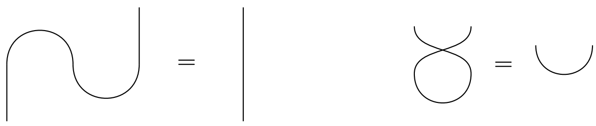
Quantum Processes and Computation

Assignment 2, Monday 28 Oct 2024

Solutions are shown after each question. Note some solutions are marked Sketch. These are intended to be instructions on how to work out the solution yourself, rather than an example of how you should answer this question on an exam.

Exercise 1: We can write the cup/cap for any dimension as a sum over ONB elements:

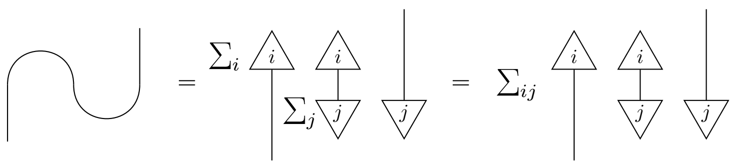
(i) Using this definition (and not the matrix form) verify the yanking equations.

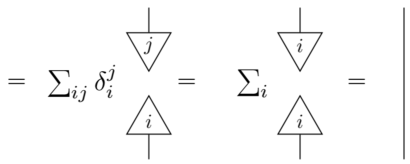


(ii) Compute the matrices for the cup and cap in 3 dimensions.

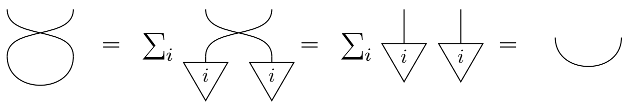
Begin Solution: . . . .

(i) These can be verified using the properties of ONBs and sums. For the first one:





For the second one:



(ii) This is a column vector, whose entries correspond to the basis elements $|i, j\rangle$ for $i, j \in \{0, 1, 2\}$. If $i = j$, the entry is a 1, otherwise it is a 0. This results in the following vector:

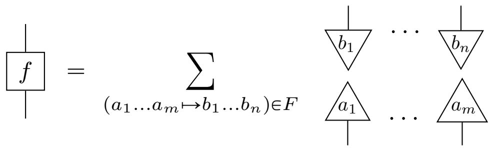
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

n.b. it is the same as the columns of a 3×3 *identity matrix, all stacked on top of each other. Cups in all dimensions have this same pattern. The cap is the transpose of the above, which is row vector:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

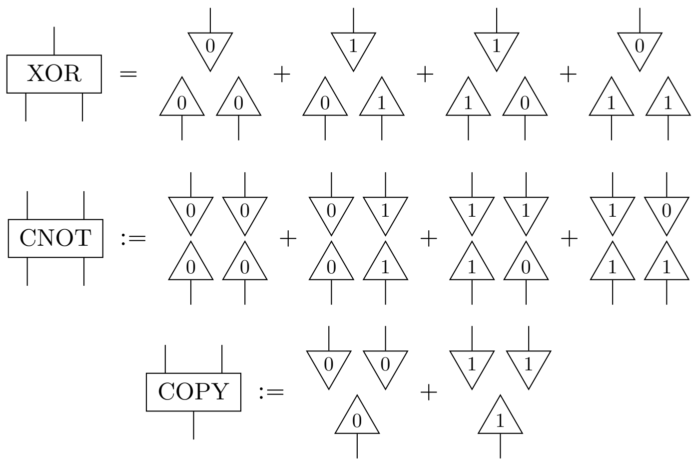
End Solution ....

Exercise 2 (5.86): This excercise is about encoding classical functions as linear maps using ONB states and effects, as explained in Section 5.3.4. For a function $F : \{0,1\}^m \to \{0,1\}^n$, we can define an associated linear map f as follows:

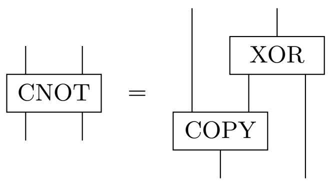


where the notation $(a_1 ... a_m \mapsto b_1 ... b_n) \in F$ means we are summing over the *graph of* F, i.e. the set of bitstrings $\{(a_1, \ldots, a_m, b_1, \ldots b_n) \mid F(a_1, \ldots, a_m) = (b_1, \ldots b_n)\}$.

Using this encoding, define:

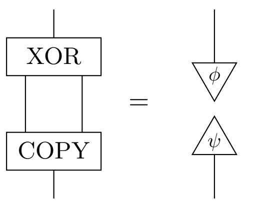


Show that



(Hint: try comparing the LHS to the RHS on all basis states, rather than writing out a big sum.)

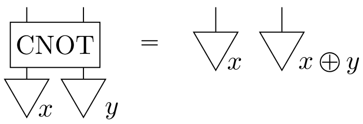
Next, find ψ and ϕ such that the following equation holds:



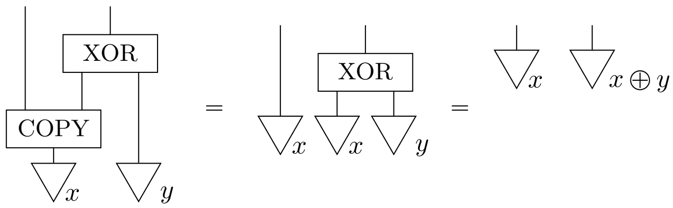
Although it might not look like much now, this equation will turn out to lie at the heart of the notion of *complementarity* which is an important part of the ZX-calculus.

Begin Solution: . . . .

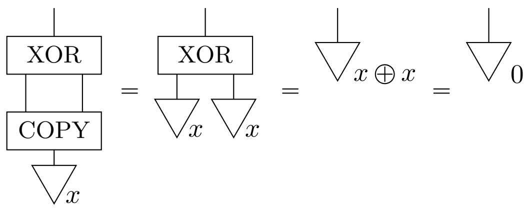
The first part can be done by plugging in basis states, and nothing that the LHS gives:



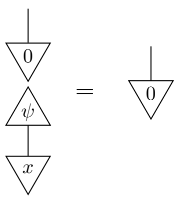
and the RHS gives:



If I evaluate the second diagram at a basis state, I get:

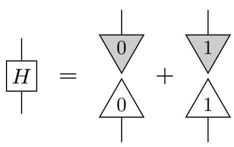


This tells me that $|\phi\rangle$ *should be* $|0\rangle$*. For* $\langle\psi|$*, I need the effect that "deletes" any basis state:* $\langle\psi| = \langle 0| + \langle 1| = \sum_i \langle i|$*. Then* $\langle\psi|0\rangle = \langle\psi|1\rangle = 1$*, so:*



End Solution . . . .

Exercise 3: Let the *Hadamard gate*, which sends the Z-basis to the X-basis be defined as follows:



where

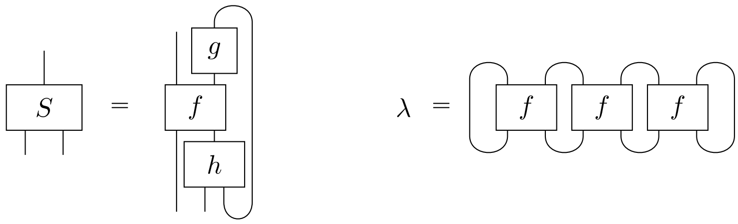
Compute the matrix of H. Show that $H = H^\dagger = H^T$. Using this fact (or otherwise) show that H also sends the X-basis back to the Z-basis.

Begin Solution: . . . .

Sketch: *The matrix can be computed by plugging in each of the 4 bras and kets of the computational basis. We can see it sends X-basis elements to Z-basis elements by applying the adjoint to both sides of the definition of H and using $H = H^\dagger$.*

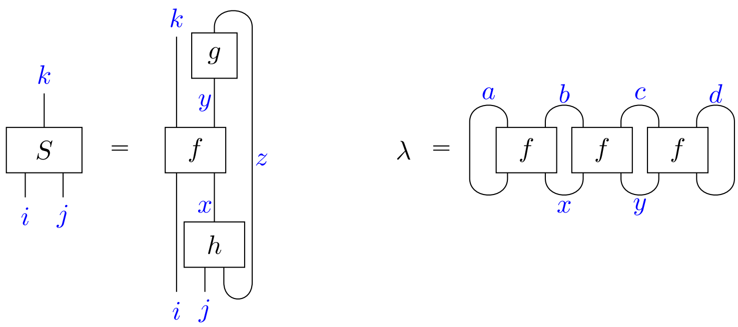
End Solution . . . .

Exercise 4: Write the following diagrams as tensor contractions, i.e. as sums over products of matrix elements f_{ij}^{kl}, etc.



Begin Solution: . . . .

Labelling the wires with some index names:



...we get:

$$S_{ij}^{k} = \sum_{xyz} f_{ix}^{ky} g_{y}^{z} h_{jz}^{x}$$

$$\lambda = \sum_{abcdxy} f_{ax}^{ab} f_{xy}^{bc} f_{yd}^{cd}$$

Note λ is a scalar, so all indices on the RHS are summed over.

End Solution . . . .