Quantum Processes and Computation

Assignment 1, Monday 21 Oct 2024

Solutions are shown after each question. Note some solutions are marked Sketch. These are intended to be instructions on how to work out the solution yourself, rather than an example of how you should answer this question on an exam.

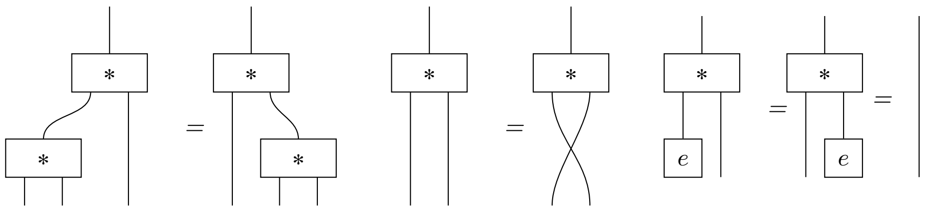
Exercise 1 (3.12): Give the diagrammatic equations of a process $*$ taking two inputs and one output that express the algebraic properties of being

1. associative: $x * (y * z) = (x * y) * z$
2. commutative: $x * y = y * x$
3. having a unit: there exists a process e (with no inputs) such that $x * e = e * x = x$

Note: x, y and z should not appear in your final diagrams. They are however useful in trying to figure out what the diagrammatic equation should be.

Begin Solution: . . . .

The diagrams corresponding to associativity, commutativity and having a unit e are respectively:

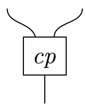


(1)

Distributivity can't be represented easily, because the lefthandside $(x + y) * z$ has 3 inputs, while $(x * z) + (y * z)$ has four inputs, 2 of them being equal (the z). This can be fixed by introducing a 'copy' operation.

End Solution . . . .

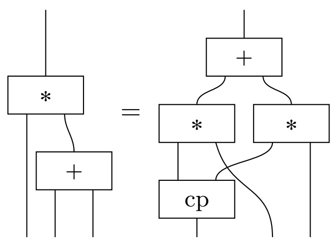
Exercise 2 (3.15): Using the copy operation:



$:: \quad x \mapsto (x, x)$

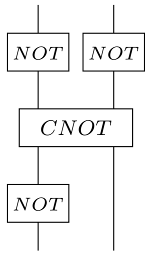
write down the diagram representing distributivity: $(x + y) * z = (x * z) + (y * z))$. Here, $+$ and $*$ are processes that take two inputs and and one output.

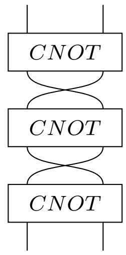
Begin Solution: . . . .



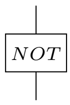
End Solution ....

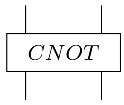
Exercise 3 (3.30): First compute the values of the following functions, then show that they can both be expressed by simpler diagrams:

(a) : 

(b) : 

where:



$:: \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$ and 

$:: \begin{cases} (0,0) \mapsto (0,0) \\ (0,1) \mapsto (0,1) \\ (1,0) \mapsto (1,1) \\ (1,1) \mapsto (1,0) \end{cases}$

Begin Solution: ....

These functions are defined in terms of how they map the elements of the set where they're defined, so it's enough to calculate it by "brute force". For the first one, calculating $(NOT \otimes NOT) \circ (CNOT) \circ (NOT \otimes Id)$ *step by step:*

$$(0,0) \mapsto (1,0) \mapsto (1,1) \mapsto (0,0) \tag{2}$$
$$(0,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,1) \tag{3}$$
$$(1,0) \mapsto (0,0) \mapsto (0,0) \mapsto (1,1) \tag{4}$$
$$(1,1) \mapsto (0,1) \mapsto (0,1) \mapsto (1,0) \tag{5}$$

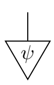
Which is equal to CNOT. For the second one, calculating CNOT,SWAP,CNOT,SWAP,CNOT:

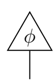
$$(0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \mapsto (0,0) \tag{6}$$
$$(0,1) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1) \mapsto (1,1) \mapsto (1,0) \tag{7}$$
$$(1,0) \mapsto (1,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,1) \mapsto (0,1) \tag{8}$$
$$(1,1) \mapsto (1,0) \mapsto (0,1) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1) \tag{9}$$

which is the SWAP map.

End Solution ....

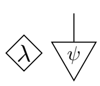
Exercise 4 (5.54): Let

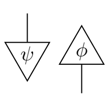


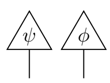
$= \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}$ and 

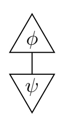
$= \begin{pmatrix} \phi_0 & \phi_1 \end{pmatrix}$

be respectively a 2-dimensional state, and 2-dimensional effect in the process theory of **linear maps**. Let λ be a number. Compute the matrices for the following processes

(i) 

(ii) 

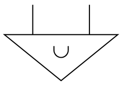
(iii) 

(iv) 

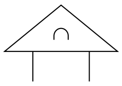
Begin Solution: ....

Sketch: *The first should be a size 2 column vector, the second a 2x2 matrix, and the third a size 4 row vector. The fourth should be a 1×1 matrix (or just a scalar) equal to $\psi^0\phi_0 + \psi^1\phi_1$.*
End Solution . . . .

Exercise 5 (5.58): The matrices for cups and caps in 2 dimensions are:

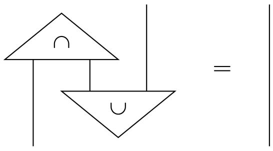


$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$



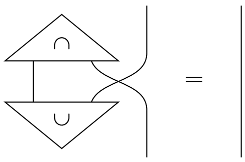
$$= \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

Verify the yanking equation

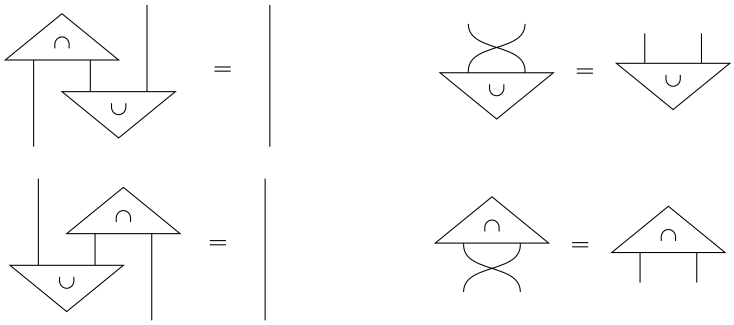


directly using the matrices of the 2-dimensional cup and cup by using the rules for sequential and parallel composition of matrices, i.e. show that $(\cap \otimes 1_{\mathbb{C}^2}) \circ (1_{\mathbb{C}^2} \otimes \cup) = 1_{\mathbb{C}^2}$ (where $1_{\mathbb{C}^2}$ is the 2×2 identity matrix).

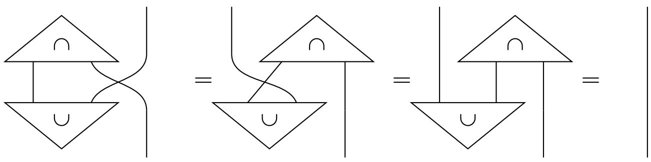
Exercise 6 (4.12): Prove that



follows from the following 4 equations:



Begin Solution: . . . .

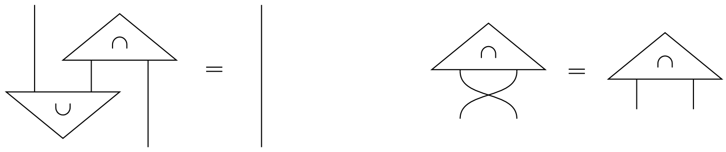


End Solution . . . .

Exercise 7 (4.14 in online version of PQP): Show that, in fact, we only need two equations for caps and cups. Namely, the following are equivalent:

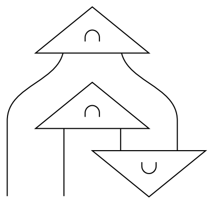
(i) a state and an effect satisfying:

(ii) a state and an effect satisfying:



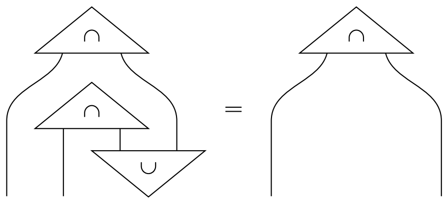
So in particular, if either eqs. (i) or eqs. (ii) hold, then all equations hold.

Hint: Most people find this (deceptively) difficult. If you are stuck proving (i) $\implies$ (ii), start by thinking about what you can do with this picture:

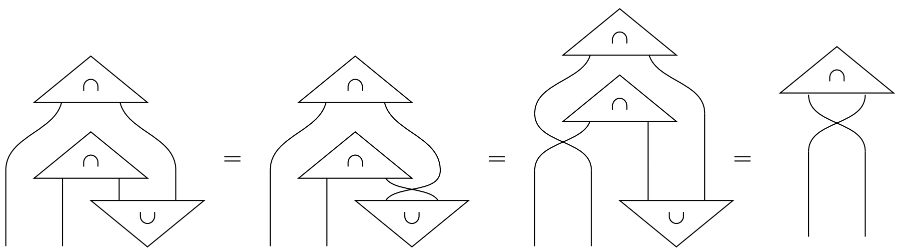


Begin Solution: ....

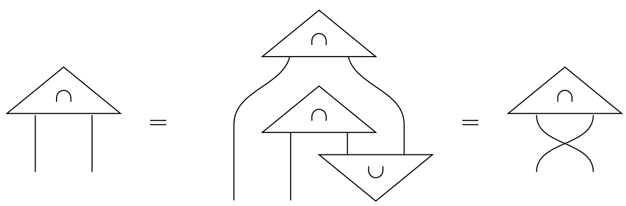
First assume the equations (i). Now, starting with the picture given as a hint, we can apply the first yanking equation to get:



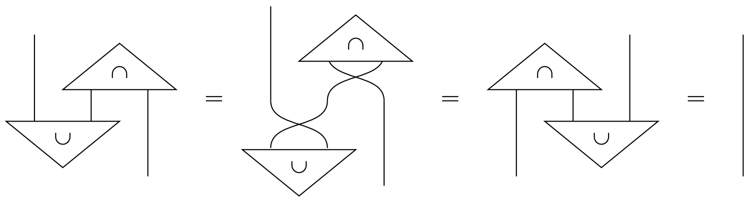
Alternatively, we can apply the second equation to introduce a swap, which allows us to the do the following:



where the second equation is just a diagram deformation (OCM) and the third equation is an application of the first yanking equation. Hence, we can conclude that:



Now that both caps and cups are invariant under swapping, it is easy to prove the reflected version of the first yanking equation:



So, we have proved that (i) implies (ii).

The proof that (ii) implies (i) follows as above, by reflecting all the diagrams vertically and interchanging cups and caps.

End Solution . . . .