Quantum Measurement Uncertainty

Reading Heisenberg's mind or invoking his spirit?

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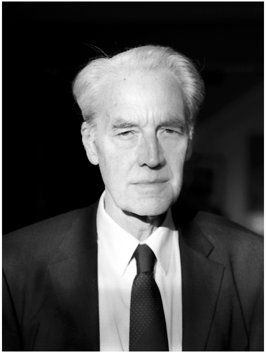
Quantum Physics and Logic – QPL 2015, Oxford, 13-17 July 2015



Peter Mittelstaedt 1929-2014

Outline

1. Introduction: *two* varieties of quantum uncertainty
2. (Approximate) Joint Measurements
3. Quantifying measurement error and disturbance
4. Uncertainty Relations for Qubits
5. Conclusion

Introduction

Heisenberg 1927

Essence of the quantum mechanical world view:

quantum uncertainty & *Heisenberg effect*

Heisenberg 1927

quantum uncertainty: limitations to what can be known about the physical world

Preparation Uncertainty Relation: PUR

For any wave function ψ:

$$(\text{Width of } Q \text{ distribution}) \cdot (\text{Width of } P \text{ distribution}) \ \sim \ \hbar$$

(Heisenberg just discusses a Gaussian wave packet.)

Later generalisation:

$$\Delta_\rho A \, \Delta_\rho B \ \geq \ \tfrac{1}{2}\left|\langle [A,B]\rangle_\rho\right|$$

(Heisenberg didn't state this...)

Heisenberg 1927

Heisenberg effect – reason for quantum uncertainty?

- any measurement disturbs the object: *uncontrollable state change*
- measurements disturb each other: *quantum incompatibility*

Measurement Uncertainty Relation: MUR

$$(\text{Error of } Q \text{ measurement}) \cdot (\text{Error of } P) \sim \hbar$$

$$(\text{Error of } Q \text{ measurement}) \cdot (\text{Disturbance of } P) \sim \hbar$$

Reading Heisenberg's thoughts?

Heisenberg allegedly claimed (and proved):

$$\varepsilon(A,\rho)\,\varepsilon(B,\rho) \;\geq\; \tfrac{1}{2}\left|\langle [A,B]\rangle_\rho\right| \qquad (???)$$

MUR made precise?

Heisenberg's thoughts – or Heisenberg's spirit?

...or: what measurement limitations are there *according to quantum mechanics*?

(combined joint measurement errors for A, B) $\geq$ (incompatibility of A, B)

True of false? **Needed:**

- precise notions of approximate measurement
- measure of approximation error
- measure of disturbance

Quantum uncertainty challenged

PRL **109,** 100404 (2012) PHYSICAL REVIEW LETTERS week ending 7 SEPTEMBER 2012

Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements

Lee A. Rozema, Ardavan Darabi, Dylan H. Mahler, Alex Hayat, Yasaman Soudagar, and Aephraim M. Steinberg
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(Received 4 July 2012; published 6 September 2012; publisher error corrected 23 October 2012)

While there is a rigorously proven relationship about uncertainties intrinsic to any quantum system, often referred to as "Heisenberg's uncertainty principle," Heisenberg originally formulated his ideas in terms of a relationship between the precision of a *measurement* and the disturbance it must create. Although this latter relationship is not rigorously proven, it is commonly believed (and taught) as an aspect of the broader uncertainty principle. Here, we experimentally observe a violation of Heisenberg's "measurement-disturbance relationship", using weak measurements to characterize a quantum system before and after it interacts with a measurement apparatus. Our experiment implements a 2010 proposal of Lund and Wiseman to confirm a revised measurement-disturbance relationship derived by Ozawa in 2003. Its results have broad implications for the foundations of quantum mechanics and for practical issues in quantum measurement.

Quantum uncertainty challenged

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Received
7 August 2012

Accepted
2 July 2013

Published
17 July 2013

Correspondence and requests for materials

Experimental violation and reformulation of the Heisenberg's error-disturbance uncertainty relation

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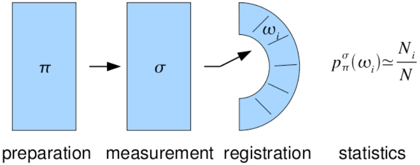
The uncertainty principle formulated by Heisenberg in 1927 describes a trade-off between the error of a measurement of one observable and the disturbance caused on another complementary observable such that their product should be no less than the limit set by Planck's constant. However, Ozawa in 1988 showed a model of position measurement that breaks Heisenberg's relation and in 2003 revealed an alternative relation for error and disturbance to be proven universally valid. Here, we report an experimental test of Ozawa's relation for a single-photon polarization qubit, exploiting a more general class of quantum measurements than the class of projective measurements. The test is carried out by linear optical devices and realizes an indirect measurement model that breaks Heisenberg's relation throughout the range of our experimental parameter and yet validates Ozawa's relation.

(Approximate) Joint Measurements

Quantum Measurement Statistics – Observables as POVMs



$$[\pi] \sim \rho, \quad [\sigma] \sim \mathsf{E} = \{\omega_i \mapsto E_i\} : \qquad p_\pi^\sigma(\omega_i) = \mathrm{tr}[\rho E_i] = p_\rho^{\mathsf{E}}(\omega_i)$$

$$\mathrm{POVM}: \qquad \mathsf{E} = \{E_1, E_2, \cdots, E_n\}, \quad 0 \le O \le E_i \le I, \quad \sum E_i = I$$

state changes: *instrument* $\omega_i, \rho \to \mathcal{I}_i(\rho)$

measurement processes: measurement scheme $\mathcal{M} = \langle \mathcal{H}_a, \phi, U, Z_a \rangle$

Signature of an observable: its statistics

$$p_\rho^{\mathsf{C}} = p_\rho^{\mathsf{A}} \quad \text{for all } \rho \quad \Longleftrightarrow \quad \mathsf{C} = \mathsf{A}$$

Minimal indicator for a measurement of C *to be a good approximate measurement of* A*:*

$$p_\rho^{\mathsf{C}} \simeq p_\rho^{\mathsf{A}} \quad \text{for all } \rho$$

Unbiased approximation – absence of systematic error:

$$\mathsf{C}[1] = \sum_j c_j C_j = \mathsf{A}[1] = \sum_i a_i A_i = A$$

... often taken as sole criterion for a good measurement

Joint Measurability/Compatibility

Definition: joint measurability (compatibility)

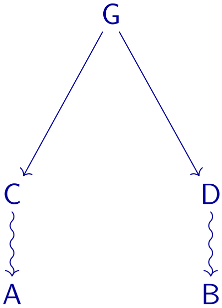
Observables $\mathsf{C} = \{C_+, C_-\}$, $\quad \mathsf{D} = \{D_+, D_-\}$ are *jointly measurable* if they are margins of an observable $\mathsf{G} = \{G_{++}, G_{+-}, G_{-+}, G_{--}\}$:

$$C_k = G_{k+} + G_{k-}, \quad D_\ell = G_{+\ell} + G_{-\ell}$$

Joint measurability in general

Pairs of **unsharp** observables may be jointly measurable
– **even when they do not commute!**

Approximate joint measurement: concept



joint observable

approximator observables (compatible)

target observable

Task: find suitable measures of *approximation errors*

Measure of disturbance: instance of joint measurement approximation error

Quantifying Measurement Error

Approximation error

(vc) *value comparison*

(*e.g.* rms) deviation of outcomes of a joint measurement: accurate reference measurement *together* with measurement to be calibrated, on *same* system

(dc) *distribution comparison*

(*e.g.* rms) deviation between distributions of *separate* measurements: accurate reference measurement and measurement to be calibrated, applied to *separate* but identically prepared ensembles

alternative measures of deviation: error bar width; relative entropy; *etc.* ...

Crucial:

Value comparison is of *limited applicability* in quantum mechanics!

Approximation error – Take 1: value comparison

Measurements/observables to be compared:

$$\mathsf{A} = \{A_1, A_2, \dots, A_m\}, \quad \mathsf{C} = \{C_1, C_2, \dots, C_n\}$$

where A is a sharp *(target)* observable
and C an *(approximator)* observable representing an approximate measurement of A
Protocol: *measure both* A *and* C *jointly on each system of an ensemble of identically prepared systems*
Proviso: This requires A and C to be compatible, hence commuting.

$$\delta_{\mathrm{vc}}(\mathsf{C}, \mathsf{A}; \rho)^2 \;=\; \sum_i (a_i - c_j)^2 \, \mathrm{tr}[\rho A_i C_j]$$

(Ozawa 1991)

Issue: $\delta_{\rm vc}$ is of limited use!
Attempted generalisation: *measurement noise* (Ozawa 2003)

$$\delta_{\rm vc}(\mathsf{C},\mathsf{A};\rho)^2 \;=\; \left\langle \mathsf{C}[2]-\mathsf{C}[1]^2\right\rangle_\rho + \left\langle (\mathsf{C}[1]-A)^2\right\rangle_\rho \;=\; \varepsilon_{\rm mn}(\mathsf{C},\mathsf{A};\rho)^2$$

where $\mathsf{C}[k]=\sum_j c_j^k C_j$, $A=\mathsf{A}[1]$ are the $k^{\rm th}$ moment operators...
...then give up assumption of commutativity of A, C

Critique (BLW 2013, 2014)

If A, C do not commute, then:

- $\delta_{\rm vc}(\mathsf{C},\mathsf{A};\rho)$ loses its meaning as rms value deviation
- and becomes unreliable as error indicator
  – *e.g.*, it is possible to have $\varepsilon_{\rm mn}(\mathsf{C},\mathsf{A};\rho)=0$ where A, C may not even have the same value sets.

Measurement noise as approximation error?

$$\varepsilon(C, A; \rho) = \left\langle (Z_\tau - A)^2 \right\rangle_{\rho\otimes\sigma}^{1/2}$$
$$= \Big[\left\langle \mathsf{C}[2] - \mathsf{C}[1]^2 \right\rangle_\rho + \left\langle (\mathsf{C}[1] - A)^2 \right\rangle_\rho \Big]^{1/2}$$

- adopted from noise concept of quantum optical theory of linear amplifiers
- first term describes intrinsic noise of POVM C, that is, its deviation from being sharp, projection valued
- second term *intended to* capture deviation between target observable A and approximator observable C
- State dependence – a virtue? Then incoherent to offer three-state method.
- $\mathsf{C}[1], A$ ma not commute: $\mathsf{C}[1] - A$ incompatible with $\mathsf{C}[1]$, A.

Ozawa and Branciard inequalities

$$\varepsilon(\mathsf{A},\rho)\,\varepsilon(B,\rho) + \varepsilon(A,\rho)\Delta_\rho B\ , + \Delta_\rho A \varepsilon(B,\rho) \ \geq\ \tfrac{1}{2}\big|\big\langle[A,B]\big\rangle_\rho,$$

$$\varepsilon(A)^2(\Delta_\rho B)^2 + \varepsilon(B)^2(\Delta_\rho A)^2 + 2\sqrt{(\Delta_\rho A)^2(\Delta_\rho B)^2 - \tfrac{1}{4}|\langle[A,B]\rangle_\rho|^2}\,\varepsilon(A)\varepsilon(B) \ \geq\ \tfrac{1}{4}|\langle[A,B]\rangle_\rho|^2$$

- Does allow for $\varepsilon(\mathsf{A};\rho)\,\varepsilon(\mathsf{B};\rho) \ <\ \frac{1}{2}|\langle[A,B]\rangle_\rho|$.
- Branciard's inequality is known to be *tight* for pure states.
- *Not purely error tradeoff relations!* (BLW 2014)

Measurement Noise – some oddities

Take two identical systems, probe in state σ,
measurement coupling $U = SWAP$, pointer $Z = A$.
Then $\mathsf{C} = \mathsf{A}$ and $\mathsf{D} = \mathsf{B}_\sigma I$.

$$\eta(\mathsf{D}, \mathsf{B}; \rho)^2 = (\Delta_\rho B)^2 + (\Delta_\sigma B)^2 + \left(\langle B\rangle_\rho - \langle B\rangle_\sigma\right)^2$$

- $\eta(\mathsf{D}, \mathsf{B}; \rho)^2$ contains a contribution from preparation uncertainty – not solely a measure of disturbance.
- For $\rho = \sigma$: $\eta(\mathsf{D}, \mathsf{B}; \sigma) = \sqrt{2}\Delta(\mathsf{B}_\sigma)$; *i.e.*, distorted observable D is statistically independent of B.
- Note $\eta(\mathsf{D}, \mathsf{B}; \sigma) \neq 0$, despite the fact that the state has not changed (no disturbance).

Approximation error – Take 2: distribution comparison

Protocol: compare distributions of A and C as they are obtained in separate runs of measurements on two ensembles of systems in state ρ

$$\delta_\gamma(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho)^\alpha \;=\; \sum_{ij}(a_i - c_j)^\alpha \gamma(i,j) \quad (1 \leq \alpha < \infty)$$

where γ is any joint distribution of the values of A and C with marginal distributions $p^{\mathsf{A}}_\rho,\;\; p^{\mathsf{C}}_\rho$

$$\Delta_\alpha(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho) \;=\; \inf_\gamma \delta_\gamma(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho)$$

Wasserstein-α distance – *scales with distances between points.*

$$\Delta_\alpha(\mathsf{C}, \mathsf{A}) \;=\; \sup_\rho \Delta_\alpha(p^{\mathsf{C}}_\rho, p^{\mathsf{A}}_\rho)$$

quantum rms error: $\alpha = 2$

Qubit Uncertainty

Qubits

$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ (Pauli matrices acting on $\mathbb{C}^2$)

- *States:* $\rho = \frac{1}{2}(I + \boldsymbol{r} \cdot \boldsymbol{\sigma}), \quad |\boldsymbol{r}| \leq 1$
- *Effects:* $A = \frac{1}{2}(a_0 I + \boldsymbol{a} \cdot \boldsymbol{\sigma}) \in [O, I], \quad 0 \leq \frac{1}{2}(a_0 \pm |\boldsymbol{a}|) \leq 1$
- *observables:* $(\Omega = \{+1, -1\})$

$$\mathrm{A}: \ \pm 1 \mapsto A_\pm = \tfrac{1}{2}(I \pm \boldsymbol{a} \cdot \boldsymbol{\sigma}) \quad |\boldsymbol{a}| = 1$$
$$\mathrm{B}: \ \pm 1 \mapsto B_\pm = \tfrac{1}{2}(I \pm \boldsymbol{b} \cdot \boldsymbol{\sigma}) \quad |\boldsymbol{b}| = 1$$
$$\mathrm{C}: \ \pm 1 \mapsto C_\pm = \tfrac{1}{2}(1 \pm \gamma)\, I \pm \tfrac{1}{2}\boldsymbol{c} \cdot \boldsymbol{\sigma} \quad |\gamma| + |\boldsymbol{c}| \leq 1$$
$$\mathrm{D}: \ \pm 1 \mapsto D_\pm = \tfrac{1}{2}(1 \pm \delta)\, I \pm \tfrac{1}{2}\boldsymbol{d} \cdot \boldsymbol{\sigma} \quad |\delta| + |\boldsymbol{d}| \leq 1$$

symmetric: $\gamma = 0$
sharp: $\gamma = 0$, $|\boldsymbol{c}| = 1$; $\quad \rightarrow \quad$ *unsharpness:* $U(\mathrm{C})^2 = 1 - |\boldsymbol{c}|^2$

Joint measurability of C, D

Symmetric case (sufficient for optimal compatible approximations):

Proposition

$\mathsf{C} = \{C_\pm = \frac{1}{2}(I \pm \boldsymbol{c}\cdot\boldsymbol{\sigma})\}$, $\mathsf{D} = \{D_\pm = \frac{1}{2}(I \pm \boldsymbol{d}\cdot\boldsymbol{\sigma})\}$ are compatible if and only if

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2.$$

Interpretation: *unsharpness* $U(\mathsf{C})^2 = 1 - |\boldsymbol{c}|^2$; $|\boldsymbol{c}\times\boldsymbol{d}| = 2\big\|[C_+, D_+]\big\|$

$$|\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| \le 2 \;\Leftrightarrow\; \big(1-|\boldsymbol{c}|^2\big)\big(1-|\boldsymbol{d}|^2\big) \ge |\boldsymbol{c}\times\boldsymbol{d}|^2$$

$$\mathsf{C},\mathsf{D}\ \text{compatible} \;\Leftrightarrow\; U(\mathsf{C})^2 \times U(\mathsf{D})^2 \ge 4\big\|[C_+, D_+]\big\|^2$$

Unsharpness Relation

Approximation error

Recall: Observable C is a good approximation to A if $p_\rho^{\mathsf{C}} \simeq p_\rho^{\mathsf{A}}$

Take here: probabilistic distance

$$d_p(\mathsf{C},\mathsf{A}) = \sup_\rho \sup_X \left| \mathrm{tr}[\rho \mathsf{C}(X)] - \mathrm{tr}[\rho \mathsf{A}(X)] \right|$$
$$= \sup_\rho \left\| p_\rho^{\mathsf{C}} - p_\rho^{\mathsf{A}} \right\|_1 = \sup_X \left\| \mathsf{C}(X) - \mathsf{A}(X) \right\|$$

Qubit case: $C_+ = \frac{1}{2}\left(c_0 I + \boldsymbol{c}\cdot\boldsymbol{\sigma}\right),\ A_+ = \frac{1}{2}\left(a_0 I + \boldsymbol{a}\cdot\boldsymbol{\sigma}\right)$

$$d_p(\mathsf{C},\mathsf{A}) = \left\| C_+ - A_+ \right\| = \frac{1}{2}|c_0 - a_0| + \frac{1}{2}|\boldsymbol{c} - \boldsymbol{a}| \ \equiv\ d_a \ \in\ [0,1].$$

Comparison 1: Wasserstein 2-distance (quantum rms error)

$$\Delta_2\left(p_\rho^{\mathsf{C}}, p_\rho^{\mathsf{A}}\right)^2 = \inf_\gamma \sum_{ij} (a_i - c_j)^2 \gamma(i,j)$$

where γ runs through all joint distributions with margins p_ρ^{C}, p_ρ^{A}.

$$\Delta_2(\mathsf{C},\mathsf{A})^2 = \sup_\rho d_2\left(p_\rho^{\mathsf{C}}, p_\rho^{\mathsf{A}}\right)^2 \equiv \Delta_a^2$$

Qubit case:

$$\begin{aligned}\Delta_a^2 = \Delta_2(\mathsf{C},\mathsf{A})^2 &= 2|c_0 - a_0| + 2|\boldsymbol{c} - \boldsymbol{a}| \\ &= 4 d_p(\mathsf{C},\mathsf{A}) = 4 d_a.\end{aligned}$$

Comparison 2: Measurement noise (Ozawa *et al*)

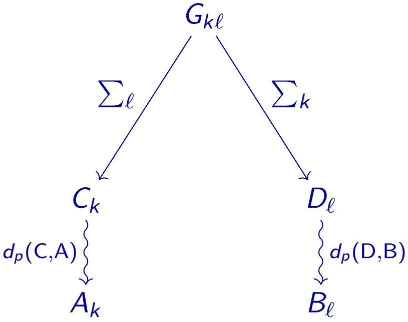
$$\varepsilon(\mathsf{C},\mathsf{A};\varphi)^2 = \langle \varphi\otimes\phi \,|\, (Z_\tau - A)^2 \varphi\otimes\phi\rangle$$
$$= \left\langle \mathsf{C}[2]-\mathsf{C}[1]^2\right\rangle_\rho + \left\langle(\mathsf{C}[1]-A)^2\right\rangle_\rho \;\equiv\; \varepsilon_a^2$$

Qubit observables, symmetric case:

$$\varepsilon_a^2 = 1 - |\boldsymbol{c}|^2 + |\boldsymbol{a}-\boldsymbol{c}|^2 = U(\mathsf{C})^2 + 4d_a^2$$

- $\varepsilon(\mathsf{A};\rho)$ double counts contribution from unsharpness.
- Virtue of state-dependence all but gone ...
- for more general approximators C, ε_a may be zero although C is quite different from A
- Branciard notices this and considers it an artefact of the definition of ε_a – you might rather consider it a fatal flaw if the aim is to identify optimal compatible approximations of incompatible observables...

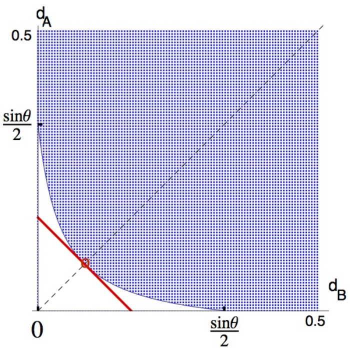
Optimising approximate joint measurements



Goal

To make errors $d_\mathrm{A} = d_p(\mathrm{C}, \mathrm{A})$, $d_\mathrm{B} = d_p(\mathrm{D}, \mathrm{B})$ simultaneously as small as possible, subject to the constraint that C, D are compatible.

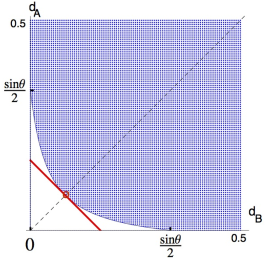
Admissible error region

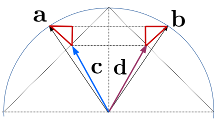


$\sin\theta = |\boldsymbol{a} \times \boldsymbol{b}|$

$(d_A, d_B) = (d_p(C, A), d_p(D, B)) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ with C, D compatible

Qubit Measurement Uncertainty Relation





$$\sin\theta = |\boldsymbol{a} \times \boldsymbol{b}|$$

PB, T Heinosaari (2008), `arXiv:0706.1415`

$$\begin{aligned} |\boldsymbol{c}+\boldsymbol{d}| + |\boldsymbol{c}-\boldsymbol{d}| &\leq 2 \\ U(\mathrm{C})^2 \times U(\mathrm{D})^2 &\geq 4\|[C_+, D_+]\|^2 \\ d_p(\mathrm{C},\mathrm{A}) + d_p(\mathrm{D},\mathrm{B}) &\geq \frac{1}{2\sqrt{2}}\left[\,|\boldsymbol{a}+\boldsymbol{b}| + |\boldsymbol{a}-\boldsymbol{b}| - 2\,\right] \end{aligned}$$

$$|\boldsymbol{a}+\boldsymbol{b}| + |\boldsymbol{a}-\boldsymbol{b}| = 2\sqrt{1+|\boldsymbol{a}\times\boldsymbol{b}|} = 2\sqrt{1+2\|[A_+, B_+]\|}$$

Qubit Measurement Uncertainty

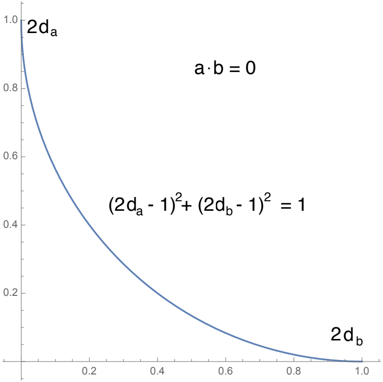
PB & T Heinosaari (2008), S Yu and CH Oh (2014)

Optimiser, case $\boldsymbol{a} \perp \boldsymbol{b}$:

$$\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a}, \quad \boldsymbol{d} = |\boldsymbol{d}|\boldsymbol{b},$$
$$2d_a = |\boldsymbol{a} - \boldsymbol{c}| = 1 - |\boldsymbol{c}|,$$
$$2d_b = |\boldsymbol{b} - \boldsymbol{d}| = 1 - |\boldsymbol{d}|,$$

Compatibility constraint:

$$|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1, \quad \textit{i.e.}, \; U(\mathrm{C})^2 + U(\mathrm{D})^2 = 1$$
$$(1 - 2d_a)^2 + (1 - 2d_b)^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$$



Ozawa–Branciard (C Branciard 2013, M Ringbauer *et al* 2014)

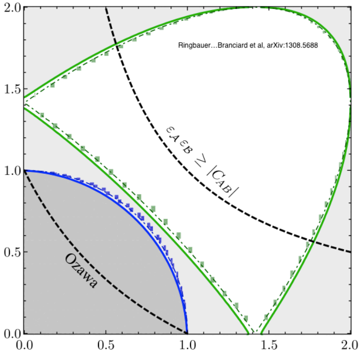
$\boldsymbol{a} \perp \boldsymbol{b}$, symmetric approximators C, D:

$$\varepsilon_a^2\left(1-\frac{\varepsilon_a^2}{4}\right) + \varepsilon_b^2\left(1-\frac{\varepsilon_b^2}{4}\right) \geq 1$$

$$\left(1-\frac{\varepsilon_a^2}{2}\right)^2 + \left(1-\frac{\varepsilon_b^2}{2}\right)^2 \leq 1$$

$$\varepsilon_a^2 \equiv 4d'_a, \quad \varepsilon_b^2 \equiv 4d'_b$$

$$(2d'_a-1)^2 + (2d'_b-1)^2 \leq 1$$



Optimiser: $\boldsymbol{c} = |\boldsymbol{c}|\boldsymbol{a}, \quad \boldsymbol{d} = |\boldsymbol{d}|\boldsymbol{b},$

Compatibility constraint: $|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$, *i.e.*, $U(\mathrm{C})^2 + U(\mathrm{D})^2 = 1$

$4d'_a = \varepsilon_a^2 = 1 - |\boldsymbol{c}|^2 + |\boldsymbol{a}-\boldsymbol{c}|^2 = 2|\boldsymbol{a}-\boldsymbol{c}| = 4d_a, \quad 4d'_b = \varepsilon_b^2 = 4d_b$

$(2d_a-1)^2 + (2d_b-1)^2 = |\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$

Experimentally confirmed!

A twist: Ozawa's error

Branciard's inequality has additional optimisers:

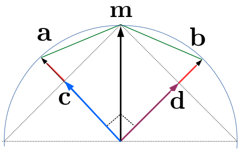
$$\varepsilon_a^2\left(1-\frac{\varepsilon_a^2}{4}\right)+\varepsilon_b^2\left(1-\frac{\varepsilon_b^2}{4}\right) \;=\; 1-|\mathbf{c}|^2+1-|\mathbf{d}|^2+|\mathbf{a}\times\mathbf{c}|^2+|\mathbf{b}\times\mathbf{d}|^2| \;\geq\; 1$$

$$\mathsf{M}=\{M_+,M_-\}=\mathsf{C}'=\mathsf{D}',\ M\pm=\tfrac{1}{2}(I\pm\mathbf{m}\cdot\boldsymbol{\sigma}),\quad |\mathbf{m}|=1:$$

Then:

$$1-|\mathbf{m}|^2+1-|\mathbf{m}|^2+|\mathbf{a}\times\mathbf{m}|^2+|\mathbf{b}\times\mathbf{m}|^2 \;=\; 1$$

$\mathbf{m}$ "between" $\mathbf{a},\mathbf{b}$



$$\varepsilon(\mathsf{M},\mathsf{A})=\varepsilon(\mathsf{M},\mathsf{B})$$
$$=\varepsilon(\mathsf{A},\mathsf{C})=\varepsilon(\mathsf{B},\mathsf{D})$$

but

$2d_p(\mathsf{C},\mathsf{A})=2d_p(\mathsf{D},\mathsf{B})=|\mathbf{a}-\mathbf{c}| \;<\; |\mathbf{a}-\mathbf{m}|=2d_p(\mathsf{M},\mathsf{A})=2d_p(\mathsf{M},\mathsf{B})$

In fact, *any* unit vector $\mathbf{m}$ will do!

Ozawa's error

Moreover, $\boldsymbol{c} = -|\boldsymbol{c}|\boldsymbol{a}$, $\boldsymbol{d} = -|\boldsymbol{d}|\boldsymbol{b}$ with $|\boldsymbol{c}|^2 + |\boldsymbol{d}|^2 = 1$ is another optimiser!

Things get worse when $\boldsymbol{a} \not\perp \boldsymbol{b}$ (T Bullock, PB 2015)

$\Rightarrow$ $\varepsilon(\mathsf{C}, \mathsf{A})$ is unreliable as a guide in finding optimal joint approximations.

But still ... a lucky coincidence that the optimisers "overlap" enough so that the experiments also confirm MUR for probabilistic errors.

Conclusion

Conclusion

(1) Heisenberg's spirit materialised

$$\big(\text{joint measurement errors for A, B}\big) \geq \big(\text{incompatibility of A, B}\big)$$
$$\big(\text{unsharpness of } \textit{compatible} \text{ C, D}\big) \geq \big(\text{noncommutativity of C, D}\big)$$

Shown for qubits; also for position and momentum (BLW 2013):

$$\Delta_2(\mathsf{C}, Q)\,\Delta_2(\mathsf{D}, P) \geq \frac{\hbar}{2}$$

Generic results: finite dimensional Hilbert spaces, arbitrary discrete, finite-outcome observables (Miyadera 2011)

(2) Importance of judicious choice of error measure

- valid MURs obtained for Wasserstein-2 distance, error bar widths, . . .
- measurement noise / value comparison – **not suited** for universal MURs

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http://demonstrations.wolfram.com/HeisenbergTypeUncertaintyRelationForQubits/