Daseinisation and quantum theory in a topos of presheaves

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Categorical Quantum Logic

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"A theory is something nobody believes, except the person who made it. An experiment is something everybody believes, except the person who made it."

(Unknown)

The Kochen-Specker theorem

Problem: Is there a realist formulation of quantum theory?

More concretely, is there a phase space (space of hidden states) for QT such that physical quantities are real-valued functions (hidden variables) on this space? Self-adjoint operators $\widehat{A}$ should be embedded into the set of these functions.

Necessary condition for the existence of a space of hidden states: existence of valuation functions $v : \mathcal{R}_{sa} \to \mathbb{R}$ such that

(1) $v(\widehat{A}) \in \mathrm{sp}(\widehat{A})$ for all $\widehat{A} \in \mathcal{R}_{sa}$ (**spectrum rule**),

(2) For all bounded Borel functions $g : \mathbb{R} \to \mathbb{R}$, we have $v(g(\widehat{A})) = g(v(\widehat{A}))$ (**FUNC principle**).

The Kochen-Specker theorem

Kochen, Specker 1967: For $\mathcal{R} = \mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} \geq 3$, there are no valuation functions and hence no phase space model of QT.

D 2005: This also holds for all unital von Neumann algebras $\mathcal{R}$ without summands of type I_1 and I_2.

The proposition "the physical quantity A has a value in the (Borel) set Δ"is written as "$A \in \Delta$".

In quantum theory, physical quantitities are represented by self-adjoint operators in $\mathcal{B}(\mathcal{H})$. The spectral theorem shows that to each proposition "$A \in \Delta$", there exists a projection $\widehat{E}[A \in \Delta] \in \mathcal{P}(\mathcal{H})$.

The KS theorem is equivalent to the fact that in quantum theory we cannot consistently assign "true"or "false"to all propositions at once (or 1 resp. 0 to the projections corresponding to the propositions).

Contexts or Weltanschauungen

- There is no model of quantum theory in which all physical quantities have values at once. This rules out any naive realist picture of QT.
- Not surprisingly, there is no problem for *abelian* algebras. The operators in an abelian C^*-algebra can be written as continuous functions on the Gel'fand spectrum.
- Abelian subalgebras of $\mathcal{B}(\mathcal{H})$ are called **contexts**.
- Some kind of *contextual* model of QT is needed (but with good control of the relations between contexts).

The context category

We consider the category $\mathcal{V}(\mathcal{H})$ of non-trivial unital abelian von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. This is a poset and is called the **context category**.

We use von Neumann algebras (rather than C^*-algebras) since

- They have enough projections; their projection lattices are complete.
- The spectral theorem holds for von Neumann algebras, which gives the connection between propositions "$A \in \Delta$"and projections.

We exclude the trivial algebra $\mathbb{C}\widehat{1}$. The minimal elements in $\mathcal{V}(\mathcal{H})$ are of the form $V_{\widehat{P}} := \{\widehat{P}, \widehat{1}\}'' = \mathbb{C}\widehat{P} + \mathbb{C}\widehat{1}$.

The spectral presheaf

To each $V \in \mathcal{V}(\mathcal{H})$, assign its Gel'fand spectrum $\underline{\Sigma}(V)$, i.e., the set of algebra homomorphisms $\lambda : V \to \mathbb{C}$. Physically, the $\lambda \in \underline{\Sigma}(V)$ are multiplicative states of the algebra V.

If $V' \subseteq V$, we have a morphism $i_{V'V} : V' \to V$ and define

$$\begin{aligned} \underline{\Sigma}(i_{V'V}) : \underline{\Sigma}(V) &\longrightarrow \underline{\Sigma}(V') \\ \lambda &\longmapsto \lambda|_{V'}. \end{aligned}$$

$\underline{\Sigma}$ is a contravariant functor from the context category $\mathcal{V}(\mathcal{H})$ to the category Set, i.e. a **presheaf over** $\mathcal{V}(\mathcal{H})$. Notation: $\underline{\Sigma} = (\underline{\Sigma}(V))_{V \in \mathcal{V}(\mathcal{H})} \in \mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$.

We regard the **spectral presheaf** $\underline{\Sigma}$ as a quantum analogue of phase space. It represents the symbol Σ of the formal language.

Reformulation of the KS theorem

Thm. (Isham, Butterfield '98): The spectral presheaf $\underline{\Sigma}$ has no global sections, i.e., there are no morphisms $\underline{1} \to \underline{\Sigma}$. This is equivalent to the Kochen-Specker theorem.

The terminal object $\underline{1}$ in $\mathrm{Set}^{\mathcal{V}(\mathcal{H})^{op}}$ is the trivial presheaf. Since mappings $\underline{1} \to \underline{\Sigma}$ can be seen as elements or **points** of $\underline{\Sigma}$, this result means that the spectral presheaf $\underline{\Sigma}$ *has no points*.

Daseinisation of projections

Let $\widehat{P} = \widehat{E}[A \in \Delta] \in \mathcal{P}(\mathcal{H})$ be the projection corresponding to the proposition "$A \in \Delta$".

In order to relate $\widehat{P}$ to *all* the contexts $V \in \mathcal{V}(\mathcal{H})$, we define a mapping

$$\begin{aligned} \delta : \mathcal{P}(\mathcal{H}) &\longrightarrow (\mathcal{P}(V))_{V \in \mathcal{V}(\mathcal{H})} \\ \widehat{P} &\longmapsto (\delta(\widehat{P})_V)_{V \in \mathcal{V}(\mathcal{H})}, \end{aligned}$$

where

$$\delta(\widehat{P})_V := \bigwedge \{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \geq \widehat{P}\}.$$

That is, we approximate $\widehat{P}$ from above in each context. If $\widehat{P} \in V$, then $\delta(\widehat{P})_V = \widehat{P}$.

Daseinisation of projections

Each projection $\delta(\widehat{P})_V \in \mathcal{P}(V)$ defines a subset of the Gel'fand spectrum $\underline{\Sigma}(V)$ by

$$\delta(\widehat{P})_V \longmapsto \{\lambda \in \underline{\Sigma}(V) \mid \lambda(\delta(\widehat{P})_V) = 1\}.$$

In fact, for each V, we get a clopen subset of $\underline{\Sigma}(V)$. One can easily show that these subsets fit together to form a subobject of $\underline{\Sigma}$. We obtain a mapping

$$\delta : \mathcal{P}(\mathcal{H}) \longrightarrow \mathrm{Sub}\,\underline{\Sigma}$$

from the projection lattice into the subobjects of $\underline{\Sigma}$, which we call the **daseinisation** of P.

Daseinisation is injective and order-preserving.

Daseinisation and the topos of presheaves

Daseinisation sends projections to subobjects of $\underline{\Sigma}$. We can now use the fact that $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$ is a **topos**, which implies that the subobjects of a given object (here $\underline{\Sigma}$) form a **Heyting algebra**.

Hence, daseinisation maps propositions about a quantum system to a distributive lattice in a contextual manner.

One can show that

$$\delta(\widehat{P} \vee \widehat{Q}) = \delta(\widehat{P}) \vee \delta(\widehat{Q}),$$

but

$$\delta(\widehat{P} \wedge \widehat{Q}) \leq \delta(\widehat{P}) \wedge \delta(\widehat{Q}).$$

Not every subobject of $\underline{\Sigma}$ comes from a projection. For example, the subobject $\delta(\widehat{P}) \wedge \delta(\widehat{Q})$ is not of the form $\delta(\widehat{R})$ for any projection $\widehat{R}$ in general.

Natural transformations from operators

We want to represent physical quantities as natural transformations from the spectral presheaf to some presheaf related to the real numbers. Whenever $V' \subseteq V$, this will give a commutative diagram

$$
\begin{array}{ccc}
\underline{\Sigma}(V) & \xrightarrow{\underline{\Sigma}(i_{V'V})} & \underline{\Sigma}(V') \\
\Big\downarrow{\scriptstyle \breve{\delta}(\widehat{A})_V} & & \Big\downarrow{\scriptstyle \breve{\delta}(\widehat{A})_{V'}} \\
\underline{\mathcal{R}}(V) & \xrightarrow[\rho(i_{V'V})]{} & \underline{\mathcal{R}}(V')
\end{array}
$$

At each stage V, the mapping

$$\breve{\delta}(\widehat{A})_V : \underline{\Sigma}(V) \longmapsto \underline{\mathcal{R}}(V)$$

will be an evaluation, sending $\lambda_V \in \underline{\Sigma}(V)$ to a real number $\lambda_V(\widehat{A}_V)$.

We construct $\widehat{A}_V$ from a given $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$ by a certain approximation, generalising the daseinisation of projections.

Daseinisation of self-adjoint operators

Let $\widehat{A} \in \mathcal{B}(\mathcal{H})_{sa}$. From the spectral family $\widehat{E}^A = (\widehat{E}^A_\lambda)_{\lambda\in\mathbb{R}}$, we obtain a new spectral family in $\mathcal{P}(V)$ by defining

$$\forall\lambda \in \mathbb{R} : \widehat{E}_\lambda^{\delta(\widehat{A})_V} := \bigvee\{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \leq \widehat{E}^A_\lambda\}.$$

This gives a self-adjoint operator $\delta(\widehat{A})_V$, which is the smallest operator in V larger than $\widehat{A}$ in the so-called spectral order.

Similarly, we can define

$$\forall\lambda \in \mathbb{R} : \widehat{E}_\lambda^{\delta^i(\widehat{A})_V} := \bigwedge\{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \geq \widehat{E}^A_\lambda\}.$$

The corresponding operator $\delta^i(\widehat{A})_V$ approximates $\widehat{A}$ from below in the spectral order.

Spectral order

Def. (Olson '71, de Groote '04): Let $\widehat{A}, \widehat{B} \in \mathcal{B}(\mathcal{H})_{sa}$ with spectral families $\widehat{E}^A, \widehat{E}^B$. The **spectral order** is defined by

$$\widehat{A} \leq_s \widehat{B} :\Longleftrightarrow \forall \lambda \in \mathbb{R} : \widehat{E}^A_\lambda \geq \widehat{E}^B_\lambda .$$

- The spectral order is a partial order on the self-adjoint operators in $\mathcal{B}(\mathcal{H})$.
- On projections, the spectral order $<_s$ and the usual order $<$ coincide.
- Equipped with the spectral order, $\mathcal{B}(\mathcal{H})_{sa}$ becomes a boundedly complete lattice.
- The spectral order is coarser than the usual order on self-adjoint operators, i.e. $\widehat{A} <_s \widehat{B} \Longrightarrow \widehat{A} < \widehat{B}$.
- If $\widehat{A}$ and $\widehat{B}$ commute, then $\widehat{A} <_s \widehat{B} \Longleftrightarrow \widehat{A} < \widehat{B}$.

The mapping

$$\delta_V : \mathcal{B}(\mathcal{H})_{sa} \longrightarrow V_{sa}$$
$$\widehat{A} \longmapsto \delta(\widehat{A})_V$$

adapts $\widehat{A}$ to the context V. The mapping δ_V is non-linear. We have

- $\mathrm{sp}(\delta(\widehat{A})_V) \subseteq \mathrm{sp}(\widehat{A})$.
- If $\widehat{A} = \widehat{P}$ is a projection, then $\delta(\widehat{A})_V$ is a projection, too, namely $\delta(\widehat{A})_V = \bigwedge\{\widehat{Q} \in \mathcal{P}(V) \mid \widehat{Q} \geq \widehat{P}\}$.
- $\delta(\widehat{A} + \alpha\widehat{I})_V = \delta(\widehat{A})_V + \alpha\widehat{I}$.
- If $\alpha \geq 0$, then $\delta(\alpha\widehat{A})_V = \alpha\delta(\widehat{A})_V$.
- $\delta(\widehat{A})_V$ is not a function of $\widehat{A}$ in general.

Analogous properties hold for δ^i_V.

The presheaf of order-reversing functions

We saw that for all $V' \subseteq V$, we have $\delta(\widehat{A})_{V'} \geq \delta(\widehat{A})_V$.

We define

$$\breve{\delta}(\widehat{A})_V : \underline{\Sigma}(V) \longrightarrow \underline{\mathcal{R}}(V)$$
$$\lambda \longmapsto \{\lambda|_{V'}(\delta(\widehat{A})_{V'}) \mid V' \subseteq V\}.$$

This is an order-reversing, real-valued function on the set $\downarrow V = \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}$.

These functions form a presheaf (Jackson '06) which we denote by $\underline{\mathbb{R}^{\succeq}}$. The restriction is simply given by restriction of the order-reversing functions.

By construction, $\breve{\delta}(\widehat{A})$ is a natural transformation from $\underline{\Sigma}$ to $\underline{\mathbb{R}^{\succeq}}$.

The k-construction on $\underline{\mathbb{R}^{\succeq}}$

The presheaf $\underline{\mathbb{R}^{\succeq}}$ is a candidate for the quantity value object for quantum theory, representing the linguistic symbol $\mathcal{R}$.

The sum of two order-reversing functions is order-reversing and the zero function acts as a neutral element for addition. Using these facts, one can see that $\underline{\mathbb{R}^{\succeq}}$ is a commutative monoid-object in $\mathrm{Set}^{\mathcal{V}(\mathcal{H})^{op}}$.

Applying Grothendieck's k-construction, we obtain an abelian group-object $k(\underline{\mathbb{R}^{\succeq}})$. One can show that $k(\underline{\mathbb{R}^{\succeq}})$ incorporates both the approximation from above (daseinisation δ) and the one from below (δ^i) that we can apply to self-adjoint operators.

For that reason, we consider the presheaf $k(\underline{\mathbb{R}^{\succeq}})$ as the quantity value-object for QT.

Subobjects from pullbacks

A proposition of the form "$A \in \Delta$" refers to the real numbers, since $\Delta \subset \mathbb{R}$. The real numbers lie *outside* the formal language.

The real numbers are a part of the representation of the formal language in Set, i.e., when we describe the system as a classical system, but they are not contained in the representation in $\mathrm{Set}^{\mathcal{V}(\mathcal{H})^{op}}$, i.e., when we choose a quantum description.

Now that we have defined $\underline{\mathbb{R}^{\succeq}}$, we can construct subobjects of $\underline{\Sigma}$ by pullback: let $\underline{\Theta}$ be a subobject of $\underline{\mathbb{R}^{\succeq}}$, then $\breve{\delta}(\widehat{A})^{-1}(\underline{\Theta})$ is a subobject of $\underline{\Sigma}$.

In this way, we get a topos-internal construction of propositions that do not refer to the real numbers. The 'meaning' of such propositions must be discussed from 'within the topos'.

Pure states and truth objects

In classical theory, a pure state is nothing but a point of phase space, i.e., a point of the state object that represents the symbol Σ.

Since the spectral presheaf $\underline{\Sigma}$ has no points, we must use another description for (pure) states, namely by certain elements of $P(P\underline{\Sigma})$. (In classical theory, both descriptions agree.)

Let ψ be a unit vector in Hilbert space. For each $V \in \mathcal{V}(\mathcal{H})$, we define

$$\begin{aligned} \mathbb{T}^{\psi}(V) &:= \{S \subseteq \underline{\Sigma}(V) \mid \langle\psi|\widehat{P}_S|\psi\rangle = 1\} \\ &= \{S \subseteq \underline{\Sigma}(V) \mid \widehat{P}_S \geq \delta(\widehat{P}_{\psi})_V\}. \end{aligned}$$

We call $\mathbb{T}^{\psi} = (\mathbb{T}^{\psi}(V))_{V\in\mathcal{V}(\mathcal{H})}$ the **truth object** corresponding to ψ.

The subobject classifier in $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$

The subobject classifier $\underline{\Omega}$ in a topos of presheaves is the presheaf of **sieves**.

A sieve in a poset like $\mathcal{V}(\mathcal{H})$ is particularly simple: let $V \in \mathcal{V}(\mathcal{H})$. A sieve α on V is a collection of subalgebras $V' \subseteq V$ such that, whenever $V' \in \alpha$ and $V'' \subset V'$, then $V'' \in \alpha$ (so α is a downward closed set).

The **maximal sieve** on V is $\downarrow V = \{V' \in \mathcal{V}(\mathcal{H}) \mid V' \subseteq V\}$.

A truth value is a global section of the presheaf $\underline{\Omega}$.

The global section consisting entirely of maximal sieves is interpreted as 'totally true', the global section consisting of empty sieves as 'totally false'.

Truth values from truth objects

We saw that subobjects of $\underline{\Sigma}$ represent propositions about the physical system under consideration, and that states are represented by truth objects.

Let $\underline{S} \in \mathrm{Sub}\,\underline{\Sigma}$ be such a subobject, and let $\mathbb{T}^\psi$ be a truth object.

Let

$$\nu(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi)_V := \{V' \subseteq V \mid \underline{S}(V') \in \mathbb{T}^\psi(V')\}.$$

One can show that this is a sieve on V. Moreover, for varying V, these sieves form a global section

$$\nu(\ulcorner \underline{S} \urcorner \in \mathbb{T}^\psi) \in \Gamma\underline{\Omega}.$$

This is the truth value of the proposition represented by $\underline{S}$, given by the truth object $\mathbb{T}^\psi$.

Open problems and goals

There are many interesting open questions. Some of the things we are working on are:

- Description of commutators within the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{op}}$.
- Topos formulation of uncertainty relations.
- Superposition of states.
- Composite systems and entanglement.
- Internal vs. external formulations.
- ...

Reference:

A.Döring, C. J. Isham, "A Topos Foundation for Theories of Physics I-IV", quant-ph/0703060, 62, 64 and 66