by equations (4.4) and (4.5). From these three qubits, the first two are copied by the classical structure represented by the red circle in the diagram. This copying is done by the relation:

$$|b_i\rangle \longmapsto |b_i b_i\rangle \tag{4.11}$$

In other words, if you have any vector (formed from 2 qubits), then the action on the vector is fully defined by the above relation, using linearity, as follows:

$$\sum_{i=1}^{4} a_i|b_i\rangle \longmapsto \sum_{i=1}^{4} a_i|b_i b_i\rangle \tag{4.12}$$

(where the a_i's are arbitrary scalars from the underlying field, which in this case is $\mathcal{C}$)

Now if we rewrite the first two qubits of eqn (4.5) in terms of the new basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$, instead of the standard computational basis using equations (4.7), (4.8), (4.9) and (4.10) , then the 3-qubit input state can be rewritten as:

$$\begin{aligned} &\tfrac{1}{\sqrt{2}}[a(\,\overline{u_{11}}|b_1\rangle + \overline{u_{12}}|b_2\rangle + \overline{u_{13}}|b_3\rangle + \overline{u_{14}}|b_4\rangle) \;+\; b(\,\overline{u_{31}}|b_1\rangle + \overline{u_{32}}|b_2\rangle + \\ &\quad \overline{u_{33}}|b_3\rangle + \overline{u_{34}}|b_4\rangle)\} \otimes |0\rangle \;+\; \{a(\,\overline{u_{21}}|b_1\rangle + \overline{u_{22}}|b_2\rangle + \overline{u_{23}}|b_3\rangle + \\ &\quad \overline{u_{24}}|b_4\rangle) \;+\; b(\,\overline{u_{41}}|b_1\rangle + \overline{u_{42}}|b_2\rangle + \overline{u_{43}}|b_3\rangle + \overline{u_{44}}|b_4\rangle)\} \otimes |1\rangle] \end{aligned} \tag{4.13}$$

After applying the copying relation- represented by the red circle- to the first two qubits, the output (a 5-qubit state) is as follows:

$$\begin{aligned}\tfrac{1}{\sqrt{2}}[a(\,\overline{u_{11}}|b_1b_1\rangle + \overline{u_{12}}|b_2b_2\rangle + \overline{u_{13}}|b_3b_3\rangle + \overline{u_{14}}|b_4b_4\rangle) \,+\, b(\,\overline{u_{31}}|b_1b_1\rangle + \overline{u_{32}}|b_2b_2\rangle + \\ \overline{u_{33}}|b_3b_3\rangle + \overline{u_{34}}|b_4b_4\rangle)\} \otimes |0\rangle \,+\, \{a(\,\overline{u_{21}}|b_1b_1\rangle + \overline{u_{22}}|b_2b_2\rangle + \overline{u_{23}}|b_3b_3\rangle + \\ \overline{u_{24}}|b_4b_4\rangle) \,+\, b(\,\overline{u_{41}}|b_1b_1\rangle + \overline{u_{42}}|b_2b_2\rangle + \overline{u_{43}}|b_3b_3\rangle + \overline{u_{44}}|b_4b_4\rangle)\} \otimes |1\rangle]\end{aligned} \tag{4.14}$$

Recall that from eqn (4.2), we can express each of the $|b_i\rangle$'s as a linear combination of the standard basis elements for two qubits: $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$. So, if we look at each column in the RHS of eqn (4.2) separately, we obtain the following four relations:

$$|b_1\rangle = \frac{1}{\sqrt{2}}(u_{11}|00\rangle + u_{21}|01\rangle + u_{31}|10\rangle + u_{41}|11\rangle) \tag{4.15}$$

$$|b_2\rangle = \frac{1}{\sqrt{2}}(u_{12}|00\rangle + u_{22}|01\rangle + u_{32}|10\rangle + u_{42}|11\rangle) \tag{4.16}$$

$$|b_3\rangle = \frac{1}{\sqrt{2}}(u_{13}|00\rangle + u_{23}|01\rangle + u_{33}|10\rangle + u_{43}|11\rangle) \tag{4.17}$$

$$|b_4\rangle = \frac{1}{\sqrt{2}}(u_{14}|00\rangle + u_{24}|01\rangle + u_{34}|10\rangle + u_{44}|11\rangle) \tag{4.18}$$

Using these four relations, we can rewrite eqn (4.14) by simply replacing the values for each of the $|b_i\rangle$'s by the corresponding relation from above. Now, in the LHS of the diagrammatic condition, after the first two qubits are copied (or co-multiplied) by the dagger-Frobenius algebra corresponding to the basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$, the next step is the multiplication of the last two qubits (of the 5-qubit state) by the classical structure corresponding to the computational basis for one qubit, i.e. $\{|0\rangle, |1\rangle\}$ This is followed by the unit operation corresponding to the same basis $\{|0\rangle, |1\rangle\}$.

In other words, we have the following operation on the last two qubits:

$$(\epsilon \circ \delta^{\dagger})(|ij\rangle) = \delta_{ij} \tag{4.19}$$

(where δ is the Kronecker Delta symbol, and i, j are the last two basis elements when the 5-qubit state is written in the computational basis).

After performing this operation, we are left with a three qubit state, which is the final state after performing the operation on the LHS of Fig(4.2). This final state can be calculated easily, and the detailed calculations are shown in the Appendix.

In order to prove the theorem, we have to determine the set of conditions on U, which will ensure that the LHS of the diagrammatic condition is equal to the RHS. Let us now recall that the RHS of our diagrammatic condition has the co-unit $\epsilon^{\dagger}$, given by the classical structure corresponding to the 'new' basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ tensored with the arbitrary input , along with a scalar multiple 1/2. In other words, the state of the three qubits on the RHS of the diagram is as follows:

$$\begin{gathered}\tfrac{1}{2}(|b_1\rangle + |b_2\rangle + |b_3\rangle + |b_4\rangle) \otimes |\psi\rangle = \tfrac{1}{2}.\tfrac{1}{\sqrt{2}}(u_{11}|00\rangle + u_{21}|01\rangle + u_{31}|10\rangle + u_{41}|11\rangle \\ +u_{12}|00\rangle + u_{22}|01\rangle + u_{32}|10\rangle + u_{42}|11\rangle + u_{13}|00\rangle + u_{23}|01\rangle + u_{33}|10\rangle + u_{43}|11\rangle + \\ u_{14}|00\rangle + u_{24}|01\rangle + u_{34}|10\rangle + u_{44}|11\rangle) \otimes (a|0\rangle + b|1\rangle)\end{gathered} \tag{4.20}$$

This can be rewritten as :

$$\begin{gathered}\tfrac{1}{2}.\tfrac{1}{\sqrt{2}}[\{(u_{11} + u_{12} + u_{13} + u_{14})|00\rangle + (u_{21} + u_{22} + u_{23} + u_{24})|01\rangle + \\ (u_{31} + u_{32} + u_{33} + u_{34})|10\rangle + (u_{41} + u_{42} + u_{43} + u_{44})|11\rangle\} \otimes (a|0\rangle + b|1\rangle)\end{gathered} \tag{4.21}$$

By calculating the final state of the three qubits on the LHS, and comparing them with eqn (4.21), we arrive at the following condition on the entries in our initial 4×4 unitary matrix U:

$$\overline{u_{1i}}u_{1i} + \overline{u_{2i}}u_{2i} \ = \ 1 \tag{4.22}$$

$$\overline{u_{3i}}u_{3i} + \overline{u_{4i}}u_{4i} \ = \ 1 \tag{4.23}$$

$$\overline{u_{1i}}u_{3i} + \overline{u_{2i}}u_{4i} \ = \ 0 \ \ (\forall i = 1, 2, 3, 4) \tag{4.24}$$

This actually translates into the fact that if we write the entries from each column of U (dropping the overall factor of $\frac{1}{\sqrt{2}}$) in the form of a 2×2 matrix, then the matrix will be unitary. In other words, if we write the entries of the i^{th} column as a 2×2 matrix like this:

$$E_i = \begin{bmatrix} u_{1i} & u_{3i} \\ u_{2i} & u_{4i} \end{bmatrix}$$

Then our theorem states that E_i must be a unitary matrix. To understand this better, notice that if we regard each column of the matrix E_i to be a vector (in a 2-dimensional Hilbert space), then equations (4.22), (4.23) and (4.24) simply say that the inner product of each vector with itself equals 1, and the inner product of two distinct vectors equals 0. This translates into the fact that $(|0\rangle, |1\rangle)\,[E_i]$ is an orthonormal basis for the 1-qubit Hilbert space, as explained in Section 2. This is exactly equivalent to saying that E_i is a unitary matrix. If we recall, we had the following equation:

$$(|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle) = \frac{1}{\sqrt{2}}(|00\rangle, |01\rangle, |10\rangle, |11\rangle) \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \tag{4.25}$$

Now, if we look at only the first column on both sides, we have,

$$|b_1\rangle = \frac{1}{\sqrt{2}}\{u_{11}|00\rangle + u_{21}|01\rangle + u_{31}|10\rangle + u_{41}|11\rangle\} \tag{4.26}$$

This equation can equivalently be written as,

$$|b_1\rangle = \frac{1}{\sqrt{2}}\{|0\rangle \otimes (u_{11}|0\rangle + u_{21}|1\rangle) + |1\rangle \otimes (u_{31}|0\rangle + u_{41}|1\rangle)\} = (1_Q \otimes E_1)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.27}$$

Here, 1_Q stands for the identity morphism on the 2 dimensional Hilbert space of one qubit.

The above relation holds for $i = 2, 3, 4$ as well, and hence our theorem has been proved. It is worth noting here that eqn (4.26) can also be rewritten as:

$$|b_1\rangle = \frac{1}{\sqrt{2}}\{(u_{11}|0\rangle + u_{31}|1\rangle) \otimes |0\rangle + (u_{21}|0\rangle + u_{41}|1\rangle) \otimes |1\rangle\} = (E_1^T \otimes 1_Q)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.28}$$

Thus, we can see that applying some unitary operation to the second qubit of the bell-state is the same as applying the transpose of that unitary to the first qubit. This property is true not only in **FdHilb**, but also at a more abstract level, in a dagger compact category, as explained in [4].

Also, since the whole matrix U must be unitary, we have that for any two columns i and j, the following equation must hold:

$$\frac{1}{2}(\overline{u_{1i}}u_{1j} + \overline{u_{2i}}u_{2j} + \overline{u_{3i}}u_{3j} + \overline{u_{4i}}u_{4j}) = \delta_{ij} \tag{4.29}$$

$$\iff Trace[U_i^{-1} \circ U_j] = \delta_{ij} \tag{4.30}$$

□

This completes the proof of the forward implication of the theorem. To see that the backward implication must also hold, notice that if the basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ consists of maximally entangled states, then the matrix U which transforms the computational basis for two qubits to this new basis (when written in the same form as used in the proof above) must satisfy equations (4.22), (4.23) and (4.24). It is quite easy to see that if these three conditions hold, then the classical structure corresponding to the basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$ must satisfy the diagrammatic conditions set out in Fig(4.2) (see Appendix). To get an idea of how the detailed calculations were done to arrive at the conditions given in equations (4.22), (4.23) and (4.24) the reader can have a look at the Appendix.

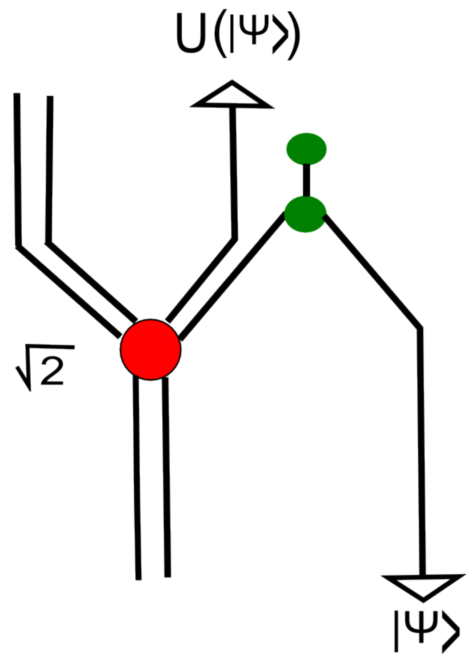
5. A closer look

Right now, the result that we have obtained in Section 4 may seem to be coming out of thin air, with no logical foundation. It may even seem like a mere coincidence. But as we shall see in this section, this is not the case, and that the diagram that we constructed to axiomatize maximally entangled states has a sound mathematical basis.

We will first explain how the operation on the third qubit (from the initial 3-qubit state in Fig (4.2)) is a unitary operation which is closely linked and dependent upon the 2-qubit state that it is placed next to. This feature is particularly useful since it can be used to simulate the act of performing unitary operations on certain qubits which are dependent on the measurement outcomes of other qubits. Then we will explain how the red copying operation that we perform in the diagram is actually representative of taking a measurement and storing *all* the possible measurement outcomes together. We can then interpret actions in many quantum informatic protocols in a new way. Finally, as an application, we will show how the entanglement swapping protocol can be represented in our diagrammatic process using classical structures.

5.1. **Unitary corrections.**

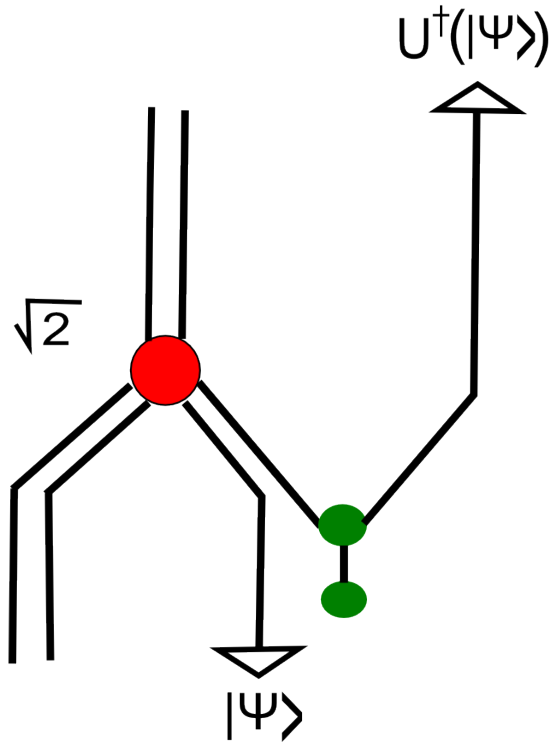
Now, we are first going to show that the operation that we performed on the third qubit (i.e. Bob's half of the shared bell-state) in the Fig(4.2) is actually a unitary operation. In other words, if $|\psi\rangle$ is a qubit in some arbitrary state, then in terms of a diagram, we have:



Fig(5.1)

Thus, the diagram says that the operation performed on the arbitrary input $|\psi\rangle$ is equivalent to performing a unitary operation on $|\psi\rangle$. We will soon see how the state of the two qubits on the left is intricately connected to the unitary operation U. But before we get to that, let us show that this operation is indeed a unitary operation.

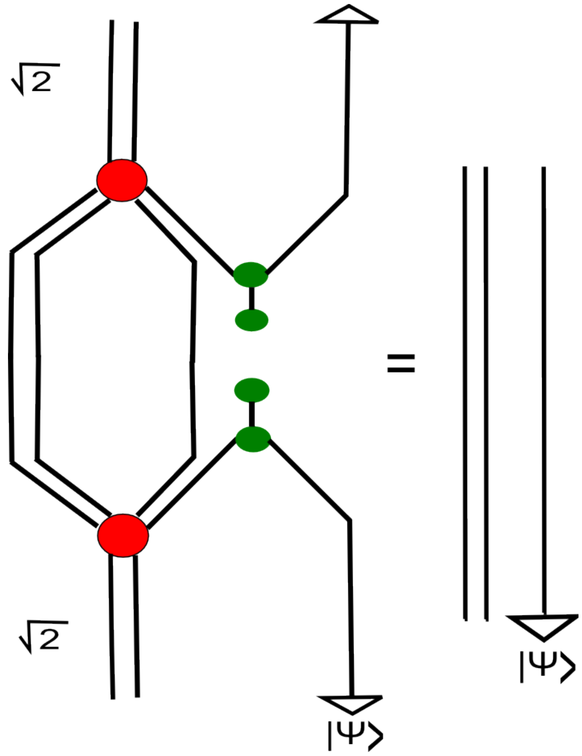
Let us remind the reader that in the diagrammatic calculus that we have followed throughout this paper, the adjoint to an operation is shown by flipping the picture for the operation upside down. Thus, if the operation above is indeed a unitary operation, then if we compose this operation with it's own reflection (with respect to a horizontal axis), then we will obtain the identity. Thus, we will actually show two things simultaneously. Firstly, we will show that Fig(5.1) represents a unitary transformation U. Secondly, we will show that the operation depicted as the reflection of Fig(5.1), which is shown below as Fig(5.2), represents the unitary transformation $U^{\dagger}$, i.e:



Fig(5.2)

The figure above represents the adjoint to Fig(5.1), and we will soon show that $U \circ U^{\dagger} = U^{\dagger} \circ U = I$. In fact, we will show something much stronger, about the nature of this unitary transformation U, and how it is related to the state of the the two qubits on the left in Fig(5.1).

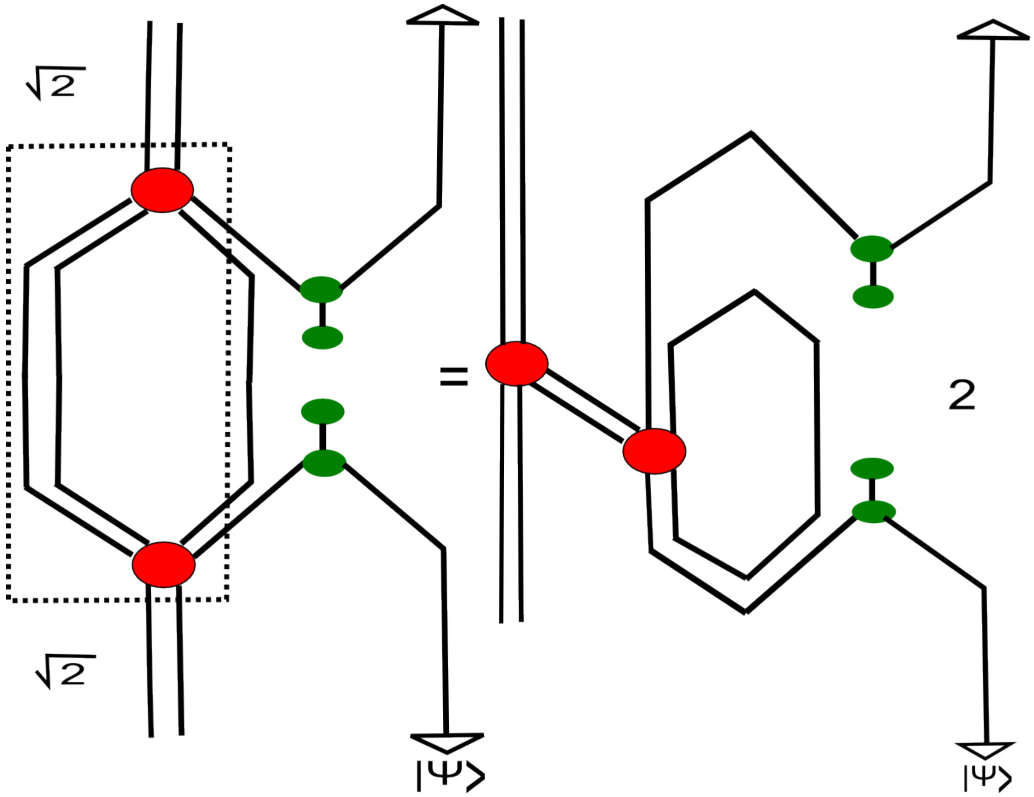
Now, to show that $U^{\dagger} \circ U = I$, we need to prove the following diagrammatic equation:



Fig(5.3)

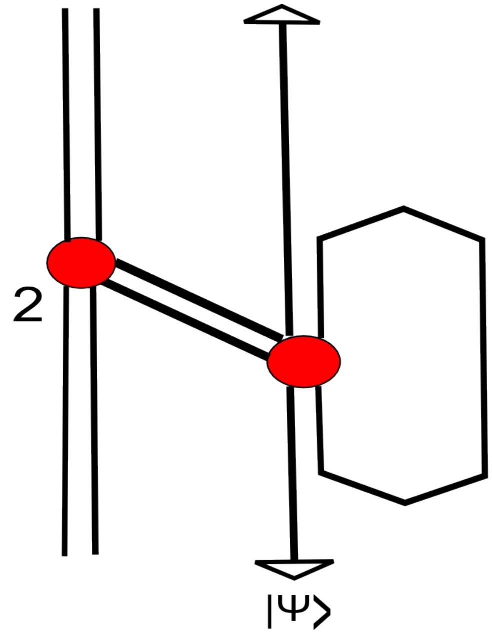
If we manage to prove this, then we would have shown that $U^\dagger \circ U = I$.

Now to see that the diagrammatic equation expressed in Fig(5.3) is indeed true, we will first use the Frobenius law expressed in Fig(3.14) for the classical structure (i.e. the special commutative dagger-Frobenius algebra) represented by the red circle. The part where we are applying the Frobenius law is indicated as being enclosed in the dotted line on the left in the following diagram (Fig(5.1)). The part on the right shows what we get after applying the Frobenius law to the enclosed area.



Fig(5.4)

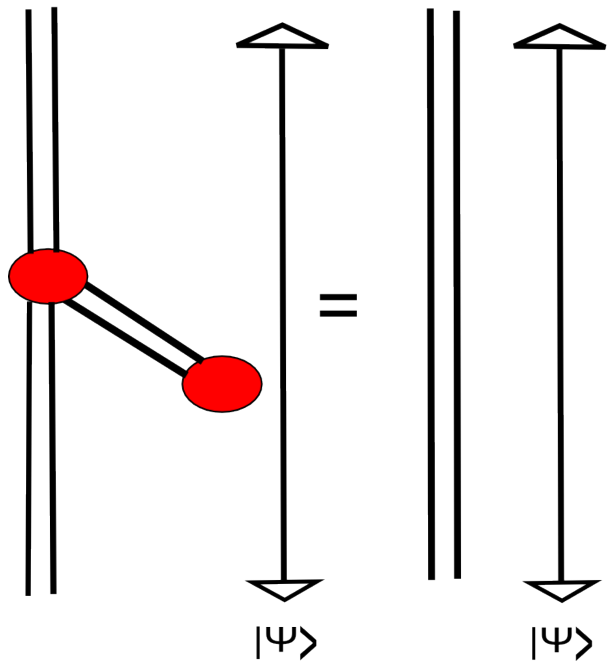
Now, in the RHS of Fig(5.3), which is obtained after applying the Frobenius equation corresponding to the classical structure denoted by the red circle, we can agin apply the Frobenius law corresponding to the *green* classical structure to obtain:



Fig(5.5)

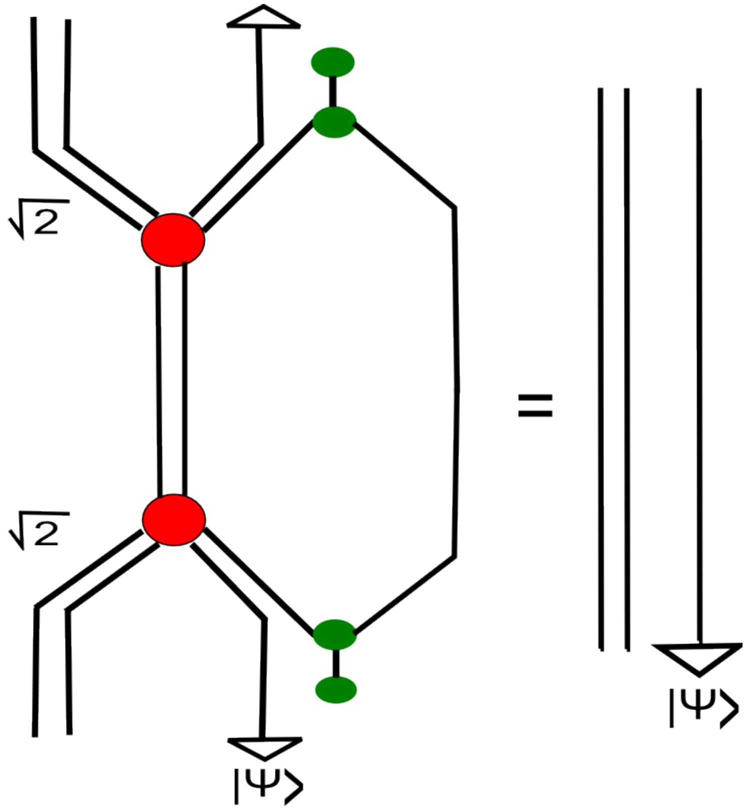
Now, in Fig(5.5), we can easily identify a structure to which we can apply the theorem that we have proved in Section 4. The red circle in the above figure represents a classical structure corresponding to a basis consisting of maximally entangled states (i.e. a basis which satisfies the conditions set out in Theorem 17), and therefore we can apply the backward implication of Theorem 17. In this structure, the enclosed hexagon represents the part which was traced out in Fig(4.2). As we had explained then, it doesn't matter which classical structure (for one qubit) is used to trace out the the relevant part, and hence, it should make no difference that the green circles are not present in the part which is traced out (if we recall, the green circles *are* present in Fig(4.2)).

Thus, after applying Theorem 17 to a part of Fig(5.5), and cancelling the factor of 2 in Fig(5.5) with the 1/2 which appears on the RHS of the diagrammatic condition laid out in Fig(4.2), we get the figure shown on the left in Fig(5.6). We can then apply the condition for classical structures depicted in Fig(3.13) to obtain the picture on the right in Fig(5.6).



Fig(5.6)

This is exactly what we needed to show in order to establish the equation shown as Fig(5.3). Thus, if we represent U as the operation shown in Fig(5.1), and $U^\dagger$ by the operation shown as Fig(5.2), then we have shown that $U^\dagger \circ U = I$. Now, in order to show that $U \circ U^\dagger = I$, we need to establish the truth of the diagrammatic eqn shown below:



Fig(5.7)

And we can easily prove the above equation in almost exactly the same way that proved the equation expressed as Fig(5.3). Thus, we can show that $U \circ U^{\dagger} = U^{\dagger} \circ U = I$.

Hence we have established the fact that the operation U, as shown in Fig(5.1) is a unitary operation, and that the operation shown in Fig(5.2) is its adjoint (and therefore its inverse).

We can actually show something far more concrete about the unitary operation shown as Fig(5.2) and how it is related to the state of the two qubits on the left. We have the following theorem:

Theorem 18. *If* $|e_1\rangle = \frac{1}{\sqrt{2}}(U_1 \otimes 1_Q)(|00\rangle + |11\rangle)$ *is an element of the basis which corresponds to the classical structure depicted as a red circle in Fig(5.1), and if the two qubits on the left in Fig(5.1) are in the state* $|e_1\rangle$*, then the arbitrary vector* $|\psi\rangle = a|0\rangle + b|1\rangle$ *(the input in Fig(5.1)) undergoes the unitary transformation* U_1 *in the operation described in Fig(5.1).*

Proof. Let us suppose that the unitary operation U_1 is given by the following matrix:

$U_1 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, then we have that

$$[U_1(|0\rangle),\, U_1(|1\rangle)] = [|0\rangle,\, |1\rangle] \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \tag{5.1}$$

Thus, the initial 3-qubit state, after multiplication by the scalar $\sqrt{2}$ in Fig(5.1) will be

$$\sqrt{2}|e_1\rangle \otimes (a|0\rangle + b|1\rangle) \tag{5.2}$$

Then after comultiplication by the red classical structure, we get

$$\sqrt{2}|e_1\rangle \otimes |e_1\rangle \otimes (a|0\rangle + b|1\rangle) \tag{5.3}$$

$$= \sqrt{2}|e_1\rangle \otimes \frac{1}{\sqrt{2}}\{(u_{11}|0\rangle + u_{21}|1\rangle) \otimes |0\rangle + (u_{12}|0\rangle + u_{22}|1\rangle) \otimes |1\rangle\} \otimes (a|0\rangle + b|1\rangle) \tag{5.4}$$

$$= |e_1\rangle \otimes (u_{11}|00\rangle + u_{21}|10\rangle + u_{12}|01\rangle + u_{22}|11\rangle) \otimes (a|0\rangle + b|1\rangle) \tag{5.5}$$

The next step in the operation described in Fig(5.1) is the multiplication of the last two qubits with respect to the classical structure corresponding to the basis $\{|0\rangle, |1\rangle\}$(the green classical structure). This is followed by the co-unit corresponding to the same classical structure. After performing these operations on the expression above, we get:

$$|e_1\rangle \otimes \{a\,(u_{11}|0\rangle + u_{21}|1\rangle) + b\,(u_{12}|0\rangle + u_{22}|1\rangle)\} \tag{5.6}$$

$$= |e_1\rangle \otimes \{a\,U_1(|0\rangle) + b\,(U_1|1\rangle)\} = |e_1\rangle \otimes U_1(a|0\rangle + b|1\rangle) \tag{5.7}$$

□

This tells us that during the operation shown as Fig(5.1), if the first two qubits are in a state $|e_1\rangle$, then the arbitrary input $|\psi\rangle$ undergoes exactly the same unitary transformation that was applied to the first qubit of the bell state in order to obtain $|e_1\rangle$.

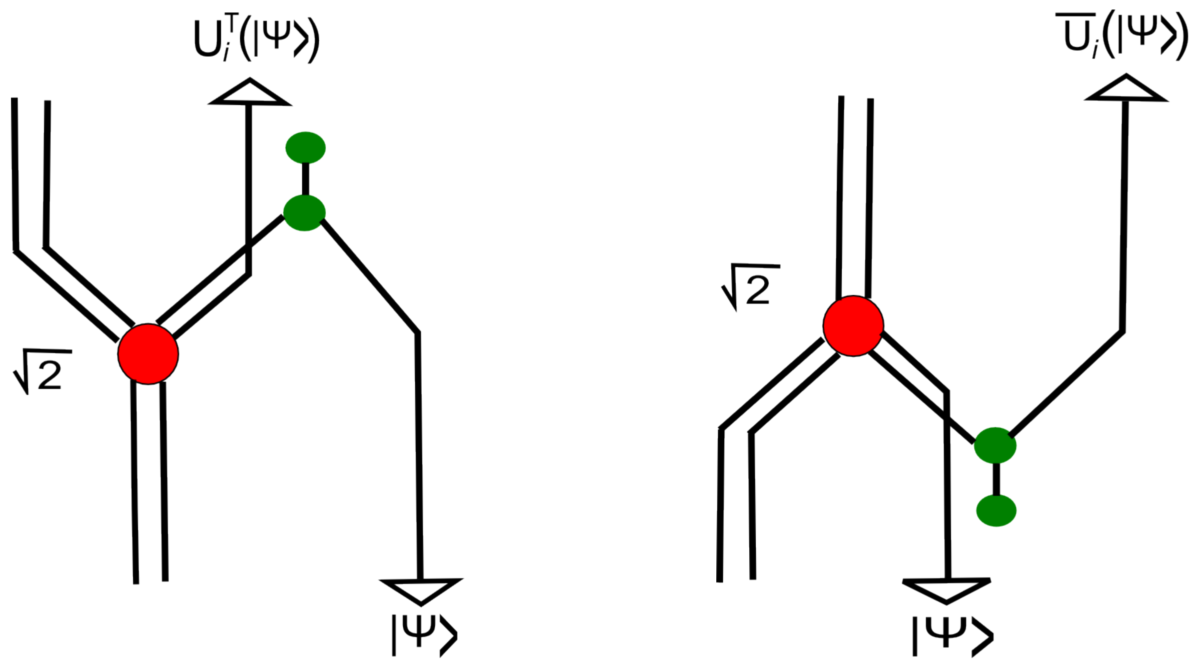
It is worth noting here, that in eqn (5.5), if we were to multiply the last and the third from last (instead of second from last) qubit, then instead of eqn (5.6), we would have obtained the final state of the three qubits to be:

$$|e_1\rangle \otimes \{a\,(u_{11}|0\rangle + u_{12}|1\rangle) + b\,(u_{21}|0\rangle + u_{22}|1\rangle)\} \tag{5.8}$$

$$= |e_1\rangle \otimes \{a\, U_1^T(|0\rangle) + b\, U_1^T(|1\rangle)\} = |e_1\rangle \otimes U_1^T(a|0\rangle + b|1\rangle) \tag{5.9}$$

Thus, by slightly modifying Fig(5.1), we can also perform the unitary transformation U^T instead of performing the operation U. This is shown diagrammatically (on the left) in Fig(5.8).

We also already know that the operation shown as Fig(5.2) is the adjoint to the operation shown in Fig(5.1). In exactly the same way that we proved Theorem 18, we can also show that in Fig(5.2), if the state of the first two qubits is $|e_i\rangle = \frac{1}{\sqrt{2}}(U_i \otimes 1_Q)(|00\rangle + |11\rangle)$, and if $|e_i\rangle$ is also a member of the basis which corresponds to the red classical structure, then the effect of the operation in Fig(5.2) is to perform the unitary transformation $U_i^\dagger$ on the arbitrary input state. In other words, if the 3-qubit input state in Fig(5.2) is $|e_i\rangle \otimes |\psi\rangle$, where $|\psi\rangle$ is some arbitrary qubit (the input), then the output of the entire protocol is $|e_i\rangle \otimes U_i^\dagger(|\psi\rangle)$. And once again, by slightly modifying this protocol, or by simply taking the adjoint of the transpose operation (shown on the left in Fig(5.8)), we can obtain the diagrammatic representation of the operation which performs $\overline{U_i} = (U_i^T)^\dagger$ on the arbitrary input when the first two qubits are in a state $|e_i\rangle$. The pictures for performing both U_i^T and $\overline{U_i}$ are side by side below. It is assumed that the two qubits on the left are in a state $|e_i\rangle = \frac{1}{\sqrt{2}}(U_i \otimes 1_Q)(|00\rangle + |11\rangle)$, and that $|e_i\rangle$ is a member of the basis which corresponds to the red classical structure.



Fig(5.8)

Also, since the red classical structure in Fig(5.1) and Fig(5.2) represents a basis in which each element is a maximally entangled state, therefore we can describe the effect of any operation of the kind in Fig(5.1), where the first two qubits are in any arbitrary state, using linearity. Thus, if the first two qubits in Fig(5.1) are in any arbitrary two qubit state, we can write it as $|\theta\rangle = u|e_1\rangle + v|e_2\rangle + w|e_3\rangle + x|e_4\rangle$ for some scalars x, y, z and w. Then the effect of the operation on the 3-qubit input state $|\theta\rangle \otimes |\psi\rangle$ would be:

$$|\theta\rangle \otimes |\psi\rangle \longmapsto u|e_1\rangle \otimes U_1(|\psi\rangle) + v|e_2\rangle \otimes U_2(|\psi\rangle) + w|e_3\rangle \otimes U_3(|\psi\rangle) + x|e_4\rangle \otimes U_4(|\psi\rangle) \tag{5.10}$$

This is extremely useful, as we will see in the next section, since it allows us to express pairs of qubits as superpositions of many measurement outcomes and also allows us to apply unitary corrections in each of the possible scenarios simultaneously, again using the superposition principle.

5.2. **Superposition of all observational branches.**

In this subsection, we are going to explain how the copying operation actually represents an encoding of all possible measurement outcomes, and thus, using a copy of the measurement outcome to perform the unitary correction amounts to performing all the unitary corrections to a superposition of all possible states of the concerned qubit simultaneously. This is actually the most important part of our entire result, since it holds the most promise for future research and findings in this area.

It will be easiest to explain what we mean by first explaining it for the protocol that we have discussed so much already- the teleportation protocol. Now, if we recall, in the actual protocol, Alice measures the first two qubits (the input $|\psi\rangle$, and her half of the bell-state that she shares with Bob). If Alice measures in the Bell-basis (or any other basis of maximally entangled states), there are four possible measurement outcomes. So far, in all accurate diagrammatic representations of teleportation, each measurement outcome and the corresponding unitary correction has to be shown separately. There are ways to represent each observational branch in the form of a tree discussed in some recent texts (such as [5, 6]), but essentially they show all the possibilities in a compact, though separated fashion.

We will show that in the representation discussed in this paper, the qubits which are measured in the protocol actually exist in a superposition of all the possible measurement outcomes, and the qubits to which unitary corrections (which are dependent on measurement outcomes) are applied exist in a superposition of the different states which would be obtained depending on which measurement outcome was obtained. We will explain this with the example of the teleportation protocol.

To see this, we will first explain how the diagrammatic equation given in Fig(4.2) can be derived using Theorem 18. Now, the LHS of Fig(4.2) can be seen equivalently as the operation shown in Fig(5.1) with the initial state of the three qubits being the same as given in eqn (4.4). If we recall, the 3-qubit state in eqn (4.4) (say, $|\phi\rangle$) was written as:

$$|\phi\rangle = a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle \tag{5.11}$$

Now, with the input state in Fig(5.1) being equal to $|\phi\rangle$, and after multiplying Fig(5.1) by a scalar factor of $\frac{1}{\sqrt{2}}$, the LHS of the diagrammatic equation given as Fig(4.2) is exactly the same as Fig(5.1). Now we will evaluate the final output after performing the operation expressed as Fig(5.1) on the 3-qubit input state $\frac{1}{\sqrt{2}}|\phi\rangle$ using Theorem 18 and show that it is exactly the same as the RHS of Fig(4.2). This will provide us with a new interpretation of the protocol represented in Fig(4.2).

In Section 4, we had rewritten the state $|\phi\rangle$ using the new basis (corresponding to the red classical structure) $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$, as eqn (4.13), which is given again here:

$$\begin{aligned}|\phi\rangle = \tfrac{1}{\sqrt{2}}[a(\,\overline{u_{11}}|b_1\rangle + \overline{u_{12}}|b_2\rangle + \overline{u_{13}}|b_3\rangle + \overline{u_{14}}|b_4\rangle) \,+\, b(\,\overline{u_{31}}|b_1\rangle + \overline{u_{32}}|b_2\rangle + \\ \overline{u_{33}}|b_3\rangle + \overline{u_{34}}|b_4\rangle)\} \otimes |0\rangle \,+\, \{a(\,\overline{u_{21}}|b_1\rangle + \overline{u_{22}}|b_2\rangle + \overline{u_{23}}|b_3\rangle + \\ \overline{u_{24}}|b_4\rangle) \,+\, b(\,\overline{u_{41}}|b_1\rangle + \overline{u_{42}}|b_2\rangle + \overline{u_{43}}|b_3\rangle + \overline{u_{44}}|b_4\rangle)\} \otimes |1\rangle]\end{aligned} \tag{5.12}$$

The above expression can be rewritten as:

$$\begin{aligned}|\phi\rangle = \tfrac{1}{\sqrt{2}}[|b_1\rangle \otimes \{a(\overline{u_{11}}|0\rangle + \overline{u_{21}}|1\rangle) + b(\overline{u_{31}}|0\rangle + \overline{u_{41}}|1\rangle)\} \,+\, |b_2\rangle \otimes \{a(\overline{u_{12}}|0\rangle + \overline{u_{22}}|1\rangle) + \\ b(\overline{u_{32}}|0\rangle + \overline{u_{42}}|1\rangle)\} \,+ |b_3\rangle \otimes \{a(\overline{u_{13}}|0\rangle + \overline{u_{23}}|1\rangle) + b(\overline{u_{33}}|0\rangle + \overline{u_{43}}|1\rangle)\} + \\ |b_4\rangle \otimes \{a(\overline{u_{14}}|0\rangle + \overline{u_{24}}|1\rangle) + b(\overline{u_{34}}|0\rangle + \overline{u_{44}}|1\rangle)\}\end{aligned} \tag{5.13}$$

In Section 4, when we derived the result that the basis corresponding to the red classical structure in Fig(4.2) consists of maximally entangled states, we had shown (in equations (4.27) and (4.28)) that each basis element $|b_i\rangle = (1_Q \otimes E_i)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = (E_i^T \otimes 1_Q)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. In this section, we are assuming that the unitary operation to obtain the maximally entangled state $|b_i\rangle$ is performed on the first qubit instead of the second. Thus, each element of the new basis (which corresponds to the red classical structure in Fig(4.2) and Fig(5.1)) can be written as $|b_i\rangle = (U_i \otimes 1_Q)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Since we earlier had

$$E_i = \begin{bmatrix} u_{1i} & u_{3i} \\ u_{2i} & u_{4i} \end{bmatrix},$$

Thus, we now get

$$U_i = E_i^T = \begin{bmatrix} u_{1i} & u_{2i} \\ u_{3i} & u_{4i} \end{bmatrix} \tag{5.14}$$

Using this definition of U_i, we can rewrite eqn (5.13) in a very nice way as:

$$\begin{aligned} |\phi\rangle = \tfrac{1}{\sqrt{2}}[|b_1\rangle \otimes U_1^{-1}(a|0\rangle + b|1\rangle) + |b_2\rangle \otimes U_2^{-1}(a|0\rangle + b|1\rangle) + \\ |b_3\rangle \otimes U_3^{-1}(a|0\rangle + b|1\rangle) + |b_4\rangle \otimes U_4^{-1}(a|0\rangle + b|1\rangle)] \end{aligned} \tag{5.15}$$

Now, in Theorem 18, we had explained that if the initial 3-qubit state, before performing the operation shown in Fig(5.1), is $|b_i\rangle \otimes (a|0\rangle + b|1\rangle)$, where $|b_i\rangle = (U_i \otimes 1_Q)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the classical structure shown as the red circle represents a basis of which $|b_i\rangle$ is a member, then the output, after performing the operation, is $|b_i\rangle \otimes U_i(a|0\rangle + b|1\rangle)$. In the present discussion, we also have to multiply the outcome by a factor of $\frac{1}{\sqrt{2}}$, (so that the LHS of Fig(4.2) is exactly the same as the LHS of Fig(5.1)).

Thus, if we have a 3-qubit input state $|\phi\rangle$, given in eqn (5.15), and if we perform our unitary operation (depicted as Fig(5.1)) on it, then using linearity we would obtain the 3-qubit output state $|\phi'\rangle$ to be:

$$\begin{aligned} |\phi'\rangle = \tfrac{1}{\sqrt{2}}.\tfrac{1}{\sqrt{2}}[|b_1\rangle \otimes U_1 \circ U_1^{-1}(a|0\rangle + b|1\rangle) + |b_2\rangle \otimes U_2 \circ U_2^{-1}(a|0\rangle + b|1\rangle) + \\ |b_3\rangle \otimes U_3 \circ U_3^{-1}(a|0\rangle + b|1\rangle) + |b_4\rangle \otimes U_4 \circ U_4^{-1}(a|0\rangle + b|1\rangle) \\ = \tfrac{1}{2}(|b_1\rangle + |b_2\rangle + |b_3\rangle + |b_4\rangle) \otimes (a|0\rangle + b|1\rangle) \end{aligned} \tag{5.16}$$

This is exactly the same as the RHS of Fig(4.2), which was given as eqn (4.21). Now, this gives us a new interpretation of the teleportation protocol shown as Fig(4.2). We can now think of the initial 3-qubit state in the teleportation protocol as being in a superposition of each possible measurement outcome tensored with the corresponding state of the third qubit. If you look at the expression for the 3-qubit state $|\phi\rangle$ given in eqn (5.15), this is exactly what it says. It says that the state $|\phi\rangle$ is in a superposition of the states $|b_i\rangle \otimes U_i^{-1}(a|0\rangle + b|1\rangle)$, where $i = 1, 2, 3, 4$. And in the protocol, when we apply the unitary correction, it takes each of the possible scenarios into account, and gives us an overall unitary correction which acts on each of the possible measurement outcomes simultaneously. This is expressed as the outcome $|\phi'\rangle$ obtained after performing the needed unitary correction on each of the superpositions.

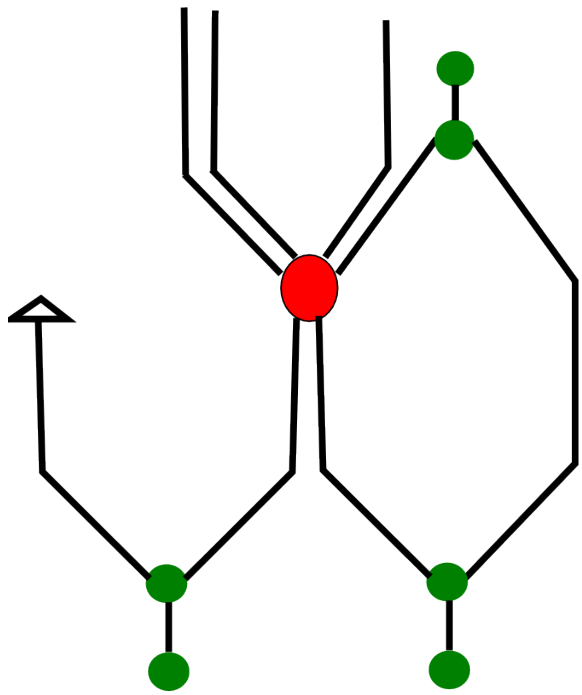
Note that after the entire protocol, we still have a red co-unit, which actually represents a superposition of all the possible states that the first two qubits (from the 3-qubit initial state) could be in after the measurement. Since the first two qubits are measured in the basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$, therefore the 2 qubits are in the state $(|b_1\rangle + |b_2\rangle + |b_3\rangle + |b_4\rangle)$, which is a supersposition of all the possible measurement outcomes.

5.3. **An application: Entanglement swapping.**

In this subsection, we are just going to illustrate how our approach can be used to simulate and verify different quantum informatic protocols. As a specific example, we are going to show how the entanglement swapping protocol can be simulated using classical structures.

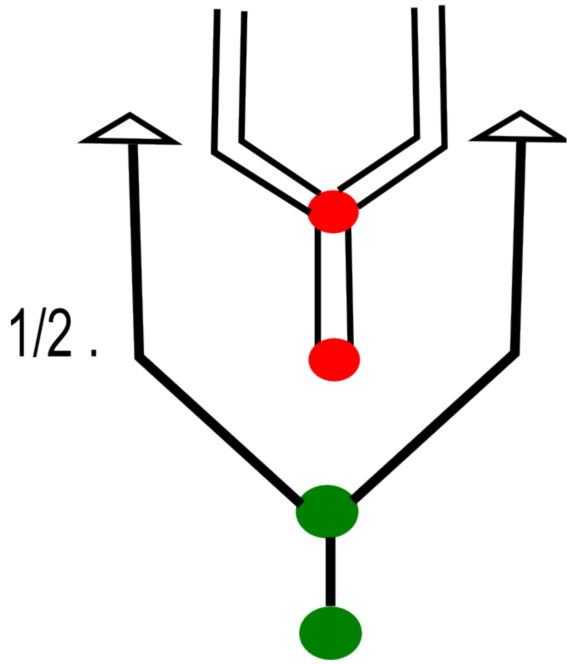
Now, the entanglement swapping protocol is just a slightly more complicated version of the teleportation protocol. Instead of one arbitrary input and one bell-state which is shared between Alice and Bob, here we have two bell states which are kept side by side. To put it more clearly, we have four qubits- A, B, C and D. A and B are together in a bell-state, and C and D are in another bell-state. Now, in the protocol, B and C are measured together in the bell-basis, and then depending on the measurement outcome, single-qubit unitary operations are performed on B and D (their transposes could be performed on C and A respectively). After this, we find that A and D share a bell-state, and B and C share another bell-state. Thus, the entanglement has been 'swapped'. This protocol, and the flow of quantum information which takes place during it are explained in detail in [5] and [6].

Now, if we are to use Classical structures to represent the entanglement swapping protocol, then instead of the LHS of Fig(4.2), we have the following situation:



Fig(5.9)

Note that in this diagram, the bell-states are not normalized. This is not a problem, since the bell-states that we get after the protocol will also be shown without normalization, making no difference to the overall picture. Once again, the red classical structure shown in the diagram corresponds to a basis of maximally entangled states for two qubits, $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$, where each element can be written as: $|b_i\rangle = (U_i \otimes 1_Q)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Now, since the bell-basis satisfies the conditions set out in Theorem 17, we can apply the backward implication of this theorem to Fig(5.9) and then co-multiply the red co-unit that we obtain in the RHS of Fig(4.2) to obtain Fig(5.10).



Fig(5.10)

Recall that in the teleportation protocol, the red co-unit represents a superposition of all the possible states that the first two qubits could be in. In the entanglement swapping protocol, it represents a superposition of all the states that the middle two qubits (B and C) could be in *before* the unitary correction is applied to qubit B (or C). However, when we apply Theorem 17, we are automatically applying the required unitary correction to qubit D. Thus, A and D already share a bell basis. Now, the unitary correction that we need to apply to B will depend upon the measurement outcome obtained after measuring B and C. As we explained earlier in this section, when we need to perform a unitary correction which is dependent on some measurement outcome, we need a *copy* of the measurement outcome next to the qubit on which the correction is to be made. This is why we are co-multipling the red co-unit in Fig(5.10)- so that we have a copy of the measurement outcome at hand, which we can then use to perform the appropriate correction.

To put it in mathematical terms, if we were to simply apply Theorem 17 to Fig(5.9), then as the final outcome we would obtain:

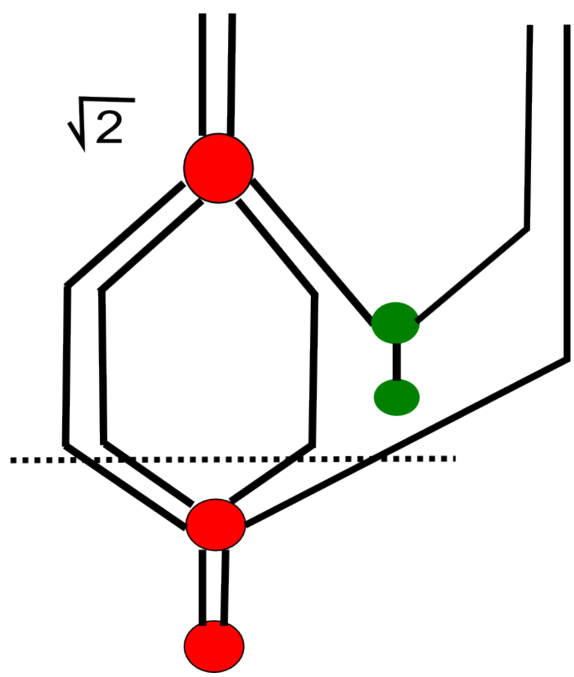
$$\frac{1}{2}\{|0\rangle \otimes (|b_1\rangle + |b_2\rangle + |b_3\rangle + |b_4\rangle) \otimes |0\rangle + |1\rangle \otimes (|b_1\rangle + |b_2\rangle + |b_3\rangle + |b_4\rangle) \otimes |1\rangle\} \tag{5.17}$$

However, to create a copy of the measurement outcome which can be used to perform the unitary correction, we copy the middle two qubits using the red classical structure and obtain:

$$\frac{1}{2}\{|0\rangle \otimes (|b_1b_1\rangle + |b_2b_2\rangle + |b_3b_3\rangle + |b_4b_4\rangle) \otimes |0\rangle + |1\rangle \otimes (|b_1b_1\rangle + |b_2b_2\rangle + |b_3b_3\rangle + |b_4b_4\rangle) \otimes |1\rangle\} \tag{5.18}$$

The state of the four qubits in the centre, in Fig(5.10), i.e. $(|b_1b_1\rangle + |b_2b_2\rangle + |b_3b_3\rangle + |b_4b_4\rangle)$, should be interpreted as a superposition of all the possible measurement outcomes together with the corresponding state of the two centre qubits (i.e. qubits B and C) in the entanglement swapping protocol. This may seem trivial, since it is quite obvious that if we measure B and C, and the measurement outcome is $|b_i\rangle$, then B and C will be in the joint state $|b_i\rangle$. However, the step of copying the red co-unit is absolutely necessary, since we need a copy of the measurement outcome in order to perform the unitary correction on C.

Since we want to end up with B and C sharing a bell-state, it is clear that if the measurement outcome is $|b_i\rangle$ (which implies that B and C are in the joint state $|b_i\rangle = (U_i \otimes 1_Q)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle))$, then we need to apply the unitary correction $U_i^\dagger$ to B in order to make sure that B and C are in the bell-state. Thus, we need to apply the operation shown as Fig (5.2) taking the 3-qubit input state to be the first three qubits of the state $(|b_1b_1\rangle + |b_2b_2\rangle + |b_3b_3\rangle + |b_4b_4\rangle)$. We can depict the application of the operation in Fig(5.2) on this particular input state as the following diagram:

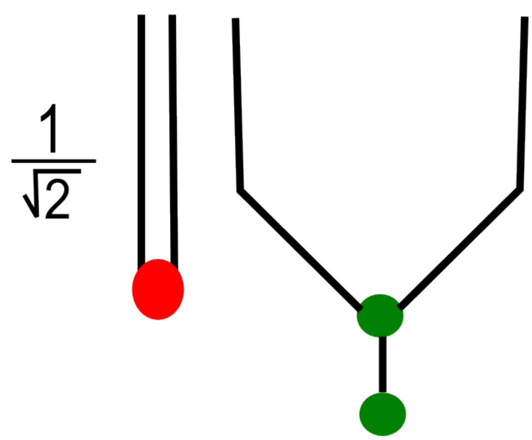


Fig(5.11)

Now, in this diagram, the first two qubits on the left represent the measurement outcome, and the next two qubits on the right represent the joint state of the qubits B and C (at the level of the dotted line). We are performing a unitary correction on B in exactly the same manner as Fig(5.2), which is why we are multiplying by a scalar factor of $\sqrt{2}$. Now, since we know that in Fig(5.11), we are applying the unitary operation $U_i^\dagger$ to the third qubit at the level of the dotted line, we get the final output state to be,

$$\begin{aligned}\textstyle\sum_{i=1}^{4} |b_i\rangle \otimes (U_i^\dagger \otimes 1_Q) \circ |b_i\rangle &= \textstyle\sum_{i=1}^{4} |b_i\rangle \otimes \frac{1}{\sqrt{2}}((U_i^\dagger \circ U_i) \otimes 1_Q)) \circ (|00\rangle + |11\rangle) \\ &= (\textstyle\sum_{i=1}^{4} |b_i\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\end{aligned} \tag{5.19}$$

which is nothing but the red co-unit tensored with the bell-state. In terms of a picture, the final state of the four qubits after performing the operation shown in Fig(5.11) is nothing but



Fig(5.12)

Here, the red co-unit represents a copy of the superposition of all the measurement outcomes. This completes the entanglement swapping protocol, since we now have B and C in a joint bell-state, and since A and D were already in a bell-state in Fig(5.10). This provides us with a new interpretation, and new way of checking the correctness of the entanglement swapping protocol.

It is also worth noting, that if we were not interested in preserving a copy of the measurement outcome at the end of the protocol- whether it is entanglement swapping or teleportation, then at the end of each protocol, we can simply apply the red unit (i.e. the unit operation corresponding to the basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$, which performs the linear operation $\epsilon \equiv X \longmapsto I \; :: \; |b_i\rangle \rightarrow 1$) to the first two qubits of the final output state, and obtain only the state of the qubit(s) that we were interested in.

For instance, if we apply the red unit to the first two qubits of the final state that we obtained after applying the teleportation protocol, i.e. to eqn (5.16), we simply obtain the state of the qubit which was the input in our teleportation protocol, i.e. $|\psi\rangle$. Similarly, if we apply the red unit to eqn (5.19), we obtain only the bell-state, which is the state that qubits B and C are left in after applying the protocol. However, in both cases, the vectors are not normalized, and we also get some scalar factors, but this does not affect the correctness of the protocol.

6. Significance and future work

In Section 4, we saw how we can use classical structures and simple diagrammatic equations to axiomatize bases consisting of maximally entangled states. We can apply the same method to axiomatize other kinds of bases having other kinds of properties, and perhaps generalise the results for a larger number of qubits. The method of using classical structures allows us to do so without having to examine each measurement outcome separately, and also allows us to represent the protocols in a compact pictographic manner. In other words, it tell us what we can actually *achieve* in a quantum informatic protocol without worrying about what will happen in each possible observational branch.

In Section 5, we have seen how our method of using classical structures to depict measurements and unitary corrections actually encodes all possible observation branches simultaneously, as a superposition of all possible measurement outcomes and the corresponding unitary corrections. We also saw how, given any 1-qubit unitary operation U, we can perform the operations U, $U^\dagger$, U^T and $\overline{U}$ on a qubit by using a classical structure corresponding to a particular basis (consisting of maximally entangled states) and choosing one of the basis elements to be the state of the first two qubits in the suitable operation. We have also seen how the preperation of bell-states can be shown easily in our diagrammatic representation. Many quantum informatic protocols can be simulated using just these basic operations, and thus our framework provides a method of verifying many of them (as we saw with teleportation and entanglement swapping). This framework also provides us with a way to discover new properties of quantum mechanical systems and hopefully design new protocols.

It would be interesting to relate the work here with other approaches using classical structures. For instance, in [9], there are diagrammatic equations to represent complimentary observables, and quantum gates such as the 2-qubit CNOT gate. In order to simulate more complicated quantum informatic protocols, it would be necessary to simulate 2-qubit Unitary gates and gates such as the CNOT gate which have the ability to entangle two qubits. We would also need to figure out the rules which will govern the interaction of different classical structures, and then apply them while simulating protocols using classical structures. We feel that this would be an interesting research avenue for the future.

In earlier work such as [1, 3, 5], and many others, the value of being able to represent quantum informatic protocols and properties in a diagrammatic manner has already been demonstrated. Here, we have attempted to further develop the "high-level" appoach to quantum mechanics that researchers have been striving for. We have explained a way to encode all possible observational paths without the use of trees or branching processes being shown explicitly. Although we have shown how the use of classical structures can help in interpreting the teleportation and the entanglement swapping protocol, a lot of work remains to be done. A more satisfactory explanation is required which also accounts for the scalar factors which appear in the different diagrammatic equations, and the approach needs to be expanded so that we can simulate protocols involving a larger number of qubits, and performing more complicated operations. These are all interesting avenues for future research.

In conclusion, we hope that this paper provides at least a small step towards finding a compact way to represent quantum informatic processes, to check the correctness of existing protocols, and most importantly to design and discover new protocols.

Appendix

In the protocol shown on the left in Fig(4.2), we start with the 3-qubit state given as eqn (4.4) or (4.5). We then copy (or comultiply) the first two qubits by the red classical structure to obtain a 5-qubit state which is given as eqn (4.14). This equation is written in terms of the basis $\{|b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle\}$, but with the help of equations (4.15), (4.16), (4.17) and (4.18), we can rewrite it in terms of the computational basis for two qubits: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The next step is the multiplication followed by the deletion (or the unit operation) of the last two qubits by the green classical structure.

On performing the above operations and simplifying the equations, we finally get a 3-qubits state. This state that we obtain is very long, and in order to conveniently compare it with eqn (4.21), we will write this final state in a four parts, where the last qubit is respectively in the state $a|0\rangle$, $a|1\rangle$, $b|0\rangle$ and $b|1\rangle$. Once we do this, we can easily compare the states for the first two qubits, and obtain the set of conditions which would make them equal.

So when the third qubit is $a|0\rangle$, then the joint state of the final three qubits is given by:

$$\begin{gathered}\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}\{((\overline{u_{11}}u_{11}+\overline{u_{21}}u_{21})u_{11}+(\overline{u_{12}}u_{12}+\overline{u_{22}}u_{22})u_{12}+(\overline{u_{13}}u_{13}+\overline{u_{23}}u_{23})u_{13}+\\ (\overline{u_{14}}u_{14}+\overline{u_{24}}u_{24})u_{14})|00\rangle+(\overline{u_{11}}u_{11}+\overline{u_{21}}u_{21})u_{21}+(\overline{u_{12}}u_{12}+\overline{u_{22}}u_{22})u_{22}+\\ (\overline{u_{13}}u_{13}+\overline{u_{23}}u_{23})u_{23}+(\overline{u_{14}}u_{14}+\overline{u_{24}}u_{24})u_{24})|01\rangle+(\overline{u_{11}}u_{11}+\overline{u_{21}}u_{21})u_{31}+\\ (\overline{u_{12}}u_{12}+\overline{u_{22}}u_{22})u_{32}+(\overline{u_{13}}u_{13}+\overline{u_{23}}u_{23})u_{33}+(\overline{u_{14}}u_{14}+\overline{u_{24}}u_{24})u_{34})|10\rangle+\\ (\overline{u_{11}}u_{11}+\overline{u_{21}}u_{21})u_{41}+(\overline{u_{12}}u_{12}+\overline{u_{22}}u_{22})u_{42}+(\overline{u_{13}}u_{13}+\overline{u_{23}}u_{23})u_{43}+\\ (\overline{u_{14}}u_{14}+\overline{u_{24}}u_{24})u_{44})|11\rangle\}\otimes|0\rangle\end{gathered}\tag{6.1}$$

When the state of the third qubit is $a|1\rangle$, then the joint state of the three qubits is:

$$\begin{gathered}\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}\{((\overline{u_{11}}u_{31}+\overline{u_{21}}u_{41})u_{11}+(\overline{u_{12}}u_{32}+\overline{u_{22}}u_{42})u_{12}+(\overline{u_{13}}u_{33}+\overline{u_{23}}u_{43})u_{13}+\\ (\overline{u_{14}}u_{34}+\overline{u_{24}}u_{44})u_{14})|00\rangle+(\overline{u_{11}}u_{31}+\overline{u_{21}}u_{41})u_{21}+(\overline{u_{12}}u_{32}+\overline{u_{22}}u_{42})u_{22}+\\ (\overline{u_{13}}u_{33}+\overline{u_{23}}u_{43})u_{23}+(\overline{u_{14}}u_{34}+\overline{u_{24}}u_{44})u_{24})|01\rangle+(\overline{u_{11}}u_{31}+\overline{u_{21}}u_{41})u_{31}+\\ (\overline{u_{12}}u_{32}+\overline{u_{22}}u_{42})u_{32}+(\overline{u_{13}}u_{33}+\overline{u_{23}}u_{43})u_{33}+(\overline{u_{14}}u_{34}+\overline{u_{24}}u_{44})u_{34})|10\rangle+\\ (\overline{u_{11}}u_{31}+\overline{u_{21}}u_{41})u_{41}+(\overline{u_{12}}u_{32}+\overline{u_{22}}u_{42})u_{42}+(\overline{u_{13}}u_{33}+\overline{u_{23}}u_{43})u_{43}+\\ (\overline{u_{14}}u_{34}+\overline{u_{24}}u_{44})u_{44})|11\rangle\}\otimes|0\rangle\end{gathered}\tag{6.2}$$

When the state of the third qubit is $b|0\rangle$, then the joint state of the three qubits is:

$$\begin{gathered}\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}\{((\overline{u_{31}}u_{11}+\overline{u_{41}}u_{21})u_{11}+(\overline{u_{32}}u_{12}+\overline{u_{42}}u_{22})u_{12}+(\overline{u_{33}}u_{13}+\overline{u_{43}}u_{23})u_{13}+\\ (\overline{u_{34}}u_{14}+\overline{u_{44}}u_{24})u_{14})|00\rangle+(\overline{u_{31}}u_{11}+\overline{u_{41}}u_{21})u_{21}+(\overline{u_{32}}u_{12}+\overline{u_{42}}u_{22})u_{22}+\\ (\overline{u_{33}}u_{13}+\overline{u_{43}}u_{23})u_{23}+(\overline{u_{34}}u_{14}+\overline{u_{44}}u_{24})u_{24})|01\rangle+(\overline{u_{31}}u_{11}+\overline{u_{41}}u_{21})u_{31}+\\ (\overline{u_{32}}u_{12}+\overline{u_{42}}u_{22})u_{32}+(\overline{u_{33}}u_{13}+\overline{u_{43}}u_{23})u_{33}+(\overline{u_{34}}u_{14}+\overline{u_{44}}u_{24})u_{34})|10\rangle+\\ (\overline{u_{31}}u_{11}+\overline{u_{41}}u_{21})u_{41}+(\overline{u_{32}}u_{12}+\overline{u_{42}}u_{22})u_{42}+(\overline{u_{33}}u_{13}+\overline{u_{43}}u_{23})u_{43}+\\ (\overline{u_{34}}u_{14}+\overline{u_{44}}u_{24})u_{44})|11\rangle\}\otimes|0\rangle\end{gathered}\tag{6.3}$$

And finally, when the state of the third qubit is $b|1\rangle$, then the joint state of the three qubits is:

$$\begin{gathered}\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}\{((\overline{u_{31}}u_{31}+\overline{u_{41}}u_{41})u_{11}+(\overline{u_{32}}u_{32}+\overline{u_{42}}u_{42})u_{12}+(\overline{u_{33}}u_{33}+\overline{u_{43}}u_{43})u_{13}+\\ (\overline{u_{34}}u_{34}+\overline{u_{44}}u_{44})u_{14})|00\rangle+(\overline{u_{31}}u_{31}+\overline{u_{41}}u_{41})u_{21}+(\overline{u_{32}}u_{32}+\overline{u_{42}}u_{42})u_{22}+\\ (\overline{u_{33}}u_{33}+\overline{u_{43}}u_{43})u_{23}+(\overline{u_{34}}u_{34}+\overline{u_{44}}u_{44})u_{24})|01\rangle+(\overline{u_{31}}u_{31}+\overline{u_{41}}u_{41})u_{31}+\\ (\overline{u_{32}}u_{32}+\overline{u_{42}}u_{42})u_{32}+(\overline{u_{33}}u_{33}+\overline{u_{43}}u_{43})u_{33}+(\overline{u_{34}}u_{34}+\overline{u_{44}}u_{44})u_{34})|10\rangle+\\ (\overline{u_{31}}u_{31}+\overline{u_{41}}u_{41})u_{41}+(\overline{u_{32}}u_{32}+\overline{u_{42}}u_{42})u_{42}+(\overline{u_{33}}u_{33}+\overline{u_{43}}u_{43})u_{43}+\\ (\overline{u_{34}}u_{34}+\overline{u_{44}}u_{44})u_{44})|11\rangle\}\otimes|0\rangle\end{gathered}\tag{6.4}$$

Now if we compare these 4 equations with eqn (4.21), then we can firstly cancel out the scalar multiple of $\frac{1}{2}.\frac{1}{\sqrt{2}}$ on both sides. First, it is quite easy to see that if the conditions given in equations (4.22), (4.23) and (4.24) hold, then the sum of the four equations above will be exactly the same as eqn (4.21). This proves the backward implication of Theorem 17.

Now, to prove the forward implication, we need to examine the four equations above and show that their sum will be equal to eqn (4.21) ***only if*** the conditions given as equations (4.22), (4.23) and (4.24) hold. Now, we know that the 'coefficients' for $a|1\rangle$ and $b|0\rangle$ should be 0 (i.e. when the third qubit is in state $a|1\rangle$ or $b|0\rangle$, then the first two qubits should be 0). And that the coefficients for $a|0\rangle$ and $b|1\rangle$should be as expressed in eqn (4.21). Since there are four possible states for the third qubit, and since there are four basis vectors for the first two qubits, this gives 16 equations to play with. To make this more clear, let me give the example of $a|1\rangle$. Now, in the 'coefficient' of $a|1\rangle$, i.e. the first two qubits of the expression involving $a|1\rangle$ are of the form: $(x_1|00\rangle + x_2|01\rangle + x_3|10\rangle + x_4|11\rangle)$, where each $x_i$is a complex number constructed from the entries in the matrix U. Now, in order for the coefficient of $a|1\rangle$ to be 0, each of these x_is should be 0. Thus, only the coefficient for $a|1\rangle$ provides us with 4 equations. And since there are four such relations, we end up with 16 equations. Now, if we manipulate these equations, we end up with the three relations given as equations (4.22), (4.23) and (4.24).

To give an idea of how the calculations were done to arrive at these three equations , we will derive the derive the first and third relation (i.e. eqn (4.22) and eqn (4.24)) for $i = 1$.

In other words, we will derive that: $\overline{u_{11}}u_{11} + \overline{u_{21}}u_{21} = 1$, and $\overline{u_{11}}u_{31} + \overline{u_{21}}u_{41} = 0$. The second relation (i.e. $\overline{u_{31}}u_{31} + \overline{u_{41}}u_{41} = 1$) can be derived in exactly the same manner as the first.

So, comparing eqn(4.21) with eqn (6.1) given above, and first looking only at the coefficient of $|00\rangle$ we get the following four equation:

(1) $(\overline{u}_{11}u_{11}+ \overline{u}_{21}u_{21}-1)u_{11}+ (\overline{u}_{12}u_{12}+ \overline{u}_{22}u_{22}-1)u_{12}+(\overline{u}_{13}u_{13}+ \overline{u}_{23}u_{23}-1)u_{13}+(\overline{u}_{14}u_{14}+ \overline{u}_{24}u_{24}-1)u_{14} = 0$

(2) $(\overline{u}_{11}u_{11}+ \overline{u}_{21}u_{21}-1)u_{21}+ (\overline{u}_{12}u_{12}+ \overline{u}_{22}u_{22}-1)u_{22}+(\overline{u}_{13}u_{13}+ \overline{u}_{23}u_{23}-1)u_{23}+(\overline{u}_{14}u_{14}+ \overline{u}_{24}u_{24}-1)u_{24} = 0$

(3) $(\overline{u}_{11}u_{11}+ \overline{u}_{21}u_{21}-1)u_{31}+ (\overline{u}_{12}u_{12}+ \overline{u}_{22}u_{22}-1)u_{32}+(\overline{u}_{13}u_{13}+ \overline{u}_{23}u_{23}-1)u_{33}+(\overline{u}_{14}u_{14}+ \overline{u}_{24}u_{24}-1)u_{34} = 0$

(4) $(\overline{u}_{11}u_{11}+ \overline{u}_{21}u_{21}-1)u_{41}+ (\overline{u}_{12}u_{12}+ \overline{u}_{22}u_{22}-1)u_{42}+(\overline{u}_{13}u_{13}+ \overline{u}_{23}u_{23}-1)u_{43}+(\overline{u}_{14}u_{14}+ \overline{u}_{24}u_{24}-1)u_{44} = 0$

Now, if we multiply the first equation throughout by $\overline{u_{11}}$, the second by $\overline{u_{21}}$, the third by $\overline{u_{31}}$ and the fourth by $\overline{u_{41}}$ and then add all four equations, we obtain the following:

$$
\begin{aligned}
&(\overline{u_{11}}u_{11} + \overline{u_{21}}u_{21} - 1)(\overline{u_{11}}u_{11} + \overline{u_{21}}u_{21} + \overline{u_{31}}u_{31} + \overline{u_{41}}u_{41}) + \\
&(\overline{u_{12}}u_{12} + \overline{u_{22}}u_{22} - 1)(\overline{u_{11}}u_{12} + \overline{u_{21}}u_{22} + \overline{u_{31}}u_{32} + \overline{u_{41}}u_{42}) + \\
&(\overline{u_{13}}u_{13} + \overline{u_{23}}u_{23} - 1)(\overline{u_{11}}u_{13} + \overline{u_{21}}u_{23} + \overline{u_{31}}u_{33} + \overline{u_{41}}u_{43}) + \\
&(\overline{u_{14}}u_{14} + \overline{u_{24}}u_{24} - 1)(\overline{u_{11}}u_{14} + \overline{u_{21}}u_{24} + \overline{u_{31}}u_{34} + \overline{u_{41}}u_{44}) = 0
\end{aligned} \tag{6.5}
$$

Since U (as defined in the proof of Theorem 17) is a unitary matrix, we already know that if we regard each of the rows as a vector, then the inner product of any two distinct rows is 0, and the inner product of a row with itself is 1. To put it more explicitly:

$$\frac{1}{2}(\overline{u_{1i}}u_{1j} + \overline{u_{2i}}u_{2j} + \overline{u_{3i}}u_{3j} + \overline{u_{4i}}u_{4j}) = \delta_{ij} \tag{6.6}$$

Thus, from equations (6.5) and (6.6), we obtain the relation: $\overline{u_{11}}u_{11} + \overline{u_{21}}u_{21} = 1$

In almost exactly the same manner, but this time choosing eqn (4.21) and eqn (6.2) given above, we obtain the following 4 equations:

(1) $(\overline{u_{11}}u_{31} + \overline{u_{21}}u_{41})u_{11} + (\overline{u_{12}}u_{32} + \overline{u_{22}}u_{42})u_{12} + (\overline{u_{13}}u_{33} + \overline{u_{23}}u_{43})u_{13} + (\overline{u_{14}}u_{34} + \overline{u_{24}}u_{44})u_{14} = 0$
(2) $(\overline{u_{11}}u_{31} + \overline{u_{21}}u_{41})u_{21} + (\overline{u_{12}}u_{32} + \overline{u_{22}}u_{42})u_{22} + (\overline{u_{13}}u_{33} + \overline{u_{23}}u_{43})u_{23} + (\overline{u_{14}}u_{34} + \overline{u_{24}}u_{44})u_{24} = 0$
(3) $(\overline{u_{11}}u_{31} + \overline{u_{21}}u_{41})u_{31} + (\overline{u_{12}}u_{32} + \overline{u_{22}}u_{42})u_{32} + (\overline{u_{13}}u_{33} + \overline{u_{23}}u_{43})u_{33} + (\overline{u_{14}}u_{34} + \overline{u_{24}}u_{44})u_{34} = 0$
(4) $(\overline{u_{11}}u_{31} + \overline{u_{21}}u_{41})u_{41} + (\overline{u_{12}}u_{32} + \overline{u_{22}}u_{42})u_{42} + (\overline{u_{13}}u_{33} + \overline{u_{23}}u_{43})u_{43} + (\overline{u_{14}}u_{34} + \overline{u_{24}}u_{44})u_{44} = 0$

In these equations again, if we perform the same trick as above, we obtain the following relation:

$$
\begin{aligned}
&(\overline{u_{11}}u_{31} + \overline{u_{21}}u_{41})(\overline{u_{11}}u_{11} + \overline{u_{21}}u_{21} + \overline{u_{31}}u_{31} + \overline{u_{41}}u_{41}) + \\
&(\overline{u_{12}}u_{32} + \overline{u_{22}}u_{42})(\overline{u_{11}}u_{12} + \overline{u_{21}}u_{22} + \overline{u_{31}}u_{32} + \overline{u_{41}}u_{42}) + \\
&(\overline{u_{13}}u_{33} + \overline{u_{23}}u_{43})(\overline{u_{11}}u_{13} + \overline{u_{21}}u_{23} + \overline{u_{31}}u_{33} + \overline{u_{41}}u_{43}) + \\
&(\overline{u_{14}}u_{34} + \overline{u_{24}}u_{44})(\overline{u_{11}}u_{14} + \overline{u_{21}}u_{24} + \overline{u_{31}}u_{34} + \overline{u_{41}}u_{44}) = 0
\end{aligned} \tag{6.7}
$$

Once again, using equations (6.6) and (6.7), we obtain the fact that $\overline{u_{11}}u_{31} + \overline{u_{21}}u_{41} = 0$

Thus, we have proven both the relations that we set out to prove. In the same way, we can prove the corresponding relations for $i = 2, 3, 4$ and thus complete the proof for the forward implication of Theorem 17.

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