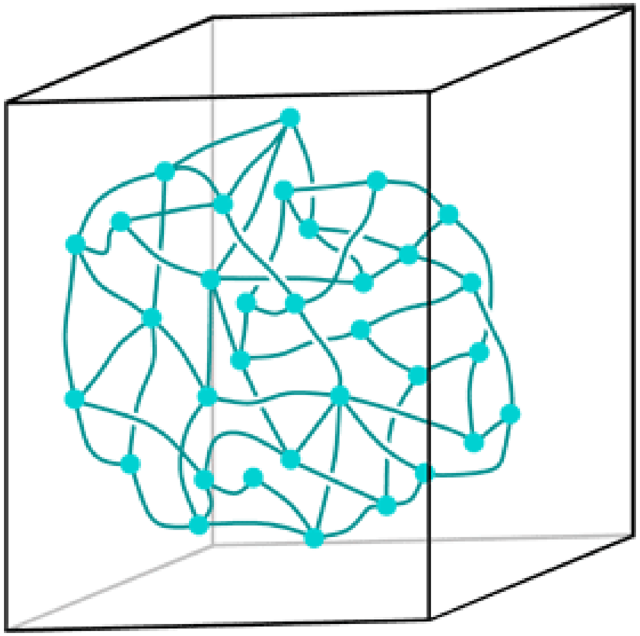
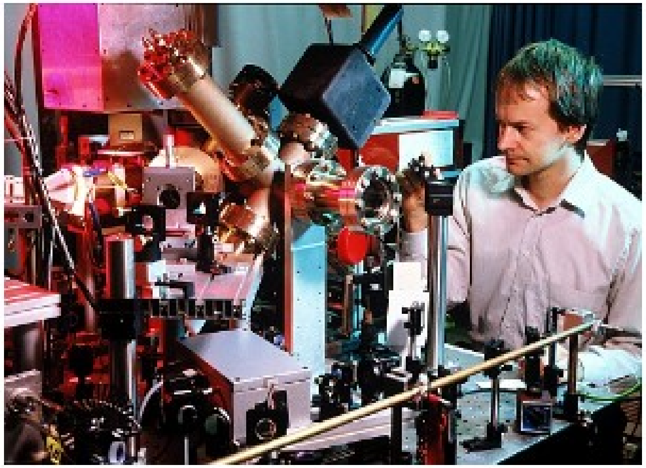
Permutational Quantum Computation



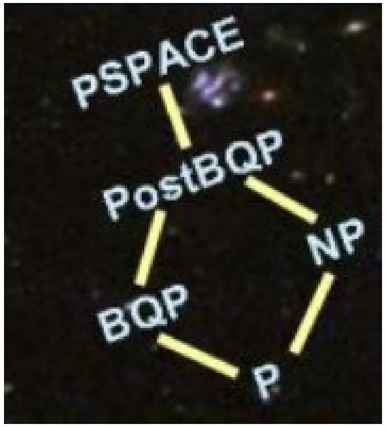
Stephen Jordan

Why formulate new models?

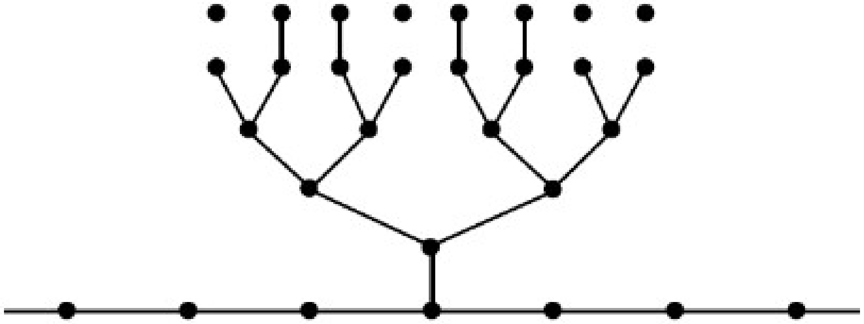
Implementation



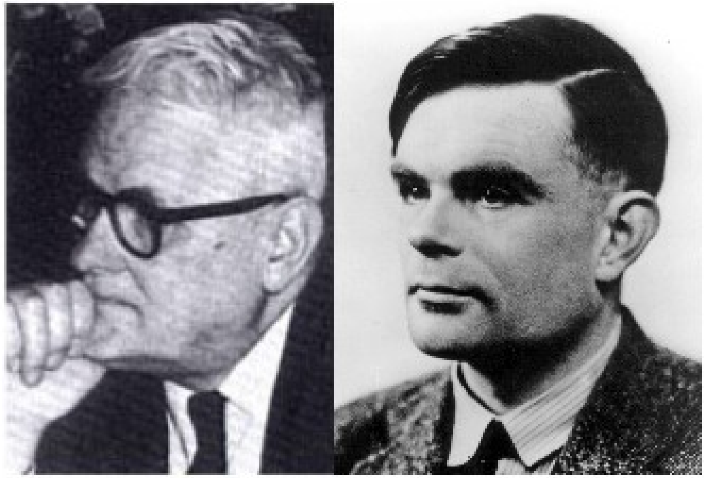
Complexity Theory



Quantum Algorithms



Church-Turing Thesis



- what is the model?
  - start with spins of known total angular momentum
  - permute the particles around
  - measure total angular momentum
  - direct analogue to topological quantum computation
- what can it do?
  - approximate irreps of S_n
  - approximate Ponzano-Regge invariant
  - give us a new complexity class?

Angular momentum of n spins

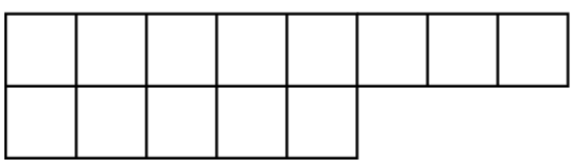
$$\vec{S}_j = \frac{1}{2}\begin{pmatrix} \sigma_x^{(j)} \\ \sigma_y^{(j)} \\ \sigma_z^{(j)} \end{pmatrix}$$

$$\vec{S} = \sum_{i=1}^{n} \vec{S}_i$$

$$S^2 = \vec{S} \cdot \vec{S}$$

$$S^2 |j\rangle = j(j+1)|j\rangle$$

- S^2 commutes with any permutation
- the eigenspaces of S^2 transform as irreducible representations of S_n
- The Young diagrams have two rows:



- The overhang is *2j*

- what about a basis for the representations?
- Example: 3 particles

$$\left.\begin{array}{l} (\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 \\ (\vec{S}_1 + \vec{S}_2)^2 \\ Z_1 + Z_2 + Z_3 \end{array}\right\}$$

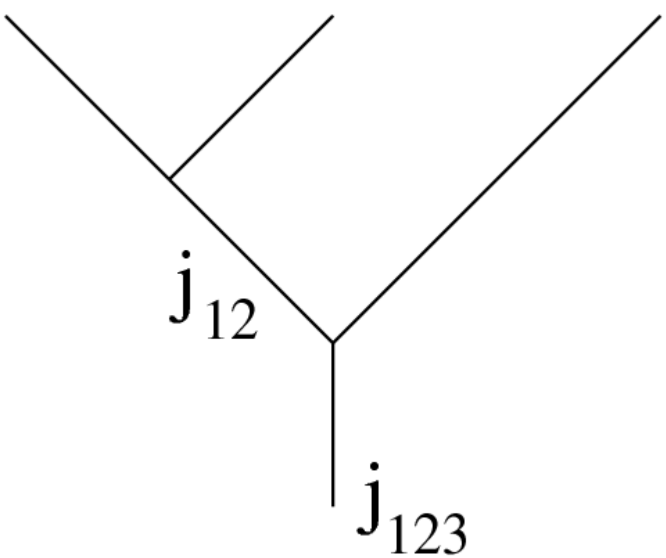
complete set of commuting observables

- How do the representations of S_3 look in this basis?
- $(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2$ tells us which irrep
- $(\vec{S}_1 + \vec{S}_2)^2$ labels the basis states within an irrep
- $(Z_1 + Z_2 + Z_3)$ is an irrelevant degree of freedom

- We have a choice between bases:

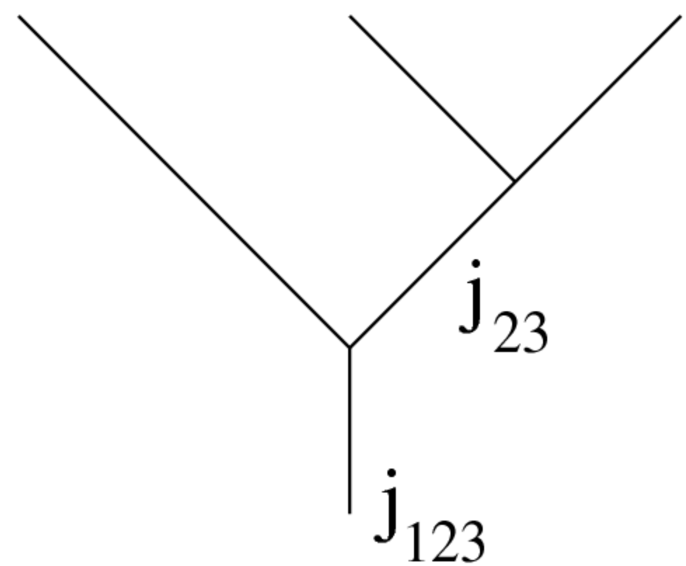
$$(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2$$

$$(\vec{S}_1 + \vec{S}_2)^2$$

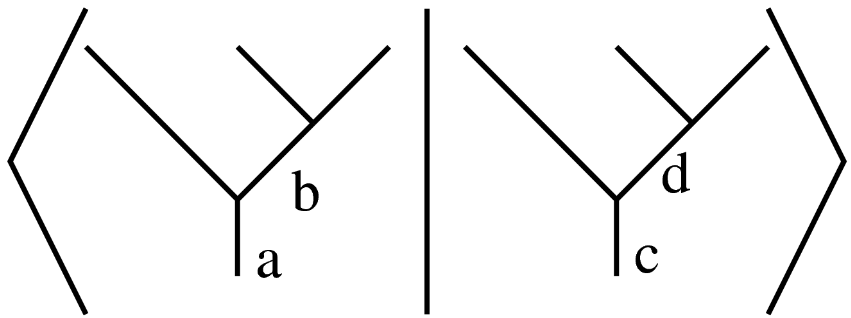


$$(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2$$

$$(\vec{S}_2 + \vec{S}_3)^2$$

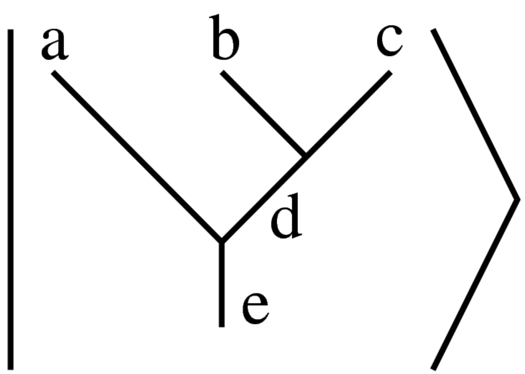


- For a given tree, different labellings correspond to orthogonal states

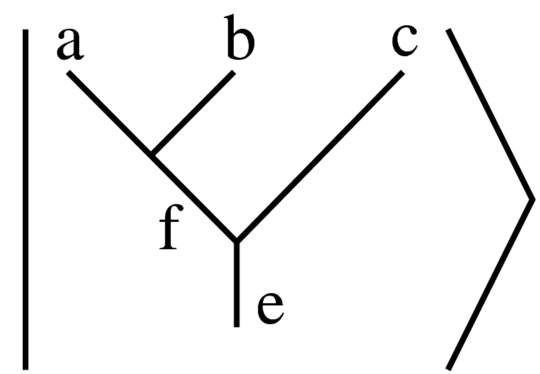


$$= \delta_{ac}\ \delta_{bd}$$

- Different trees are related by recoupling tensors



$$= \sum_{f} \begin{bmatrix} a & b & f \\ c & e & d \end{bmatrix}$$



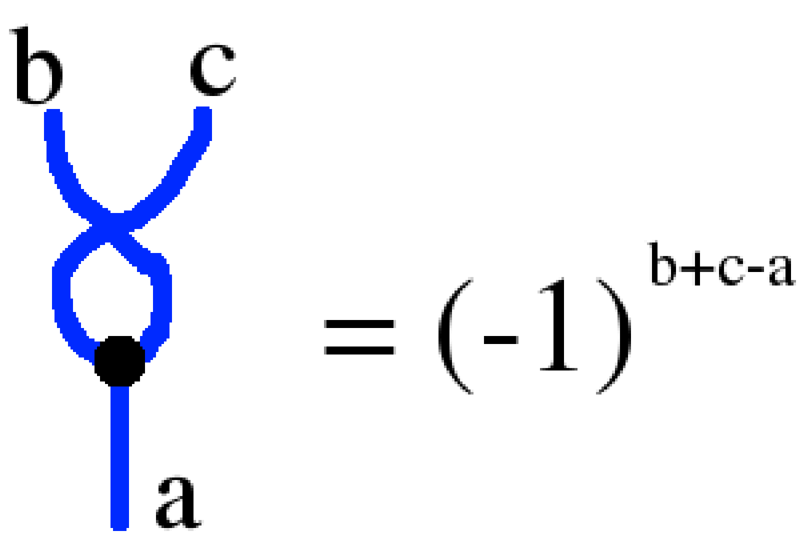
- The recoupling tensors are:

$$\left[\begin{array}{ccc} a & b & f \\ c & e & d \end{array}\right] = (-1)^{a+b+c+f}\sqrt{(2d+1)(2f+1)}\left\{\begin{array}{ccc} a & b & f \\ c & e & d \end{array}\right\}$$

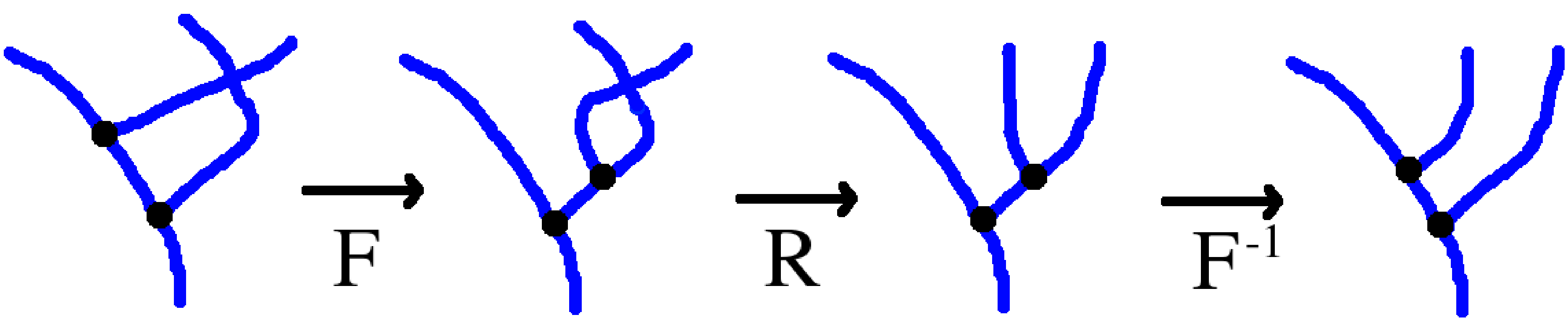
- The *6j* symbols $\left\{\begin{array}{ccc} a & b & f \\ c & e & d \end{array}\right\}$ can be computed

  in $\mathrm{poly}(a+b+c+d+e+f)$ time using the Racah formula.

- These states have this exchange symmetry:



- This plus recoupling tells us everything about permutation.



- We now have a model of computation:

  1) Prepare a basis state from some complete set of commuting angular momentum operators.

  2) Permute the qubits.

  3) Measure some other complete set of commuting angular momentum operators

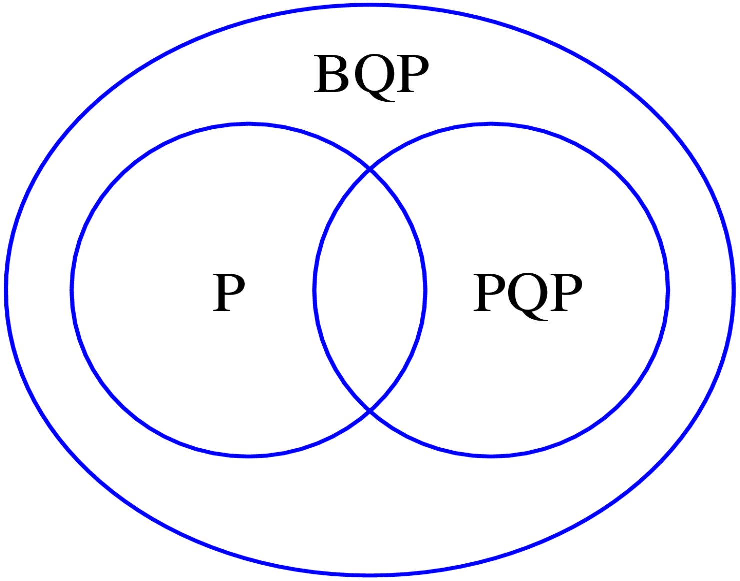
- Variant: include phases

[See Also Marzuoli et al.]

Topological	Permutational
Anyons	Spin-1/2
Braid (B_n)	Permute (S_n)
Fuse	Measure Angular Momentum
Braided Tensor Category	Racah-Wigner Tensor Category

How Powerful Is It?

- What I think:



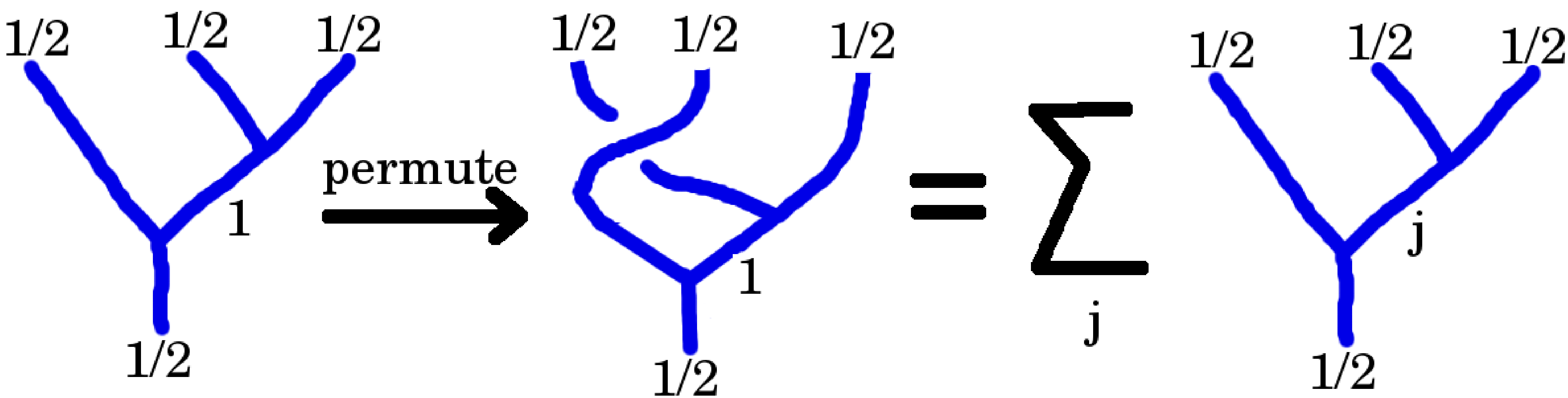
- What I know:

$$\mathrm{PQP} \subset \mathrm{BQP}$$

Can approximate irreps of S_n and simulate certain special cases of Ponzano-Regge.

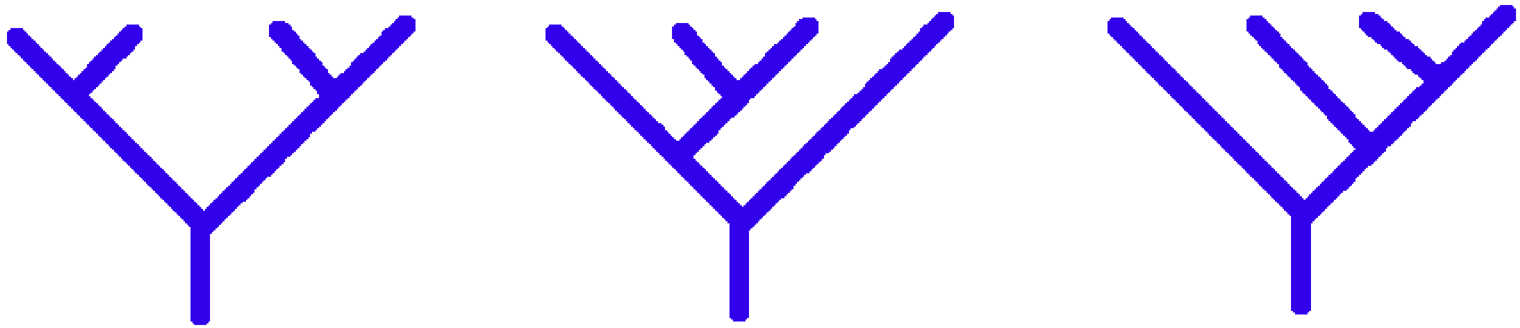
Algorithm for Symmetric Group

- permutation induces a linear transformation:

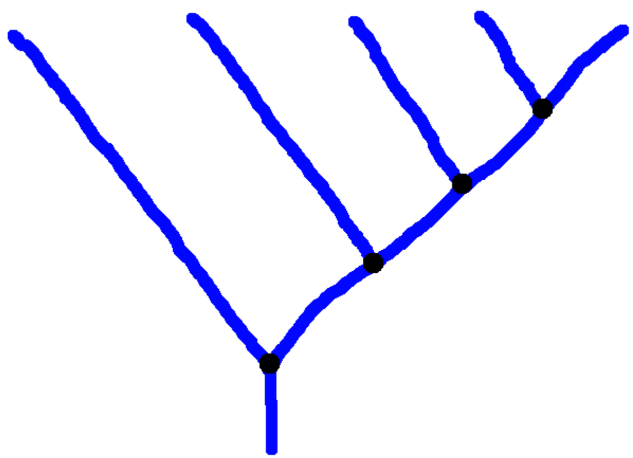


- This map from permutations to linear transformations is a representation of S_n

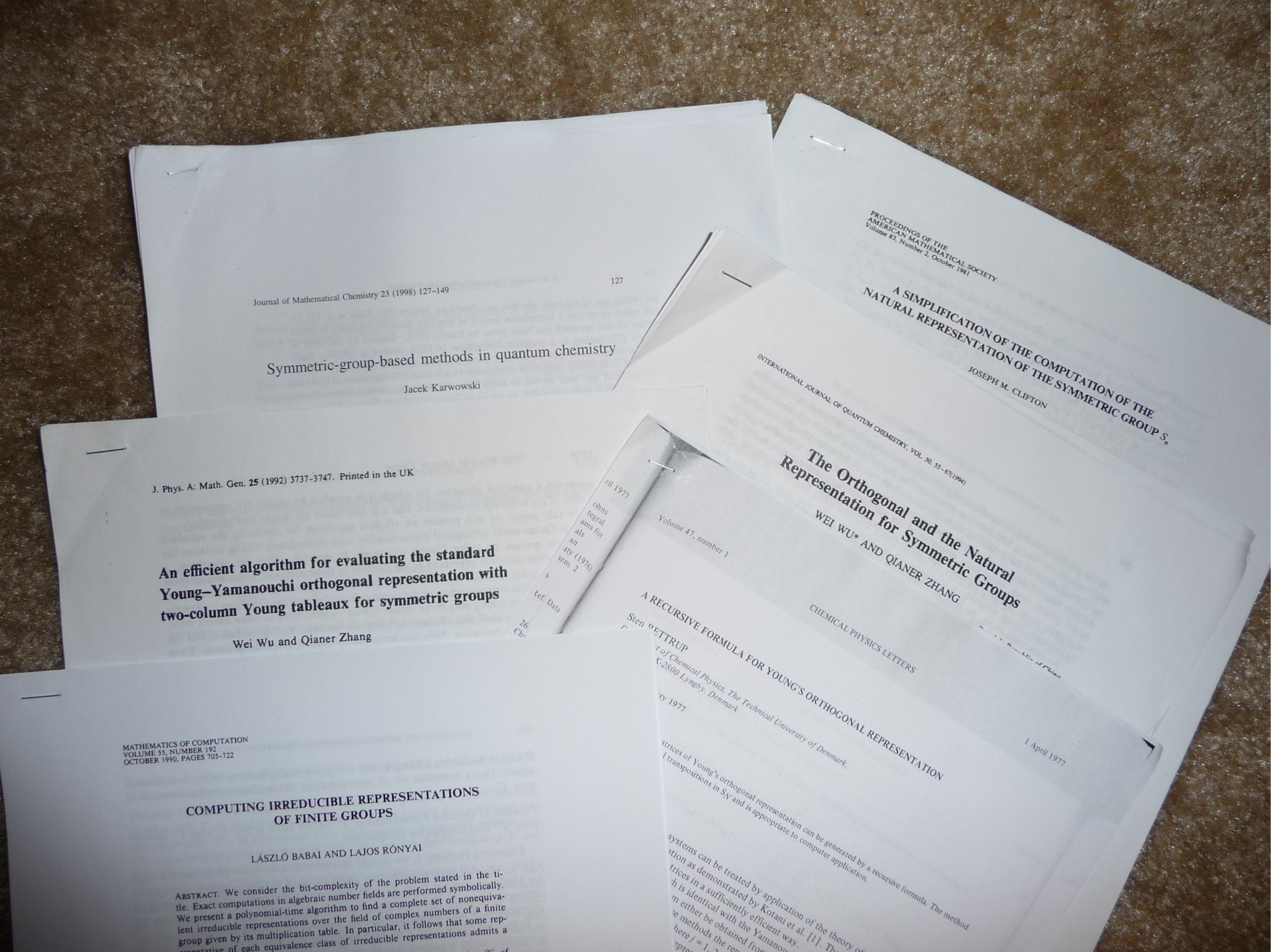
A choice of tree is a choice of basis.



If we choose this type of tree



then the representation of S_n is in Young's orthogonal form.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Symmetric-group-based methods in quantum chemistry
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NATURAL REPRESENTATION OF THE SYMMETRIC GROUP S_n
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The Orthogonal and the Natural
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WEI WU* AND QIANER ZHANG
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An efficient algorithm for evaluating the standard
Young–Yamanouchi orthogonal representation with
two-column Young tableaux for symmetric groups
Wei Wu and Qianer Zhang
Volume 47, number 1
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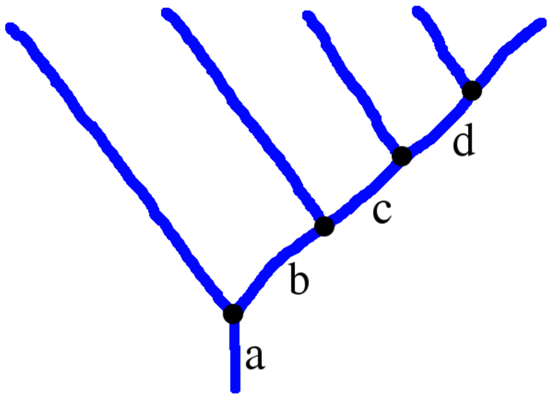
1 April 1977
A RECURSIVE FORMULA FOR YOUNG'S ORTHOGONAL REPRESENTATION
MATHEMATICS OF COMPUTATION
VOLUME 55, NUMBER 192
OCTOBER 1990, PAGES 705–722
COMPUTING IRREDUCIBLE REPRESENTATIONS
OF FINITE GROUPS
LÁSZLÓ BABAI AND LAJOS RÓNYAI
ABSTRACT. We consider the bit-complexity of the problem stated in the ti-
tle. Exact computations in algebraic number fields are performed symbolically.
We present a polynomial-time algorithm to find a complete set of nonequiva-
lent irreducible representations over the field of complex numbers of a finite
group given by its multiplication table. In particular, it follows that some rep-

$\mathrm{PQP} \subset \mathrm{BQP}$

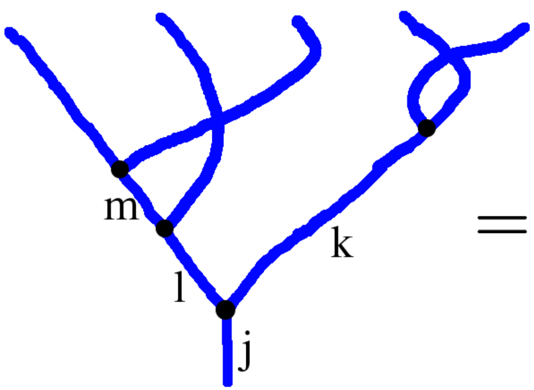
Proof Sketch:

work in this basis:



$= |abcd\rangle$

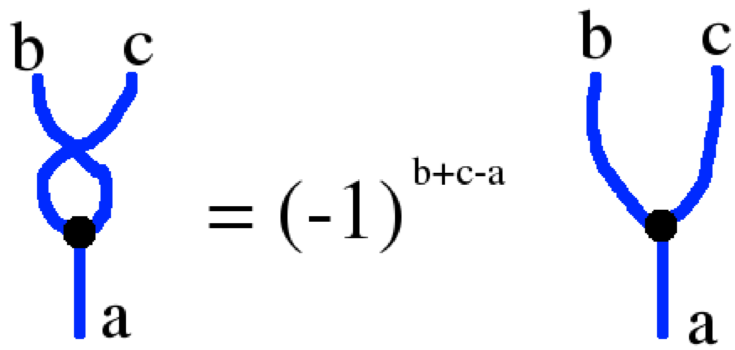
make any PQP state by polynomially many F and R moves



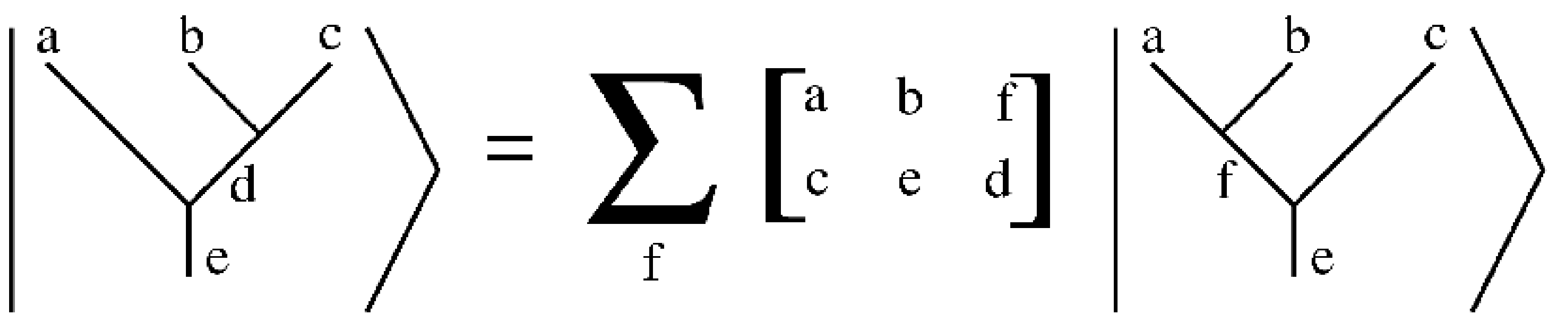
$= \sum_{abcd} \psi(abcd)|abcd\rangle$

use Hadamard test

- R is easy to implement: just a phase



- How about F?



  - it is sparse
  - we can efficiently compute the nonzero entries using the Racah formula

- We know how to implement any sparse row-computable Hamiltonian.
- From this we can implement any sparse row- and column-computable unitary.

$$H = \begin{bmatrix} 0 & U \\ U^\dagger & 0 \end{bmatrix}$$

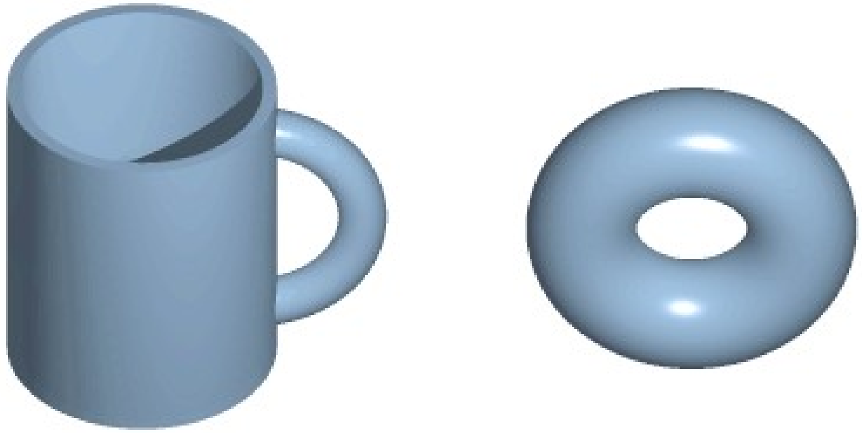
$$e^{iH\pi/2} = i \begin{bmatrix} 0 & U \\ U^\dagger & 0 \end{bmatrix}$$

■ End of Proof Sketch.

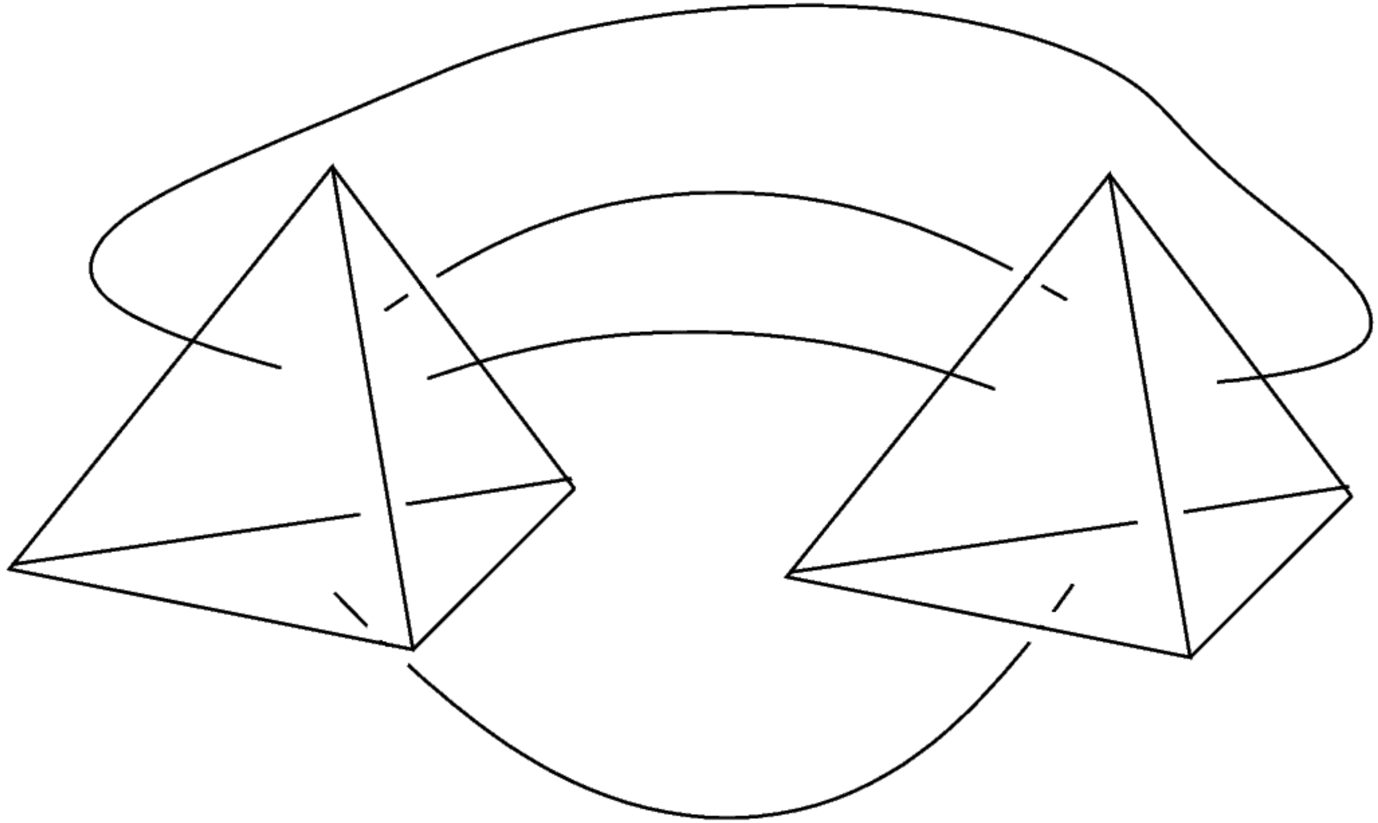
[Joint work with Pawel Wocjan]

Permutational Algorithms for 3-manifold invariants

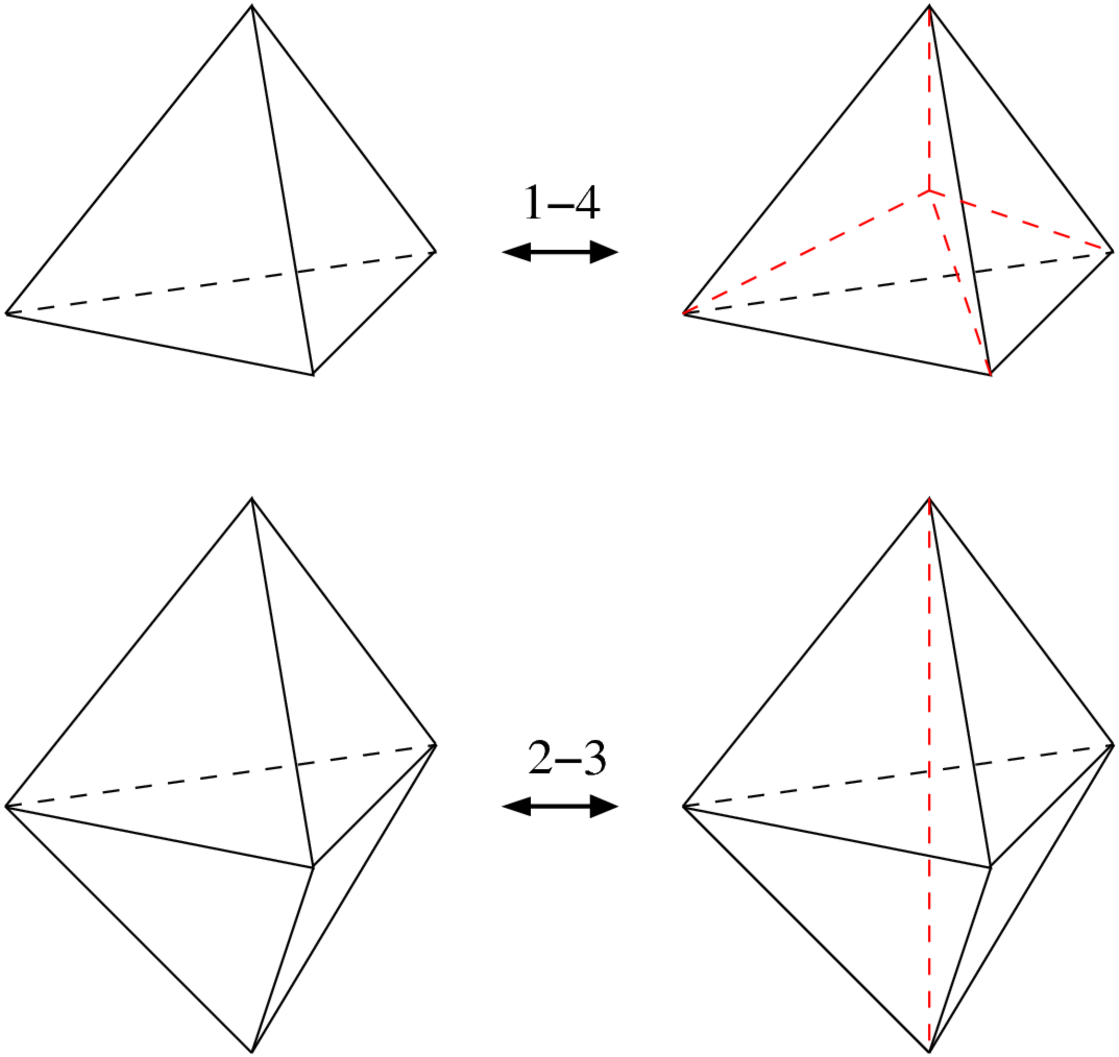
- 3-manifold: topological space locally like $\mathbb{R}^3$
- homeomorphism: a bijective, continuous map between manifolds whose inverse is continuous
- if a homeomorphism exists between a pair of manifolds we consider them equivalent



- How do we describe a 3-manifold to a computer?
- one way is to use a triangulation:
  - a set of tetrahedra
  - a gluing of the faces



- two triangulations yield equivalent 3-manifolds iff they are connected by a finite sequence of Pachner moves



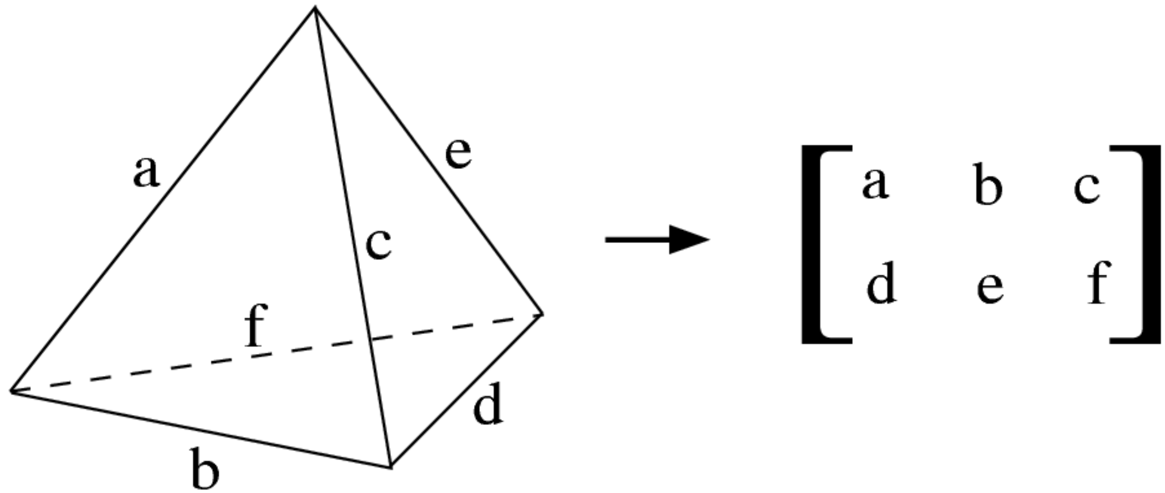
- deciding equivalence of manifolds is not easy!

	equivalence
2–manifolds	in P
3–manifolds	computable
4–manifolds	uncomputable

- partial solution:
  manifold invariant – if manifolds A and B are diffeomorphic then f(A) = f(B)

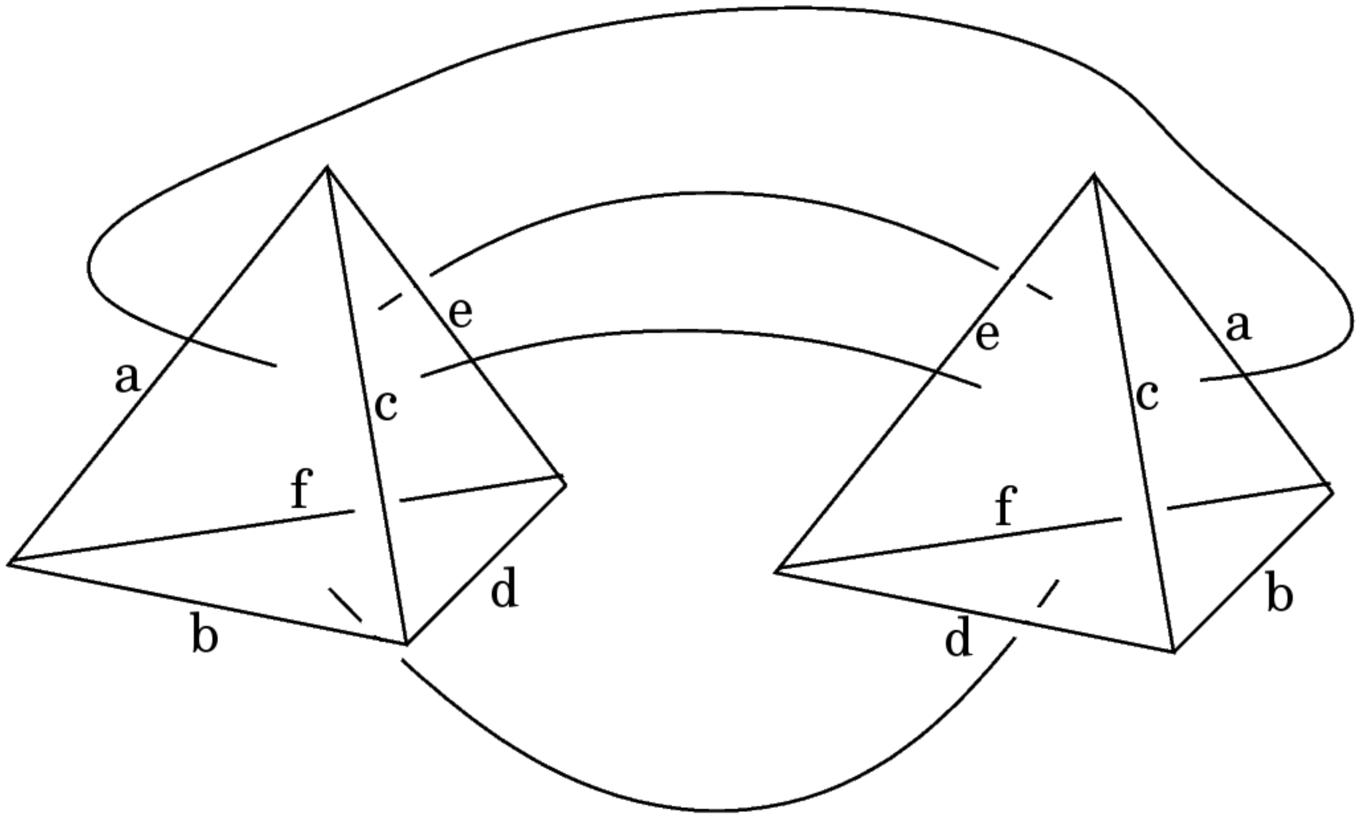
Ponzano-Regge Invariant

- to each tetrahedron, associate one recoupling tensor (one index to each edge)

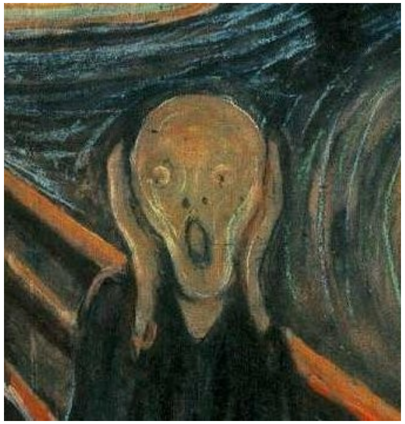


- for each glued face, contract (sum over) the corresponding indices

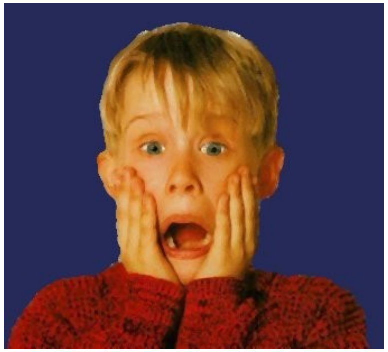
Example



$$\sum_{abcdef} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} c & b & a \\ f & e & d \end{bmatrix}$$



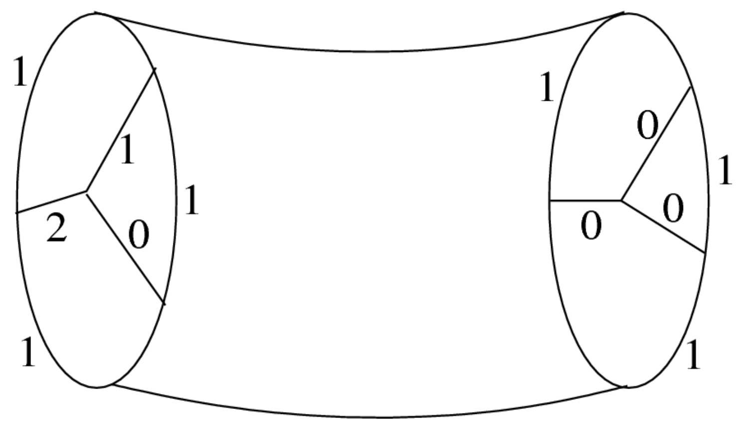
Divergences



- For some triangulations the Ponzano-Regge tensor network is an infinite sum
- Any pair of triangulations of a given manifold such that the sum has finitely many terms yield the same value for the Ponzano-Regge invariant

[Barrett & Naish-Guzman, '08]

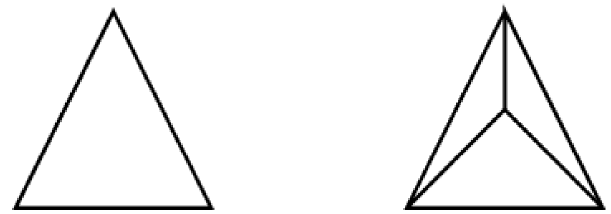
- Boundary is triangulated surface
- we have *j* labels on edges of triangulation
- these specify a geometry



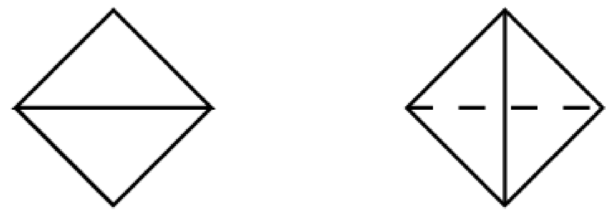
- the value of Ponzano-Regge tensor network is a transition amplitude between geometries
- sum over cobordisms, obtain model of topological quantum gravity

- Gluing of tetrahedra induces a change of triangulation:

subdivide

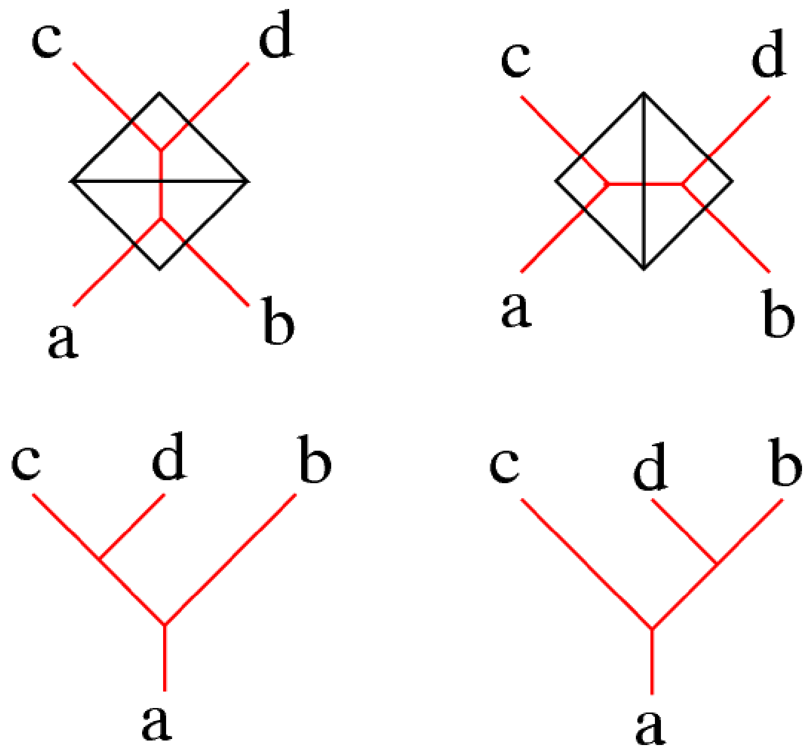


flip

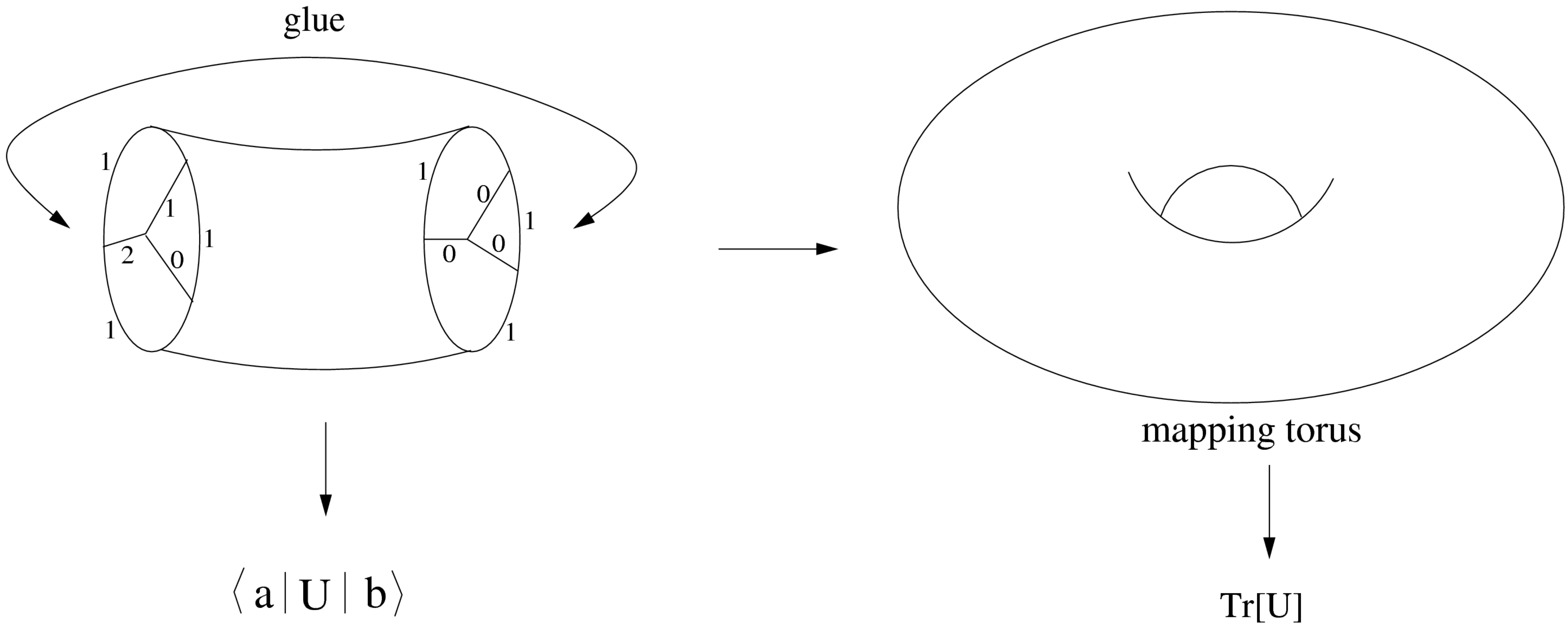


- The Ponzano-Regge amplitude corresponding to the flip move is approximable in PQP

- flip move is F move on the dual:



- Hence, in PQP we approximate Ponzano-Regge amplitude for 3-manifolds such that:
  - dual triangulation of boundaries are trees
  - tetrahedra glued two-faces at a time (flip moves)


glue
1
1
1
2
0
1
1
0
1
0
0
1
⟨ a | U | b ⟩
mapping torus
Tr[U]

One Clean Qubit

	B_n	S_n
Matrix Elements	BQP-complete	$\subset \mathrm{BQP}$
Characters	DQC1-complete	$\subset \mathrm{BPP}$

Normalized Characters of S_n in BPP Proof:

Theorem 1 (Roichman) *For any partitions* $\mu = (\mu_1, \ldots, \mu_l)$ *and* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *of* n*, the corresponding irreducible character of* S_n *is given by*

$$\chi_\mu^\lambda = \sum_\Lambda W_\mu(\Lambda)$$

where the sum is over all standard Young tableaux Λ *of shape* λ *and*

$$W_\mu(\Lambda) = \prod_{\substack{1 \le i \le k \\ i \notin B(\mu)}} f_\mu(i, \Lambda)$$

where $B(\mu) = \{\mu_1 + \ldots + \mu_r | 1 \le r \le l\}$ *and*

$$f_\mu(i,\Lambda) = \begin{cases} -1 & \text{box } i+1 \text{ of } \Lambda \text{ is in the southwest of box } i \\ 0 & i+1 \text{ is northeast of } i,\ i+2 \text{ is southwest of } i+1, \text{ and } i+1 \notin B(\mu) \\ 1 & \text{otherwise} \end{cases}$$

Theorem 2 (Greene, Nijenhuis, and Wilf) *With polynomial resources, one can sample uniformly from the standard Young Tableaux corresponding to a given shape (n-box Young diagram) using the Hook walk algorithm.*

End of Proof.

Is it Universal?

- So far we have seen:
  - $\mathrm{PQP} \subset \mathrm{BQP}$
  - probably $\mathrm{PQP} \not\subset \mathrm{P}$
- $\mathrm{PQP} = \mathrm{BQP}$?
- My intuition: No
  - no density
  - one clean qubit version is in BPP

Fault Tolerance

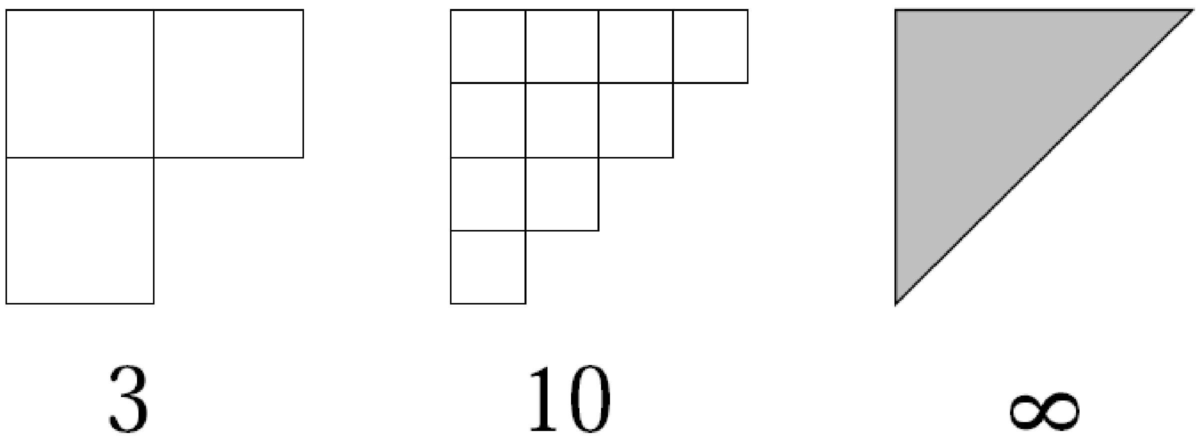
- Model is inherently discrete
  - like topological model
  - unlike circuit model
- Computation occurs in a noiseless subsystem for uniform magnetic fields
  - total angular momentum operators commute with magnetic field operators

Some Open Questions

- Implementation of phase measurements?
- *q*-deformed version of Ponzano-Regge algorithm (Turaev-Viro)
- How computationally powerful are spin-foam models for general topologies?

- For an exponentially large unitary matrix the average magnitude of the matrix elements is exponentially small.
- We approximate to polynomial precision?
- Is this trivial?
  - For random instances: yes.
  - In worst case: probably not.

- The normalized character tells us the average diagonal element.
- In certain cases this is large.



$$\frac{\chi_{\lambda_n}(\pi)}{d_{\lambda_n}} = C_\pi(\omega) n^{-|\pi|/2} + O(n^{-|\pi|/2-1})$$

[Biane]

Young's Orthogonal Form

$$\rho_\lambda(\sigma_i)\Lambda = \frac{1}{\tau_i^\Lambda}\Lambda + \sqrt{1 - \frac{1}{(\tau_i^\Lambda)^2}}\Lambda'$$

$$\rho_{\tiny\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}}(\sigma_2)\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = -\frac{1}{2}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \frac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

1	2	4
3	5	6