DISTRIBUTED SELFISH LOAD BALANCING*

PETRA BERENBRINK†, TOM FRIEDETZKY‡, LESLIE ANN GOLDBERG§,
PAUL W. GOLDBERG§, ZENGJIAN HU†, AND RUSSELL MARTIN§

Abstract. Suppose that a set of m tasks are to be shared as equally as possible amongst a set of n resources. A game-theoretic mechanism to find a suitable allocation is to associate each task with a "selfish agent", and require each agent to select a resource, with the cost of a resource being the number of agents to select it. Agents would then be expected to migrate from overloaded to underloaded resources, until the allocation becomes balanced.

Recent work has studied the question of how this can take place within a distributed setting in which agents migrate selfishly without any centralized control. In this paper we discuss a natural protocol for the agents which combines the following desirable features: It can be implemented in a strongly distributed setting, uses no central control, and has good convergence properties. For $m \gg n$, the system becomes approximately balanced (an ϵ-Nash equilibrium) in expected time $O(\log\log m)$. We show using a martingale technique that the process converges to a perfectly balanced allocation in expected time $O(\log\log m + n^4)$. We also give a lower bound of $\Omega(\max\{\log\log m, n\})$ for the convergence time.

1. Introduction. Suppose that a consumer learns the price she would be charged by some domestic power supplier other than the one she is currently using. It is plausible that if the alternative price is lower than the price she is currently paying, then there is some possibility that she will switch to the new power supplier. Furthermore, she is more likely to switch if the ratio of current price to new price is large. If there is only a small saving, then it becomes unattractive to make the switch, since an influx of new business (oneself and other consumers) may drive up the price of the new power supplier and make it no longer competitive.

We study a simple mathematical model of the above natural rule, in the context of a load balancing (or task allocation) scenario that has received a lot of recent attention. We assume the presence of many individual users who may assign their tasks to chosen resources. The users are *selfish* in the sense that they attempt to optimize their own situation, i.e., try to assign their tasks to minimally loaded resources, without trying to optimize the global situation. In general, a *Nash equilibrium* among a set of selfish users is a state in which no user has the incentive to change her current decision. In our setting, this corresponds to no user having an incentive to reallocate their task to some other resource. An ϵ-Nash equilibrium is a standard notion of an approximate Nash equilibrium, and is a state in which no user can change her cost by a multiplicative factor of less than $1-\epsilon$ by changing action. Here we do not focus on the *quality* of equilibria, but rather on the (perhaps more algorithmic) question of convergence time to such a state.

We assume a strongly distributed and concurrent setting, i.e., there is no centralized control mechanism whatsoever, and all users may choose to reallocate their tasks at the same time. Thus, we do not (and cannot) use the traditional *Elementary Step System*, where the assumption is that at most one user may reallocate her task at any given stage [15, 8].

Throughout we let m denote the number of tasks (in the above discussion, customers) and n the number of resources (power suppliers). As hinted in the above discussion, we assume that typically $m \gg n$. In a single time step (or round) each task does the following. Let i be the resource currently being used by the task. Select j uniformly at random from $\{1, \ldots, n\}$ and find the load of resource j. Let X_i and X_j be the loads of resources i and j respectively. If $X_j < X_i$, migrate from i to j with a probability of $1 - X_j/X_i$; the transition from round t to round $t+1$ is given in Figure 1.1. Notice that if we had unconditional migrations, i.e., without an additional coin flip (move only with probability $1 - X_j(t)/X_i(t)$), then this may lead to an unstable system; consider for example the case $m = 2$ with initially most tasks assigned to one of the resources: the overload would oscillate between the two resources, with a load ratio tending towards 2:1. (This observation about the risk of oscillation has also been made in similar contexts in [13, 12], and we will not elaborate on it further.)

It can easily be seen that, if all tasks use the above policy, then the expected load of every resource at

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†School of Computing Science, Simon Fraser University, Canada
‡Department of Computer Science, University of Durham, U.K.
§Department of Computer Science, University of Liverpool, U.K.

For each task b **do** in parallel
  Let i_b be the current resource of task b
  Choose resource j_b uniformly at random
  Let $X_{i_b}(t)$ be the current load of resource i
  Let $X_{j_b}(t)$ be the current load of resource j
  If $X_{i_b}(t) > X_{j_b}(t)$ **then**
    Move task b from resource i_b to j_b with probability $1 - X_{j_b}(t)/X_{i_b}(t)$

FIG. 1.1. *The protocol with "neutral moves" allowed.*

the next step is m/n:

OBSERVATION 1.1. *Regardless of the load distribution at time step t, the expected load of every resource at the next step is m/n.*

Proof. To see this, assume that the loads $X_i(t)$ are arranged in descending order so that $X_j(t) \geq X_{j+1}(t)$ and note that

$$\mathbb{E}[X_i(t+1)] = X_i(t) + \sum_{\ell=1}^{i-1} \frac{1}{n} X_\ell(t)\Big(1 - \frac{X_i(t)}{X_\ell(t)}\Big) - \sum_{\ell=i+1}^{n} \frac{1}{n} X_i(t)\Big(1 - \frac{X_\ell(t)}{X_i(t)}\Big)$$

$$= X_i(t) + \frac{1}{n}\sum_{\ell=1}^{i-1}(X_\ell(t) - X_i(t)) - \frac{1}{n}\sum_{\ell=i+1}^{n}(X_i(t) - X_\ell(t))$$

$$= X_i(t) + \frac{1}{n}\sum_{\ell=1}^{n}(X_\ell(t) - X_i(t)) = \frac{1}{n}\sum_{\ell=1}^{n} X_\ell(t) = \frac{m}{n}.$$

□

This provides a compelling motivation for the policy, which is that as a result, no task has an incentive to deviate unilaterally from this policy. This implies that in the terminology of [9] it is a *Nash rerouting policy*. It is also a simple regret-minimizing policy in the sense of [2] since the average cost of resources used by an agent is no higher than the best choice of a single resource to be used repeatedly. Although the above rule is very natural and has the nice properties described above, we show that it may take a long time to converge to a perfectly balanced allocation of tasks to resources. Define a *neutral move* to be a task migration from a resource with load ℓ at time t to a resource with load $\ell - 1$ at time t (so, if no other task migrates, then the cost to the migrating task is unchanged.) We consider a modification in which neutral moves are specifically disallowed (see Figure 2.1). That seemingly-minor change is necessary to ensure fast convergence from an almost balanced state to a perfectly-balanced state. To summarize, here are the most important features of the modified protocol:

- We do not need any global information whatsoever (apart from the number of available resources); in particular, a task does not need to know the total number of tasks in the system. Also, it is strongly distributed and concurrent. If additional tasks were to enter the system, it would rapidly converge once again, with no outside intervention.
- A migrating task needs to query the load of only one other resource (thus, doing a constant amount of work in each round).
- When a task finds a resource with a significantly smaller load (that is, a load that is smaller by at least two), the migration policy is *exactly* the same as that used by the Nash rerouting policy of Figure 1.1, so the incentive is to use that probability.
- When a task finds a resource with a load that is smaller by exactly one unit, the migration policy is sufficiently close to the Nash rerouting policy that the difference in expected load is at most one, and there is little incentive to deviate.
- The protocol is simple (as well as provably efficient) enough to convince users to actually stick to it.

1.1. Related Work. We are studying a simple kind of *congestion game*. In their general form, congestion games specify a set of agents, and a set of resources, and for each agent, a set of allowed strategies, where a strategy is the selection of a subset of the resources (in this paper, a singleton subset). The cost of a resource is a non-decreasing function of the number of agents using it, and the cost for an agent is the sum of the costs of resources it uses. A classical result due to Rosenthal [27] is that pure Nash equilibria (NE) always exist for congestion games, and this is shown by exhibiting a potential function; they are a type of *potential game* [25]. The potential function also establishes that pure NE can be found via sequences of "better-response" moves, in which agents repeatedly switch to lower-cost strategies. The potential function we use later in this paper is the one of [27], modulo a linear re-scaling.

These results do not show how to find Nash equilibrium efficiently, the problem being that in the worst case, sequences of these self-improving moves may be exponentially-long. The following questions arise: when can NE can be found by any efficient algorithm, and if so, whether it can be found via an algorithm that purports to be a realistic model of agents' behavior. Regarding the first of these questions, the answer is no in the general setting (the problem is PLS-complete for general congestion games [10], see also [1, 3]). PLS-completeness (introduced in [18]) is a generally-accepted criterion for intractability of computational problems in which we seek a local optimum of a given objective function.

However, due to the basic fact of [27, 25] that pure NE are sure to result from a sufficiently long better-response sequence, many algorithms for finding them are based on such sequences. An important sub-class is the *Elementary step system* (ESS), proposed in Orda et al. [26], which consists of best-response moves (where a migrating agent switches not to any improved choice, but to one that is optimal at the time of migration). For matroid games (a class of congestion games that includes the ones we consider here), Ackermann et al. [1] show that best-response sequences must have length polynomial in the number of players, resources, and maximal rank of the matroids. In this paper we consider the special case of singleton congestion games (where players' strategies are always single resources, thus the ranks of the matroids is 1). For these games, Ieong et al. [17] give polynomial bounds for best-response and better-response sequences. Chien and Sinclair [3] study a version of the ESS in the context of approximate Nash equilibria, and show that in some cases the *ϵ-Nash dynamics* may find an ϵ-NE where finding an exact NE is PLS-complete. Mirrokni and Vetta [23] study the convergence rate of the ESS to solutions, and the quality of the approximation after limited iterations.

While best- and better-response dynamics are a plausible model of selfish behaviour, algorithms typically require that migrations be done one-by-one, and another common assumption is that best (not better) responses are always selected. This means that to some extent, agents are being assumed to be governed by a centralized algorithm that finds a NE, and raises the question of what sort of *distributed* algorithms can do so, especially if agents have limited information about the state of the system (and so may not be able to find best responses). That issue is of central importance to us in this paper. Goldberg [15] studied situations where simple better-response approaches can be realised as weakly distributed algorithms (where each agent looks for moves independently of the others, but it is assumed that moves take place consecutively, not simultaneously). In a strongly distributed setting (as we study here), where moves may occur simultaneously, we need to address the possibility that a change of strategy may increase an agent's cost. It may happen that after a best response has been identified, it is not optimal at the time it is executed. Even-Dar and Mansour [9] consider concurrent, independent rerouting decisions where tasks are allowed to migrate from overloaded to underloaded resources. Their rerouting process terminates in expected $O(\log\log m + \log n)$ rounds when the system reaches a Nash equilibrium. Note that their convergence rate as a function of the number n of resources is faster than the one we obtain in this paper. The reason is that is requires agents to have a certain amount of global knowledge. A task is required to know whether its resource is overloaded (having above-average load) and tasks on underloaded resources do not migrate at all. Our rerouting policy does not require that agents know anything other that their current resource load, and the load of a randomly-chosen alternative. Even-Dar and Mansour also present a general framework that can be used to show a logarithmic convergence rate for a wide class of rerouting strategies. Our protocol does not fall into that class, since we do not require migrations to occur only from overloaded resources. Note that our lower bound is linear in n (thus, more than logarithmic).

Distributed algorithms have been studied in the Wardrop setting (the limit of infinitely many agents), in which recent work has also extensively studied the coordination ratio [29, 28]. Fischer et al. [12] investigate convergence to Wardrop equilibria for games where agents select paths through a shared network to route their traffic. (Singleton games correspond to a network of parallel links.) Their re-routing strategies are

slightly different to ours — they assume that in each round, an agent queries a path with probability proportional to the traffic on that path. Here we assume paths (individual elements of a set of parallel links) are queried uniformly at random, so that agents can be assumed to have minimal knowledge. As in this paper, the probability of switching to a better path depends on the latency difference, and care has to be taken to avoid oscillation. Also in the Wardrop setting, Blum et al. [2] show that approximate NE is the outcome of *regret-minimizing* rerouting strategies, in which an agent's cost, averaged over time, should approximate the cost of the best individual link available to that agent.

Certain generalisations of singleton games have also been considered. These generalisations are not strictly congestion games according to the standard definition we gave above, but many ideas carry over. A widely-studied version we introduced by Koutsoupias and Papadimitriou [19], and subsequently studied extensively in different contexts (for example [21, 7, 14, 5, 29]). In this generalisation, each task may have a numerical *weight* (sometimes called traffic, or demand) and each resource has a *speed* (or capacity). The cost of using a resource is the total weight of tasks using it, divided by its speed. Even-Dar et al. [8] give a generalized version of the potential function of [27] that applies to these games, and which was subsequently used in [15]. For these games however, it seems harder to find polynomial-length best-response sequences. Feldman et al. [11] show how a sequence of steps may lead to NE, under the weaker condition that the maximal cost experienced by agents must not increase, but individual steps need not necessarily be "selfish". They also note that poorly-chosen better-response moves may lead to an exponential convergence rate. Another generalisation of singleton games is *Player-specific* cost functions [22], which allow different agents to have different cost functions for the same resource. In this setting there is no potential function and better-response dynamics may cycle, although it remains the case that pure NE always exist.

Our rerouting strategy is also related to reallocation processes for balls into bins games. The goal of a balls into bins game is to allocate m balls as evenly as possible into n bins. It is well-known that a fairly even distribution can be achieved if every ball is allowed to randomly choose d bins and then the ball is allocated to the least loaded amongst the chosen bins (see [24] for an overview). Czumaj et al. [6] consider such an allocation where each ball initially chooses two bins. They show that, in a polynomial number of steps, the reallocation process ends up in a state with maximum load at most $\lceil m/n \rceil + 1$. Sanders et al. [30] show that a maximum load of $\lceil m/n \rceil + 1$ is optimal if every ball is restricted to two random choices.

In conclusion, this paper sits at one end of a spectrum in which we study a very simple load-balancing game, but we seek solutions in a very adverse setting in which agents have, at any point in time, a minimal amount of information about the state of their environment, and carry out actions simultaneously in a strongly distributed sense.

1.2. Overview of our results. Section 4 deals with upper bounds on convergence time. The main result, Theorem 4.1, is that the protocol of Figure 2.1 converges to a Nash equilibrium within expected time $O(\log\log m + n^4)$.

The proof of Theorem 4.1 shows that the system becomes *approximately* balanced very rapidly. Specifically, Corollary 4.11 shows that if $n \leq m^{1/3}$, then for all ϵ, either version of the distributed protocol (with or without neutral moves allowed) attains an ϵ-Nash equilibrium (where all load ratios are within $[1-\epsilon, 1+\epsilon]$; we use ϵ to denote a multiplicative factor as in [3]) in expected $O(\log\log m)$ rounds. The rest of Section 4 analyses the protocol of Figure 2.1. It is shown that within an additional $O(n^4)$ rounds the system becomes optimally balanced.

In Section 5, we provide two lower bound results. The first one, Theorem 5.1, shows that the first protocol (of Figure 1.1, including moves that do not necessarily yield a strict improvement for an individual task but allow for simply "neutral" moves as well, results in exponential (in n) expected convergence time. Finally, in Theorem 5.2 we provide a general lower bound (regardless of which of the two protocols is being used) on the expected convergence time of $\Omega(\log\log m)$. This lower bound matches the upper bound as a function of m.

2. Notation. There are m tasks and n resources. An assignment of tasks to resources is represented as a vector $(x_1, \ldots, x_n)$ in which x_i denotes the number of tasks that are assigned to resource i. In the remainder of this paper, $[n]$ denotes $\{1, \ldots, n\}$. The assignment is a Nash equilibrium if for all $i \in [n]$ and $j \in [n]$, $|x_i - x_j| \leq 1$. We study a distributed process for constructing a Nash equilibrium. The states of the process, $X(0), X(1), \ldots$, are assignments. The transition from state $X(t) = (X_1(t), \ldots, X_n(t))$ to state $X(t+1)$ is given by the greedy distributed protocol in Figure 2.1.

For each task b **do** in parallel
  Let i_b be the current resource of task b
  Choose resource j_b uniformly at random
  Let $X_{i_b}(t)$ be the current load of resource i
  Let $X_{j_b}(t)$ be the current load of resource j
  If $X_{i_b}(t) > X_{j_b}(t) + 1$ **then**
    Move task b from resource i_b to j_b with probability $1 - X_{j_b}(t)/X_{i_b}(t)$

FIG. 2.1. *The modified protocol, with "neutral moves" disallowed.*

Note that if $X(t)$ is a Nash equilibrium, then $X(t+1) = X(t)$ so the assignment stops changing. Here is a formal description of the transition from a state $X(t) = x$. Independently, for every $i \in [n]$, let $(Y_{i,1}(x), \ldots, Y_{i,n}(x))$ be a random variable drawn from a multinomial distribution with the constraint $\sum_{j=1}^{n} Y_{i,j}(x) = x_i$. ($Y_{ij}$ represents the number of migrations from i to j in a round.) The corresponding probabilities $(p_{i,1}(x), \ldots, p_{i,n}(x))$ are given by

$$p_{i,j}(x) = \begin{cases} \frac{1}{n}\left(1 - \frac{x_j}{x_i}\right) & \text{if } x_i > x_j + 1, \\ 0 & \text{if } i \neq j \text{ but } x_i \leq x_j + 1, \\ 1 - \sum_{j \neq i} p_{i,j}(x) & \text{if } i = j. \end{cases}$$

Then $X_i(t+1) = \sum_{\ell=1}^{n} Y_{\ell,i}(x)$.

For any assignment $x = (x_1, \ldots, x_n)$, let $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Similar to [8, 15, 9] we define a potential function $\Phi(x) = \sum_{i=1}^{n} (x_i - \overline{x})^2$. Note that $\Phi(x) = \sum_{i=1}^{n} x_i^2 - n\overline{x}^2$.

3. Technical tools. In this section we present a version of the Chernoff bounds, which can be found, for example, in [16]. Let $n \geq 1$ and let $p_i \in [0,1]$ for $i = 1, \ldots, n$. Let $X_1, X_2, \ldots, X_n$ be independent Bernoulli random variables with $P[X_i = 1] = p_i$ for $i = 1, \ldots, n$ and let $X = X_1 + \cdots + X_n$. Then we have $\mathbb{E}[X] = \sum_{i=1}^{n} p_i$ and

$$\Pr(X \geq (1+\epsilon) \cdot \mathbb{E}[X]) \leq \exp\left(-\frac{\epsilon^2 \cdot \mathbb{E}[X]}{2}\right) \tag{3.1}$$

for $\epsilon \geq 0$, and

$$\Pr(X \leq (1-\epsilon) \cdot \mathbb{E}[X]) \leq \exp\left(-\frac{\epsilon^2 \cdot \mathbb{E}[X]}{3}\right) \tag{3.2}$$

for $0 \leq \epsilon \leq 1$.

4. Upper bound on convergence time. Our main result is the following.

THEOREM 4.1. *Let T be the number of rounds taken by the protocol of Figure 2.1 to reach a Nash equilibrium for the first time. Then $\mathbb{E}[T] = O(\log\log m + n^4)$.*

The proof of this theorem proceeds as follows. First (Lemma 4.6) we give an upper bound on $\mathbb{E}[\Phi(X(t))]$ which implies (Corollary 4.10) that there is a $\tau = O(\log\log m)$ such that, with high probability, $\Phi(X(\tau)) = O(n)$. We also show (Observation 4.5 and Corollary 4.14) that $\Phi(X(t))$ is a super-martingale and (Lemma 4.15) that it has enough variance. Using these facts, we obtain the upper bound on the convergence time.

Definition: Let $S_i(x) = \{j \mid x_j < x_i - 1\}$. $S_i(x)$ is the set of resources that are significantly smaller than resource i in state x (in the sense that their loads are at least two tasks smaller than the load of resource i). Similarly, let $L_i(x) = \{j \mid x_j > x_i + 1\}$ and let $d_i(x) = \frac{1}{n}\sum_{j:|x_i - x_j| \leq 1}(x_i - x_j)$.

OBSERVATION 4.2. $\mathbb{E}[X_i(t+1) \mid X(t) = x] = \overline{x} + d_i(x)$.

Proof.

$$\begin{aligned}
\mathbb{E}[X_i(t+1) \mid X(t)=x] &= \sum_{\ell=1}^{n} \mathbb{E}[Y_{\ell,i}(x)] = \sum_{\ell=1}^{n} x_\ell p_{\ell,i}(x) \\
&= \sum_{\ell \in L_i(x)} x_\ell \frac{1}{n}\left(1-\frac{x_i}{x_\ell}\right) + x_i\left(1-\sum_{j\in S_i(x)}\frac{1}{n}\left(1-\frac{x_j}{x_i}\right)\right) \\
&= x_i + \frac{1}{n}\left(\sum_{\ell\in L_i(x)}(x_\ell - x_i) - \sum_{j\in S_i(x)}(x_i - x_j)\right) = x_i + \frac{1}{n}\sum_{\ell\in L_i(x)\cup S_i(x)}(x_\ell - x_i) \\
&= x_i + \frac{1}{n}\sum_{\ell=1}^{n}(x_\ell - x_i) - \frac{1}{n}\sum_{\ell\notin L_i(x)\cup S_i(x)}(x_\ell - x_i) \\
&= \overline{x} - \frac{1}{n}\sum_{\ell\notin L_i(x)\cup S_i(x)}(x_\ell - x_i) \\
&= \overline{x} + \frac{1}{n}\sum_{\ell\notin L_i(x)\cup S_i(x)}(x_i - x_\ell).
\end{aligned}$$

□

Observation 4.3. $\sum_{i=1}^{n}\left(\mathbb{E}[X_i(t+1) \mid X(t)=x]\right)^2 = n\overline{x}^2 + \sum_{i=1}^{n} d_i(x)^2$.

Proof. Using Observation 4.2,

$$\sum_{i=1}^{n}(\mathbb{E}[X_i(t+1) \mid X(t)=x])^2 = \sum_{i=1}^{n}(\overline{x}+d_i(x))^2 = n\overline{x}^2 + 2\overline{x}\sum_{i=1}^{n} d_i(x) + \sum_{i=1}^{n} d_i(x)^2,$$

and the second term is zero since $d_i(x) = \mathbb{E}[X_i(t+1) \mid X(t)=x] - \overline{x}$. □

Observation 4.4. $\mathrm{var}[X_i(t+1) \mid X(t)=x] \le \frac{1}{n}\sum_{\ell\in L_i(x)}(x_\ell - x_i) + \frac{1}{n}\sum_{j\in S_i(x)}(x_i - x_j)$.

Proof.

$$\begin{aligned}
\mathrm{var}(X_i(t+1) \mid X(t)=x) &= \sum_{\ell=1}^{n}\mathrm{var}(Y_{\ell,i}(x)) = \sum_{\ell=1}^{n} x_\ell p_{\ell,i}(x)(1-p_{\ell,i}(x)) \\
&= \sum_{\ell\in L_i(x)} x_\ell \frac{1}{n}\left(1-\frac{x_i}{x_\ell}\right)(1-p_{\ell,i}(x)) + x_i p_{i,i}(x)\left(\sum_{j\in S_i(x)}\frac{1}{n}\left(1-\frac{x_j}{x_i}\right)\right) \\
&= \frac{1}{n}\sum_{\ell\in L_i(x)}(x_\ell - x_i)(1-p_{\ell,i}(x)) + p_{i,i}(x)\frac{1}{n}\sum_{j\in S_i(x)}(x_i - x_j) \\
&\le \frac{1}{n}\sum_{\ell\in L_i(x)}(x_\ell - x_i) + \frac{1}{n}\sum_{j\in S_i(x)}(x_i - x_j).
\end{aligned}$$

□

Definition: For any assignment x, let $s_i(x) = |\{j \mid x_j = x_i - 1\}|$ and $l_i(x) = |\{j \mid x_j = x_i + 1\}|$. Let $u_1(x) = \sum_{i=1}^{n}\sum_{j\in[n]:|x_i - x_j|>1}|x_i - x_j|$ and $u_2(x) = \sum_{i=1}^{n}\left(s_i(x) - l_i(x)\right)^2$. Let $u(x) = u_1(x)/n + u_2(x)/n^2$. We will show that $u(x)$ is on upper bound on the expected potential after one step, starting from state x. The quantity $u_1(x)$ corresponds to the contribution arising from the sum of the variances of the individual loads and $u_2(x)$ corresponds to the rest.

Observation 4.5. $\mathbb{E}[\Phi(X(t+1)) \mid X(t)=x] \le u(x)$.

Proof.

$$\mathbb{E}[\Phi(X(t+1)) \mid X(t)=x] + n\overline{x}^2 = \sum_{i=1}^{n} \mathbb{E}[X_i(t+1)^2 \mid X(t)=x]$$
$$= \sum_{i=1}^{n} (\mathbb{E}[X_i(t+1) \mid X(t)=x])^2 + \sum_{i=1}^{n} \mathrm{var}(X_i(t+1) \mid X(t)=x).$$

Using Observations 4.3 and 4.4, this is at most $n\overline{x}^2 + \sum_{i=1}^{n} d_i(x)^2 + u_1(x)/n$. But

$$d_i(x) = \frac{1}{n} \sum_{j:|x_i - x_j| \le 1} (x_i - x_j) = \frac{1}{n}(s_i(x) - \ell_i(x)),$$

so the result follows. ∎

LEMMA 4.6. $\mathbb{E}[\Phi(X(t+1)) \mid X(t)=x] \le n + 2n^{1/2}\Phi(x)^{1/2}$.

Proof. In the proof of Observation 4.5, we established that $\mathbb{E}[\Phi(X(t+1)) \mid X(t)=x] \le \sum_{i=1}^{n} d_i(x)^2 + u_1(x)/n$. Upper-bounding $u_1(x)$ and using $d_i(x) \le 1$, we have

$$\mathbb{E}[\Phi(X(t+1)) \mid X(t)=x] \le n + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} |x_i - x_j|,$$

and since $|x_i - x_j| \le |x_i - \overline{x}| + |x_j - \overline{x}|$, this is at most $n + 2\sum_{i=1}^{n} |x_i - \overline{x}|$. By Cauchy-Schwarz, $\left(\sum_i |x_i - \overline{x}| \cdot 1\right)^2 \le \sum_i |x_i - \overline{x}|^2 \sum_i 1$ so

$$\mathbb{E}[\Phi(X(t+1)) \mid X(t)=x] \le n + 2\left(n\sum_{i=1}^{n} |x_i - \overline{x}|^2\right)^{1/2}.$$

∎

COROLLARY 4.7. $\mathbb{E}[\Phi(X(t+1))] \le n + 2n^{1/2}(\mathbb{E}[\Phi(X(t))])^{1/2}$.

Proof. Using Lemma 4.6, $\mathbb{E}[\Phi(X(t+1))] \le n + 2n^{1/2}\mathbb{E}[f^{1/2}]$ where f denotes the random variable $\Phi(X(t))$. By Jensen's inequality $\mathbb{E}[f^{1/2}] \le (\mathbb{E}[f])^{1/2}$ since the square-root function is concave, so we get $\mathbb{E}[\Phi(X(t+1))] \le n + 2n^{1/2}(\mathbb{E}[f])^{1/2}$. ∎

LEMMA 4.8. *Either there is a $t' < t$ s.t. $\mathbb{E}[\Phi(X(t'))] \le 18n$ or $\mathbb{E}[\Phi(X(t))] \le 9^{1-2^{-t}} n^{1-2^{-t}} \Phi(X(0))^{2^{-t}}$.*

Proof. The proof is by induction on t. The base case is $t = 0$. For the inductive step, note that $1 - 2^{-t} = \sum_{k=1}^{t} 2^{-k}$. Suppose that for all $t' < t$, $\mathbb{E}[\Phi(X(t'))] > 18n$ (otherwise we are finished). Then by Corollary 4.7,

$$\mathbb{E}[\Phi(X(t))] = n + 2n^{1/2}(\mathbb{E}[\Phi(X(t-1))])^{1/2} \le 3n^{1/2}(\mathbb{E}[\Phi(X(t-1))])^{1/2}.$$

Applying the inductive hypothesis,

$$\mathbb{E}[\Phi(X(t))] \le 3n^{1/2}\left(3^{2(1-2^{-(t-1)})} n^{1-2^{-(t-1)}} \Phi(X(0))^{2^{-(t-1)}}\right)^{1/2}.$$

∎

COROLLARY 4.9. *There is a $\tau \le \lceil \lg\lg \Phi(X(0)) \rceil$ such that $\mathbb{E}[\Phi(X(\tau))] \le 18n$.*

Proof. Take $t = \lceil \lg\lg \Phi(X(0)) \rceil$. Either there is a $\tau < t$ with $\mathbb{E}[\Phi(X(\tau))] \le 18n$ or, by the lemma,

$$\mathbb{E}[\Phi(X(t))] \le 9n\Phi(X(0))^{2^{-t}} \le 18n.$$

∎

COROLLARY 4.10. *There is a $\tau \le \lceil \lg\lg \Phi(X(0)) \rceil$ such that $\Pr(\Phi(X(\tau)) > 720n) \le 1/40$.*

Proof. Consider the (non-negative) random variable $Y = \Phi(X(\tau))$ where τ is the quantity from Corollary 4.9. Markov's inequality says that for any $a > 0$, $\Pr(Y \ge a) \le \mathbb{E}[Y]/a$. Now use Corollary 4.9 with $a = 720n$. ∎

Corollary 4.11. *For all $\epsilon > 0$, provided that $n < m^{1/3}$, the expected time to reach ϵ-Nash equilibrium is $O(\log\log m)$.*

Proof. Since the bound is asymptotic as a function of m for fixed ϵ, we can assume without loss of generality that $m > (60/\epsilon)^2$ and that $\epsilon m/(2n)$ is an integer. We show that for any starting assignment $X(0)$, there exists $\tau \leq \log\log(m^2)$ such that $\Pr(X(\tau) \text{ is } \epsilon\text{−Nash}) > \frac{39}{40}$. This implies the statement of the result since the number of blocks of τ steps needed to reach an ϵ-Nash equilibrium is at most

$$1 + \left(\frac{1}{40}\right) + \left(\frac{1}{40}\right)^2 + \cdots = \frac{40}{39} < 2.$$

Suppose assignment x is not ϵ-Nash. If $X(t) = x$ there exist resources i, j with $X_i(t) - X_j(t) > \epsilon m/n$. We use the following notation. Let $\Delta = \epsilon m/(2n)$. Let $\beta = X_i(t) - X_j(t) - 2\Delta$. Note $\beta > 0$. If $X(t+1)$ is obtained from $X(t)$ by transferring Δ tasks from i to j, then

$$\begin{aligned}
\Phi(X(t)) - \Phi(X(t+1)) &= X_i(t)^2 + X_j(t)^2 - X_i(t+1)^2 - X_j(t+1)^2 \\
&= (2\Delta + \beta + X_j(t))^2 + X_j(t)^2 - (\Delta + \beta + X_j(t))^2 - (\Delta + X_j(t))^2 \\
&= 2\Delta(\Delta + \beta + X_j(t)) + \Delta^2 - \left(2\Delta X_j(t) + \Delta^2\right) \\
&= 2\Delta(\Delta + \beta) \geq \Delta^2 = (\epsilon m/2n)^2.
\end{aligned}$$

It follows that $\Phi(X(t)) \geq (\epsilon m/2n)^2$.

From Corollary 4.10, $\Pr(\Phi(X(\tau)) < 720n) > \frac{39}{40}$, for $\tau = \log\log(\Phi(0)) = O(\log\log m)$.

An assignment $X(\tau)$ with $\Phi(X(\tau)) \leq 720n$ must be ϵ-Nash if $(\epsilon m/2n)^2 > 720n$. Note that $m > n^3$ and $m > (60/\epsilon)^2$. Hence, from $\epsilon^2(60/\epsilon)^2 n^3 > 4.720.n^3$, we can deduce $\epsilon^2 m^2 > 4.720.n^3$, hence $(\epsilon m/2n)^2 > 720n$. ∎

Corollary 4.10 tells us that $\Phi(X(\tau))$ is likely to be $O(n)$. We want to show that $\Phi(X(t))$ quickly gets even smaller (all the way to a Nash equilibrium) and to this end, we show that $\Phi(X(t))$ is a super-martingale. By Observation 4.5, it suffices to show $u(x) \leq \Phi(x)$, and we proceed with this. In the following, we shall consider the cases $|x_i - \overline{x}| < 2.5$ for all $i \in [n]$ (Lemma 4.12) and $\exists i \in [n] : |x_i - \overline{x}| \geq 2.5$ (Lemma 4.13) separately.

Lemma 4.12. *Suppose that assignment $x = (x_1, \ldots, x_n)$ satisfies $|x_i - \overline{x}| < 2.5$ for all $i \in [n]$. Then $u(x) \leq \Phi(x)$.*

Proof. For all $i \in [n]$ and $j \in [n]$ we have $|x_i - x_j| \leq |x_i - \overline{x}| + |x_j - \overline{x}| < 5$. Let $z = \min_i x_i$ so every $x_i \in \{z, \ldots, z+4\}$. Let $n_i = |\{j \mid x_j = z + i\}|$. Then

$$n^2\Phi(x) = n^2 \sum_{i=1}^{n} x_i^2 - n\left(\sum_{i=1}^{n} x_i\right)^2 = n^2\left(\sum_{j=0}^{4} n_j(z+j)^2\right) - \left(\sum_{j=0}^{4} n_j(z+j)\right)^2.$$

Also, $n^2 u(x) = n u_1(x) + u_2(x)$, where

$$u_1(x) = n_0(2n_2 + 3n_3 + 4n_4) + n_1(2n_3 + 3n_4) + n_2(2n_0 + 2n_4) + n_3(3n_0 + 2n_1) + n_4(4n_0 + 3n_1 + 2n_2)$$

and

$$u_2(x) = n_0 n_1^2 + n_1(n_0 - n_2)^2 + n_2(n_1 - n_3)^2 + n_3(n_2 - n_4)^2 + n_4 n_3^2.$$

Plugging in these expressions and simplifying, we get

$$\begin{aligned}
n^2\Phi(x) - n^2 u(x) =& 4n_0n_1n_2 + 3n_0^2n_3 + 4n_0n_1n_3 + 4n_0n_2n_3 + 4n_1n_2n_3 + 3n_0n_3^2 + 8n_0^2n_4 + 12n_0n_1n_4 \\
&+ 3n_1^2n_4 + 8n_0n_2n_4 + 4n_1n_2n_4 + 12n_0n_3n_4 + 4n_1n_3n_4 + 4n_2n_3n_4 + 8n_0n_4^2 + 3n_1n_4^2,
\end{aligned}$$

which is clearly non-negative since all coefficients are positive. ∎

Lemma 4.13. *Suppose that assignment $x = (x_1, \ldots, x_n)$ satisfies $|x_n - \overline{x}| \geq 2.5$ and, for all $i \in [n]$, $|x_i - \overline{x}| \leq |x_n - \overline{x}|$. Let $w = (w_1, \ldots, w_{n-1})$ be the assignment with $w_i = x_i$ for $i \in [n-1]$. Then $\Phi(x) - u(x) \geq \Phi(w) - u(w)$, that is, the lower bound on the potential drop for x is at least as big as that for w.*

Proof. Let $k = |x_n - \overline{x}|$. We will show

(1) $\Phi(x) - \Phi(w) \geq k^2$, and
(2) $u(x) - u(w) \leq 2k + 1$.
Then

$$\Phi(x) - u(x) - (\Phi(w) - u(w)) \geq k^2 - (2k+1),$$

which is non-negative since $k \geq 2.5 \geq 1 + \sqrt{2}$.
First, we prove (1). Let $f(z) = \sum_{i=1}^{n-1} (x_i - z)^2$. Note that the derivative of $f(z)$ is

$$f'(z) = 2(n-1)z - 2\sum_{i=1}^{n-1} x_i = 2(n-1)z - 2(n-1)\overline{w}.$$

Furthermore the second derivative is $f''(z) = 2(n-1) \geq 0$. Thus, $f(z)$ is minimized at $z = \overline{w}$. Now note that

$$\Phi(x) - \Phi(w) = k^2 + \sum_{i=1}^{n-1} (x_i - \overline{x})^2 - \sum_{i=1}^{n-1} (x_i - \overline{w})^2 \geq k^2.$$

Now we finish the proof by proving (2). Assume first that $x_n = \overline{x} + k$. Then

$$u_1(x) - u_1(w) = 2 \sum_{i \in [n]: |x_i - x_n| > 1} |x_i - x_n| \leq 2\sum_{i=1}^{n} |x_i - x_n| = 2\sum_{i=1}^{n} (x_n - x_i) = 2nk.$$

Let $z_j = |\{\ell \mid x_\ell = j\}|$. Clearly $z_j = 0$ for $j > x_n$. Let $\xi = \lceil x_n - 2k \rceil$. For $\ell \in [n]$ we have $x_\ell \geq \overline{x} - k = x_n - 2k$ so $z_j = 0$ for $j < \xi$. Now $u_2(x) = \sum_{j=\xi}^{x_n} z_j (z_{j-1} - z_{j+1})^2$. The representation of w in terms of z_js is the same as the representation of x except that z_{x_n} is reduced by one. Therefore,

$$\begin{aligned} u_2(x) - u_2(w) &= z_{x_n - 1}\left((z_{x_n-2} - z_{x_n})^2 - (z_{x_n-2} - z_{x_n} + 1)^2\right) + +(z_{x_n-1} - z_{x_n+1})^2 \\ &= z_{x_n-1}(-2z_{x_n-2} + 2z_{x_n} + z_{x_n-1} - 1) \leq z_{x_n-1}(2z_{x_n} + z_{x_n-1}). \end{aligned}$$

But since $z_{x_n} \leq n - z_{x_n-1}$, the upper bound on the right-hand side is at most

$$z_{x_n-1}(2n - 2z_{x_n-1} + z_{x_n-1}) = 2z_{x_n-1}(n - z_{x_n-1}/2),$$

which is at most n^2 since the right-hand side is maximized at $z_{x_n-1} = n$. To finish the proof of (2), use the definition of u to deduce that

$$u(x) - u(w) \leq \frac{u_1(x) - u_1(w)}{n} + \frac{u_2(x) - u_2(w)}{n^2}.$$

The proof of (2) when $x_n = \overline{x} - k$ is similar. ◻

Corollary 4.14. *For any assignment $x = (x_1, \ldots, x_n)$, $\Phi(x) - u(x) \geq 0$.*

Proof. The proof is by induction on n. The base case, $n = 1$, follows from Lemma 4.12. Suppose $n > 1$. Neither $\Phi(x)$ nor $u(x)$ depends upon the order of the components in x, so assume without loss of generality that $|x_i - \overline{x}| \leq |x_n - \overline{x}|$ for all i. If $|x_n - \overline{x}| < 2.5$ then apply Lemma 4.12. Otherwise, use Lemma 4.13 to find an assignment $w = (w_1, \ldots, w_{n-1})$ such that $\Phi(x) - u(x) \geq \Phi(w) - u(w)$. By the inductive hypothesis, $\Phi(w) - u(w) \geq 0$. ◻

Together, Observation 4.5 and Corollary 4.14 tell us that $\mathbb{E}[\Phi(X(t+1)) \mid X(t) = x] \leq \Phi(x)$. The next lemma will be used to give a lower bound on the variance of the process. Let $V = 0.4n^{-2}$.

Lemma 4.15. *Suppose that $X(t) = x$ and that x is not a Nash equilibrium. Then*

$$\Pr(\Phi(X(t+1)) \neq \Phi(x) \mid X(t) = x) \geq V.$$

Proof. Choose s and ℓ such that for all $i \in [n]$, $x_s \leq x_i \leq x_\ell$. Since x is not a Nash equilibrium, $x_\ell > x_s + 1$. Assuming $X(t) = x$, consider the following experiment for choosing $X(t+1)$.

The intuition behind the experiment is as follows. We wish to show that the transition from $X(t)$ to $X(t+1)$ has some variance in the sense that $\Phi(X(t+1))$ is sufficiently likely to differ from $\Phi(X(t))$. To do this, we single out a "least loaded" resource s and a "most loaded" resource ℓ as above. In the transition from $X(t)$ to $X(t+1)$ we make transitions from resources other than resource ℓ in the usual way. We pay special attention to transitions from resource ℓ (and particular attention to transitions from resource ℓ which could either go to resource s or stay at resource ℓ). It helps to be very precise about how the random decisions involving tasks that start at resource ℓ are made. In particular, for each task b that starts at resource ℓ, we first make a decision about whether b would *accept* the transition from resource ℓ to resource s *if b happened to choose resource s*. Then we make the decision about which resource task b should choose. Of course, we can't cheat and we have to sample from the original required distribution. Here are the details.

Independently, for every $i \neq \ell$, choose $(Y_{i,1}(x), \ldots, Y_{i,n}(x))$ from the multinomial distribution described in Section 2. (In the informal description above, this corresponds to making transitions from resources other than resource ℓ in the usual way.) Now, for every task $b \in x_\ell$, let $z_b = 1$ with probability $1 - x_s/x_\ell$ and $z_b = 0$ otherwise. (In the informal description above, this corresponds to deciding whether b would *accept* the transition to s if resource s were (later) chosen.) Let x_ℓ^+ be the number of tasks b with $z_b = 1$ and let x_ℓ^- be the number of tasks b with $z_b = 0$. Choose $(Y_{\ell,1}^+(x), \ldots, Y_{\ell,n}^+(x))$ from a multinomial distribution with the constraint $\sum_{j=1}^n Y_{\ell,j}^+(x) = x_\ell^+$ and probabilities given by

$$p_{\ell,j}^+(x) = \begin{cases} \frac{1}{n} & \text{if } j = s, \\ \frac{1}{n}\left(1 - \frac{x_j}{x_\ell}\right) & \text{if } j \neq s \text{ and } x_\ell > x_j + 1, \\ 0 & \text{if } \ell \neq j \text{ but } x_\ell \leq x_j + 1, \\ 1 - \sum_{j \neq \ell} p_{\ell,j}(x) & \text{if } \ell = j. \end{cases}$$

Similarly, choose $(Y_{\ell,1}^-(x), \ldots, Y_{\ell,n}^-(x))$ from a multinomial distribution with the constraint $\sum_{j=1}^n Y_{\ell,j}^-(x) = x_\ell^-$ and probabilities given by

$$p_{\ell,j}^-(x) = \begin{cases} 0 & \text{if } j = s, \\ \frac{1}{n}\left(1 - \frac{x_j}{x_\ell}\right) & \text{if } j \neq s \text{ and } x_\ell > x_j + 1, \\ 0 & \text{if } \ell \neq j \text{ but } x_\ell \leq x_j + 1, \\ 1 - \sum_{j \neq \ell} p_{\ell,j}(x) & \text{if } \ell = j. \end{cases}$$

For all j, let $Y_{\ell,j}(x) = Y_{\ell,j}^+(x) + Y_{\ell,j}^-(x)$. Informally, the $p_{\ell,j}^+$ transition probabilities are set up so that packets which decided that they would accept a transition to s behave appropriately and the $p_{\ell,j}^-$ transition probabilities are set up so that packets which decided that they would *not* accept a transition to s behave appropriately. By combining the probabilities, we see that $X(t+1)$ is chosen from the correct distribution in this way.

Now, consider the transition from x to $X(t+1)$. Condition on the choice for $(Y_{i,1}(x), \ldots, Y_{i,n}(x))$ for all $i \neq \ell$. Suppose $x_\ell^+ > 2$. Condition on the choice for $(Y_{\ell,1}^-(x), \ldots, Y_{\ell,n}^-(x))$. Flip a coin for each of the first $x_b^+ - 2$ tasks with $z_b = 1$ to determine which of $Y_{\ell,1}^+(x), \ldots, Y_{\ell,n}^+(x)$ the task contributes to. Condition on these choices. Consider the following options:

(1) Let x_1 be the resulting value of $X(t+1)$ when we add both of the last two tasks to $Y_{\ell,\ell}^+(x)$.
(2) Let x_2 be the resulting value of $X(t+1)$ when we add one of the last two tasks to $Y_{\ell,\ell}^+(x)$ and the other to $Y_{\ell,s}^+(x)$.
(3) Let x_3 be the resulting value of $X(t+1)$ when we add both of the last two tasks to $Y_{s,s}^+(x)$.

Note that, given the conditioning, each of these choices occurs with probability at least n^{-2}. Also, $\Phi(x_1)$, $\Phi(x_2)$ and $\Phi(x_3)$ are not all the same. Thus, $\Pr(\Phi(X(t+1) \neq \Phi(x) \mid X(t) = x, x_\ell^+ > 2) \geq n^{-2}$. Also,

$$\Pr(x_\ell^+ > 2) = 1 - \left(\frac{x_s}{x_\ell}\right)^{x_\ell} - x_\ell\left(1 - \frac{x_s}{x_\ell}\right)\left(\frac{x_s}{x_\ell}\right)^{x_\ell - 1}.$$

Since the derivative with respect to x_s is negative, this is minimized by taking x_s as large as possible, namely $x_\ell - 2$, so $\Pr(x_\ell^+ > 2) \geq 1 - 7e^{-2} \geq 0.4$, and the result follows. □

In order to finish our proof of convergence, we need the following observation about $\Phi(x)$.

Observation 4.16. *For any assignment x, $\Phi(x) \le m^2$. Let $r = m \bmod n$. Then $\Phi(x) \ge r(1 - r/n)$, with equality if and only if x is a Nash equilibrium.*

Proof. Suppose that in assignment x there are resources i and j such that $x_i - x_j \ge 2$. Let x' be the assignment constructed from x by transferring a task from resource i to resource j. Then

$$\begin{aligned}\Phi(x) - \Phi(x') &= x_i^2 - {x_i'}^2 + x_j^2 - {x_j'}^2 = x_i^2 - (x_i^2 - 2x_i + 1) + x_j^2 - (x_j^2 + 2x_j + 1)\\ &= 2x_i - 2x_j - 2 = 2(x_i - x_j) - 2 > 0.\end{aligned}$$

Now suppose that, in some assignment x', resources i and j satisfy $x_i' \ge x_j' > 0$. Let x be the assignment constructed from x' by transferring a task from resource j to resource i. Since $(x_i' + 1) - (x_j' - 1) \ge 2$, the above argument gives $\Phi(x) > \Phi(x')$. We conclude that an assignment x with maximum $\Phi(x)$ must have all of the tasks in the same resource, with $\Phi(x) = m^2$.

Furthermore, an assignment x with minimum $\Phi(x)$ must have $|x_i - x_j| \le 1$ for all i, j. In this case there must be r resources with loads of $q + 1$ and $n - r$ resources with loads of q, where $m = qn + r$. So

$$\Phi(x) = r(q + 1 - \bar{x})^2 + (n - r)(q - \bar{x})^2 = r\Big(1 - \frac{r}{n}\Big)^2 + (n - r)\Big(\frac{r}{n}\Big)^2 = r\left(1 - \frac{r}{n}\right).$$

Note that x is a Nash assignment if and only if $|x_i - x_j| \le 1$ for all i and j. ∎

Combining Observation 4.16 and Corollary 4.10 we find that there is a $\tau \le \lceil \lg \lg m^2 \rceil$ such that $\Pr(\Phi(X(\tau)) > 720n) \le 1/40$. Let $B = 7200n + \left\lceil \frac{m^2}{n} \right\rceil - \frac{m^2}{n}$. Let $t' = \tau + \lceil 10B^2/V \rceil$.

Lemma 4.17. *Given any starting state $X(0) = x$, the probability that $X(t')$ is a Nash equilibrium is at least $3/4$.*

Proof. The proof is based on a standard martingale argument, see [20]. Suppose that $\Phi(X(\tau)) \le 720n$. Let $W_t = \Phi(X(t+\tau)) - r(1 - r/n)$ and let $D_t = \min(W_t, B)$. Note that $D_0 \le 720n$. Together, Observation 4.5 and Corollary 4.14 tell us that W_t is a supermartingale. This implies that D_t is also a supermartingale since

$$\mathbb{E}[D_{t+1} \mid D_t = x < B] \le \mathbb{E}[W_{t+1} \mid W_t = x < B] \le W_t = D_t,$$

and

$$\mathbb{E}[D_{t+1} \mid D_t = B] \le B = D_t.$$

Together, Lemma 4.15 and Observation 4.16 tell us that if $x > 0$, $\Pr(W_{t+1} \ne W_t \mid W_t = x) \ge V$. Thus, if $0 < x < B$,

$$\begin{aligned}\Pr(D_{t+1} \ne D_t \mid D_t = x) &= \Pr(\min(W_{t+1}, B) \ne W_t \mid W_t = x)\\ &\ge \Pr(W_{t+1} \ne W_t \wedge B \ne W_t \mid W_t = x) = \Pr(W_{t+1} \ne W_t \mid W_t = x) \ge V.\end{aligned}$$

Since $D_{t+1} - D_t$ is an integer, $\mathbb{E}[(D_{t+1} - D_t)^2 \mid 0 < D_t < B] \ge V$. Let T be the first time at which either (a) $D_t = 0$ (i.e., $X(t + \tau)$ is a Nash equilibrium), or (b) $D_t = B$. Note that T is a stopping time. Define $Z_t = (B - D_t)^2 - Vt$, and observe that $Z_{t \wedge T}$ is a sub-martingale, where $t \wedge T$ denotes the minimum of t and T. Let p be the probability that (a) occurs. By the optional stopping theorem $\mathbb{E}[D_T] \le D_0$, so $(1 - p)B = \mathbb{E}[D_T] \le D_0$ and $p \ge 1 - D_0/B \ge \frac{9}{10}$. Also, by the optional stopping theorem

$$pB^2 - V\mathbb{E}[T] = \mathbb{E}[(B - D_T)^2] - V\mathbb{E}[T] = \mathbb{E}[Z_T] \ge Z_0 = (B - D_0)^2 > 0,$$

so $\mathbb{E}[T] \le pB^2/V$. Conditioning on (a) occurring, it follows that $\mathbb{E}[T \mid D_T = 0] \le B^2/V$. Hence $\Pr(T > 10B^2/V \mid D_T = 0) \le \frac{1}{10}$. So, if we now run for $10B^2/V$ steps, then the probability that we do not reach a Nash equilibrium is at most $\frac{1}{40} + 2 \cdot \frac{1}{10} < 1/4$. ∎

Now we can give the proof of Theorem 4.1.

Proof. Subdivide time into intervals of t' steps. The probability that the process has not reached a Nash equilibrium before the $(j + 1)$st interval is at most $(1/4)^{-j}$. ∎

5. Lower Bounds. The following theorem gives an exponential lower bound for the expected convergence time of the process in Figure 1.1.

THEOREM 5.1. *Let $X(t)$ be the process in Figure 1.1 with $m = n$. Let $X(0)$ be the assignment given by $X(0) = (n, 0, \ldots, 0)$. Let T be the first time at which $X(t)$ is a Nash equilibrium. Then $\mathbb{E}[T] = \exp(\Theta(\sqrt{n}))$.*

Proof.

For an assignment x, let $n_0(x)$ denote the number of resources i with $x_i = 0$. Thus, $n_0(X(0)) = n - 1$. The (unique) Nash equilibrium x assigns one task to each resource, so $n_0(x) = 0$. Let $k = \lfloor\sqrt{n}\rfloor$. We will show that for any assignment x with $n_0(x) \geq k$,

$$\Pr(n_0(X(t)) < k \mid X(t-1) = x) \leq \exp(-\Theta(\sqrt{n})).$$

This implies the result.

Suppose $X(t-1) = x$ with $n_0(x) \geq k$. For convenience, let n_0 denote $n_0(x)$. Let x' denote $X(t)$, and let n_0' denote $n_0(x')$. We will show that, with probability at least $1 - \exp(-\Theta(\sqrt{n}))$, $n_0' \geq k$. During the course of the proof, we will assume, where necessary, that n is sufficiently large. This is without loss of generality given the Θ notation in the statement of the result.

Case 1. $n_0 > 8k$.

Consider the protocol in Figure 1.1. Let $U = \{b \mid x_{j_b} = 0\}$. $\mathbb{E}[|U|] = n_0$, so by a Chernoff bound (Equation (3.2), $\Pr(|U| \leq \lceil\frac{n_0}{2}\rceil + \lceil\frac{3n_0}{8}\rceil) \leq \Pr(|U| \leq \frac{8}{9}n_0) = \exp\left(-\Theta(\sqrt{n})\right)$. Thus, $|U| \geq \lceil n_0/2\rceil + \lceil 3n_0/8\rceil$ with probability at least $1 - \exp(-\Theta(\sqrt{n}))$. Suppose this is the case. Partition U into U_1 and U_2 with $|U_1| = \lceil n_0/2\rceil$. Let $W = \cup_{b\in U_1}\{j_b\}$. First, suppose $|W| \leq \frac{3}{8}n_0$. In that case

$$|\{j \mid x_j' > 0\}| \leq n - |U_1| + \frac{3}{8}n_0 = n - \lceil n_0/2\rceil + \frac{3}{8}n_0 \leq n - k,$$

so $n_0' \geq k$. Otherwise, let $U' = \{b \in U_2 \mid j_b \in W\}$.

$$\mathbb{E}[|U'|] = |U_2|\frac{|W|}{n_0} \geq \frac{9}{64}n_0 > \frac{9}{8}k,$$

so by a Chernoff bound (Eqn. 3.2), $\Pr(|U'| \leq k) = \Pr(|U'| \leq (1 - \frac{1}{9})\mathbb{E}[|U'|]) = \exp\left(-\Theta(\sqrt{n})\right)$, recalling that $k = \lfloor\sqrt{n}\rfloor$. Thus $|U'| \geq k$ with probability at least $1 - \exp(-\Theta(\sqrt{n}))$, which implies $n_0' \geq k$.

Case 2. $k \leq n_0 \leq 8k$.

Consider the protocol in Figure 1.1. Let L be the set of "loners" defined by $L = \{i \mid x_i = 1\}$ and let $\ell = |L|$. The number of resources i with $x_i > 1$ is $n - n_0 - \ell$ and this is at most half as many as the number of tasks assigned to such resources (which is $n - \ell$), so $\ell \geq n - 2n_0$. Let $U = \{b \mid i_b \in L \text{ and } x_{j_b} = 0\}$. $\mathbb{E}[|U|] = \ell\frac{n_0}{n} \geq \frac{(n-2n_0)n_0}{n} = \Theta(\sqrt{n})$, so by a Chernoff bound (Eqn. 3.2), $\Pr(|U| \leq 2\lceil\frac{1}{4}\ell\frac{n_0}{n}\rceil) \leq \Pr(|U| \leq \frac{2}{3}\mathbb{E}[|U|]) \leq \exp\left(-\Theta(\sqrt{n})\right)$. Thus, $|U| \geq 2\lceil\frac{1}{4}\ell\frac{n_0}{n}\rceil$ with probability at least $1 - \exp(-\Theta(\sqrt{n}))$. Suppose this is the case. Let U_1 and U_2 be disjoint subsets of U of size $\lceil\frac{1}{4}\ell\frac{n_0}{n}\rceil$. Order tasks in U arbitrarily and let $S = \{b \in U \mid \text{for some } b' \in U \text{ with } b' < b,\ j_{b'} = j_b.\}$. (Note that $|S|$ does not depend on the ordering.) Let $W = \cup_{b\in U_1}\{j_b\}$.

Note that if $|W| \leq \frac{1}{5}\ell\frac{n_0}{n}$ then $|S| \geq \frac{1}{20}\ell\frac{n_0}{n} > \frac{n_0}{40}\left(\frac{\ell}{n}\right)^2$. Otherwise, let $U' = \{b \in U_2 \mid j_b \in W\}$.

$$\mathbb{E}[|U'|] = |U_2|\frac{|W|}{n_0} \geq \frac{n_0}{20}\left(\frac{\ell}{n}\right)^2,$$

so, by a Chernoff bound (Eqn. 3.2), $\Pr(|U'| \leq \frac{1}{2}\frac{n_0}{20}\left(\frac{\ell}{n}\right)^2) \leq \exp\left(-\Theta(\sqrt{n})\right)$ (recall that $n_0\left(\frac{\ell}{n}\right)^2 \geq n_0\left(\frac{n-2n_0}{n}\right)^2 \geq k\left(\frac{n-16k}{n}\right)^2 = \Theta(\sqrt{n})$), and thus $|U'| \geq \frac{n_0}{40}\left(\frac{\ell}{n}\right)^2$ with probability at least $1 - \exp(-\Theta(\sqrt{n}))$, so $|S| \geq \frac{n_0}{40}\left(\frac{\ell}{n}\right)^2$.

Suppose then that $|S| \geq \frac{n_0}{40}\left(\frac{\ell}{n}\right)^2$. Assuming that n is sufficiently large, $|S| \geq k/41$. Let $B_0 = \cup_{b\in U}\{j_b\}$ and $B_1 = \cup_{b\in L-U}\{i_b\}$. Note that every resource in $B_0 \cup B_1$ is used in x' for some task $b \in L$. Thus, $|B_0 \cup B_1| \leq \ell - |S|$. Let $R = \{i \mid x_i = 0\} \cup L - B_0 - B_1$. Then $|R| \geq n_0 + \ell - (\ell - |S|) \geq n_0 + |S| \geq (1 + \frac{1}{41})k$.

Let $T = \{b \mid i_b \notin L, j_b \in R\}$. $\mathbb{E}[T] = (n - \ell)\frac{|R|}{n}$ and

$$\Pr\left(T \geq \frac{|R|}{100}\right) \leq \binom{n-\ell}{\frac{|R|}{100}}\left(\frac{|R|}{n}\right)^{|R|/100} \leq \left(\frac{2n_0 e 100}{n}\right)^{|R|/100},$$

so with probability at least $1-\exp(-\Theta(\sqrt{n}))$, $T < |R|/100$. In that case, $n_0' \geq |R|(1-\frac{1}{100}) \geq k$. ∎

The following theorem provides a lower bound on the expected convergence time regardless of which of the two protocols is being used.

THEOREM 5.2. *Suppose that m is even. Let $X(t)$ be the process in Figure 2.1 with $n=2$. Let $X(0)$ be the assignment given by $X(0)=(m,0)$. Let T be the first time at which $X(t)$ is a Nash equilibrium. Then $\mathbb{E}[T]=\Omega(\log\log m)$. The same result holds for the process in Figure 1.1.*

Proof. Note that both protocols have the same behaviour since m is even and, therefore, the situation $x_1 = x_2+1$ cannot arise. For concreteness, focus on the protocol in Figure 2.1.

Let $y(x) = \max_i x_i - m/2$ and let $y_t = y(X(t))$ so $y_0 = m/2$ and, for a Nash equilibrium x, $y(x)=0$. We will show that for any assignment x, $\Pr(y_{t+1} > y(x)^{1/10} \mid X(t)=x) \geq 1-y_t^{-1/4}$. (There is nothing very special about the exact value "1/10" – this value is being used as part of an explicit "lack of concentration" inequality in the proof, noting that for a lower bound we essentially want to lower-bound the variances of the load distributions. This seems to require a somewhat ad-hoc approach, in contrast with the usage of concentration inequalities.)

Suppose $X(t)=x$ is an assignment with $x_1 \geq x_2$. As we have seen in Section 2, $Y_{1,2}(x)$ (the number of migrations from resource 1 to resource 2 in the round) is a binomial random variable

$$B\left(x_1, \frac{1}{2}\left(1-\frac{x_2}{x_1}\right)\right) = B\left(\frac{m}{2}+y_t, \frac{2y_t}{m+2y_t}\right).$$

In general, let T_t be the number of migrations from the most-loaded resource in $X(t)$ to the least-loaded resource and note that the distribution of T_t is $B\left(\frac{m}{2}+y_t, \frac{2y_t}{m+2y_t}\right)$ with mean y_t. If $T_t = y_t+\ell$ or $T_t = y_t-\ell$ then $y_{t+1}=\ell$. Thus $\Pr(y_{t+1} > y_t^{1/10}) = \Pr(|T_t - \mathbb{E}[T_t]| > y_t^{1/10})$. We continue by showing that this binomial distribution is sufficiently "spread out" in the region of its mode, that we can find an upper bound on $\Pr(y_{t+1} \leq y_t^{1/10})$. This will lead to our lower bound on the expected time for $(y_t)_t$ to decrease below some constant (we use the constant 16).

$$\Pr(T_t = y_t) = \binom{\frac{1}{2}m+y_t}{y_t}\left(\frac{2y_t}{m+2y_t}\right)^{y_t}\left(\frac{m}{m+2y_t}\right)^{\frac{1}{2}m}$$

$$\Pr(T_t = y_t+j) = \binom{\frac{1}{2}m+y_t}{y_t+j}\left(\frac{2y_t}{m+2y_t}\right)^{y_t+j}\left(\frac{m}{m+2y_t}\right)^{\frac{1}{2}m-j}$$

Suppose $j>0$.

$$\begin{aligned}\frac{\Pr(T_t=y_t+j)}{\Pr(T_t=y_t)} &= \left(\frac{2y_t}{m+2y_t}\right)^j\left(\frac{m}{m+2y_t}\right)^{-j}\left(\frac{y_t!(\frac{1}{2}m)!}{(y_t+j)!(\frac{1}{2}m+y_t-(y_t+j))!}\right)\\ &= \left(\frac{2y_t}{m}\right)^j\left(\prod_{\ell=1}^{j}\frac{\frac{1}{2}m+1-\ell}{y_t+\ell}\right) = \left(\frac{2y_t}{m}\right)^j\left(\prod_{\ell=1}^{j}\frac{m+2-2\ell}{2y_t+2\ell}\right)\\ &> \left(\frac{2y_t}{m}\right)^j\left(\prod_{\ell=1}^{j}\frac{m-2j}{2y_t+2j}\right) = \left[\left(\frac{2y_t}{m}\right)\left(\frac{m-2j}{2y_t+2j}\right)\right]^j.\end{aligned}$$

Similarly, for $j<0$,

$$\begin{aligned}\frac{\Pr(T_t=y_t+j)}{\Pr(T_t=y_t)} &= \left(\frac{2y_t}{m}\right)^j\left(\prod_{\ell=1}^{|j|}\frac{y_t+1-\ell}{\frac{1}{2}m+\ell}\right) = \left(\frac{m}{2y_t}\right)^{|j|}\left(\prod_{\ell=1}^{|j|}\frac{2y_t+2-2\ell}{m+2\ell}\right)\\ &> \left(\frac{m}{2y_t}\right)^{|j|}\left(\frac{2y_t-2|j|}{m+2|j|}\right)^{|j|} = \left[\left(\frac{m}{2y_t}\right)\left(\frac{2y_t-2|j|}{m+2|j|}\right)\right]^{|j|}\\ &= \left[\left(\frac{2y_t}{m}\right)\left(\frac{m-2j}{2y_t+2j}\right)\right]^j.\end{aligned}$$

So for all j,

$$\frac{\Pr(T_t = y_t + j)}{\Pr(T_t = y_t)} > \Big[\Big(\frac{2y_t}{m}\Big)\Big(\frac{m-2j}{2y_t+2j}\Big)\Big]^j = \Big[\Big(\frac{y_t}{y_t+j}\Big)\Big(\frac{m-2j}{m}\Big)\Big]^j.$$

So, for all j with $|j| \le y_t^{1/4}$, where $y_t^{1/4}$ is the positive fourth root of y_t, this is at least

$$\begin{aligned}
&\Big(\frac{y_t}{y_t + y_t^{1/4}}\Big)^{y_t^{1/4}} \Big(\frac{m - 2y_t^{1/4}}{m}\Big)^{y_t^{1/4}} \\
&\ge \Big(\frac{y_t}{y_t + y_t^{1/4}}\Big)^{y_t^{1/4}} \Big(\frac{2y_t - 2y_t^{1/4}}{2y_t}\Big)^{y_t^{1/4}} = \Big(\frac{y_t - y_t^{1/4}}{y_t + y_t^{1/4}}\Big)^{y_t^{1/4}} = \left(\frac{y_t + y_t^{1/4} - 2y_t^{1/4}}{y_t + y_t^{1/4}}\right)^{y_t^{1/4}} \\
&= \left(1 - \frac{2y_t^{1/4}}{y_t + y_t^{1/4}}\right)^{y_t^{1/4}} \ge \left(1 - \frac{2y_t^{1/4}}{y_t}\right)^{y_t^{1/4}} = \Big(1 - 2y_t^{-3/4}\Big)^{y_t^{1/4}} \\
&\ge 1 - 2y_t^{-3/4} y_t^{1/4} = 1 - 2y_t^{-1/2} \ge \frac{1}{2}
\end{aligned}$$

where the last inequality just requires $y_t \ge 16$.

Note that the mode of a binomial distribution is one or both of the integers closest to the expectation, and the distribution is monotonically decreasing as you move away from the mode. But, for $|j| \le y_t^{1/4}$, $\Pr(T_t = y_t + j) \ge \frac{1}{2}\Pr(T_t = y_t)$, hence $\Pr(T_t = y_t) \le 2/(1 + 2y_t^{1/4})$. Since $\Pr(T_t = y_t + j) \le \Pr(T_t = y_t)$, it follows that

$$\Pr(T_t \in [y_t - y_t^{1/10}, y_t + y_t^{1/10}]) \le (2y_t^{1/10} + 1)\Pr(T_t = y_t) < 3y_t^{-3/20}.$$

We say that the transition from y_t to y_{t+1} is a "fast round" if $y_{t+1} \le y_t^{1/10}$ (equivalently, it is a fast round if $T_t \in [y_t - y_t^{1/10}, y_t + y_t^{1/10}]$). Otherwise it is a slow round. Recall that $y_0 = m/2$. Let

$$r = \left\lfloor \log_{10}\left(\frac{\log(y_0)}{\log(12^{20/3})}\right)\right\rfloor.$$

If the first j rounds are slow then $y_j \ge y_0^{10^{-j}}$. If $j \le r$ then $y_0^{10^{-j}} \ge 12^{20/3}$ so the probability that the transition from y_j to y_{j+1} is the first fast round is at most $3\Big(y_0^{10^{-j}}\Big)^{-3/20} \le 1/4$.

Also, if $j < r$ then these probabilities increase geometrically so that the ratio of the probability that the transition to y_{j+1} is the first fast round and the probability that the transition to y_j is the first fast round is

$$\frac{3\Big(y_0^{10^{-(j+1)}}\Big)^{-3/20}}{3\big(y_0^{10^{-j}}\big)^{-3/20}} = \Big(y_0^{10^{-j} - 10^{-(j+1)}}\Big)^{3/20} \ge \Big(y_0^{10^{-(j+1)}}\Big)^{3/20} \ge 12 \ge 2,$$

so $\sum_{j=0}^{r-1} \Pr(\text{transition from } y_j \text{ to } y_{j+1} \text{ is the first fast round}) \le 2 \cdot 1/4 = \frac{1}{2}$. Therefore, with probability at least $1/2$, all of the first r rounds are slow. In this case, $\arg\min_t(y_t \le 16) = \Omega(\log\log(m))$, which proves the theorem. ∎

We also have the following observation.

Observation 5.3. *Let $X(t)$ be the process in Figure 2.1 with $m = n$. Let $X(0)$ be the assignment given by $X(0) = (2, 0, 1, \ldots, 1)$. Let T be the first time at which $X(t)$ is a Nash equilibrium. Then $\mathbb{E}[T] = \Omega(n)$.*

The observation follows from the fact that the state does not change until one of the two tasks assigned to the first resource chooses the second resource.

6. Summary. We have analyzed a very simple, strongly distributed rerouting protocol for m tasks on n resources. We have proved an upper bound of $(\log\log m + n^4)$ on the expected convergence time (convergence to a Nash equilibrium), and for $m > n^3$ an upper bound of $O(\log\log m)$ on the time to reach an approximate Nash equilibrium. Our lower bound of $\Omega(\log\log m + n)$ matches the upper bound as function of m. We have also shown an exponential lower bound on the convergence time for a related protocol that allows "neutral moves".

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