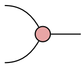
Quantum Software

Assignment 4, Michaelmas 2024

Solutions are shown after each question. Note some solutions are marked Sketch. These are intended to be instructions on how to work out the solution yourself, rather than an example of how you should answer this question on an exam.

Exercise 1 (3.2): In *Picturing Quantum Software*, it was shown that the X-spider with two inputs and one output gives XOR:

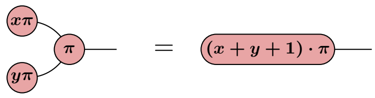


$$= \frac{1}{\sqrt{2}} \left(|0\rangle\langle 00| + |0\rangle\langle 11| + |1\rangle\langle 01| + |1\rangle\langle 10| \right)$$

What classical map would we get if we instead took the X-spider with 2 inputs, 1 output and a π phase?

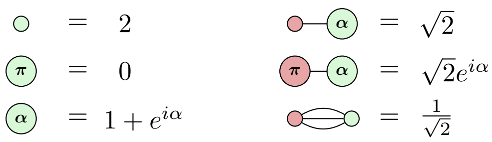
Solution: ....

The classical map is $\mathrm{NOT}(x \oplus y) = x \oplus y \oplus 1$, a.k.a. "$x = y$". That is, it sends $|xy\rangle$ to $|1\rangle$ iff $x = y$. This can be proven by concrete calculation, or (more easily) by expressing $|0\rangle$ and $|1\rangle$ as X-spiders with a zero and π phase, respectively.



End Solution ....

Exercise 2: In the lectures we often ignore scalar factors in ZX-diagrams. We can however represent any scalar we want with a ZX-diagram. For instance, we have:

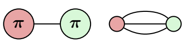


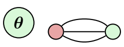
(1)

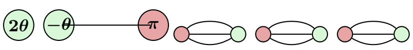
By combining the diagrams from (1), find a ZX-diagram to represent the following scalar values z:

1. $z = -1$.
2. $z = e^{i\theta}$ for any θ.
3. $z = \frac{1}{2}$.
4. $z = \cos\theta$ for any value θ.
5. Find a general description or algorithm to construct the ZX-diagram for any complex number z.

Solution: ....

1. 

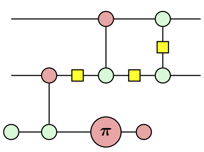
2. 

3. 
4. 

5. First fix a k such that for $z' = 1/\sqrt{2}^k z$ we have $|z'| \leq 1$. Then we can find phases α, β such that $z' = e^{i\alpha}\cos\beta$. Since we know how to write these three components as diagrams, we are then done.

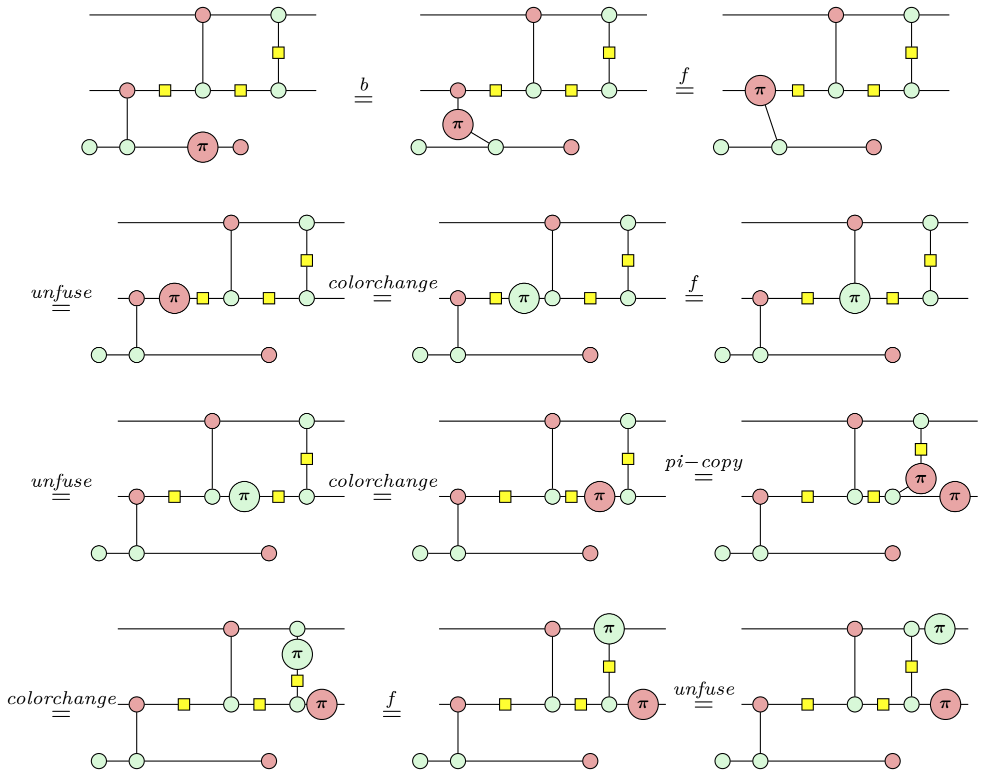
End Solution ....

Exercise 3: Using ZX-calculus rewrites, help the poor trapped π phase find it's way to an exit (i.e. an output).



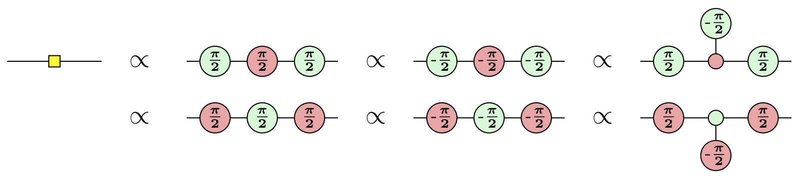
Note that it might be leaving with friends.

Solution: ....
Here is one solution:



End Solution ....

Exercise 4: The Euler decomposition from the lecture is just one possible way to write the Hadamard in terms of spiders. There is in fact an entire family of representations that will also be useful to note:

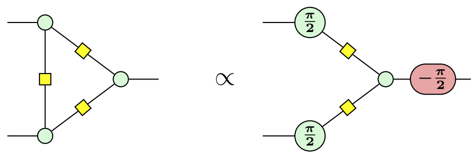


(2)

Prove that all the equations of (2) hold in the ZX-calculus, by using the top-left decomposition and the other rewrite rules of the ZX-calculus we have seen so far.

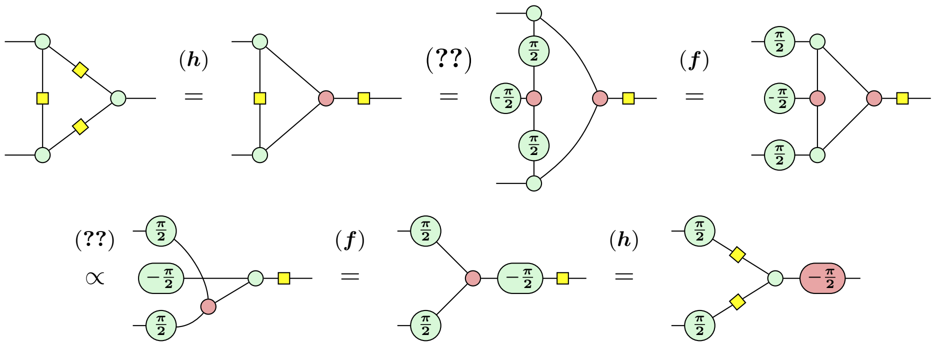
Solution: ....
Sketch: Graphical proofs, using colour change, spider, and π-copy rules. It's probably useful to prove a lemma relating the Z phase state of $\pi/2$ to the X phase state of $-\pi/2$.
End Solution ....

Exercise 5: Prove the following base-case of the local complementation lemma using the ZX-calculus:



(3)

Hint: Push the top Hadamards up and decompose the middle Hadamard using one of Eq. (2) to reveal a place where you can apply strong complementarity.
Solution: ....



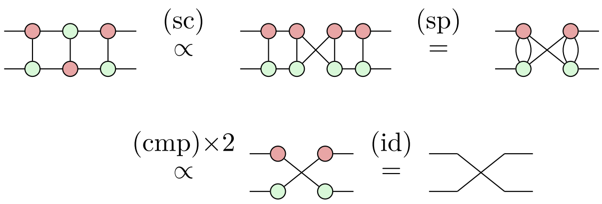
End Solution . . . .

How this result can be used to prove general local complementation is shown in Picturing Quantum Software.

Exercise 6 (3.7): In Section 3.2.4 of *Picturing Quantum Software*, it is shown how to prove 3 CNOT gates equals a SWAP gate, by applying the strong complementarity rule:

in the reverse direction (i.e. by replacing the RHS with the LHS). Show that there is an alternative proof, applying strong complementarity in the forward direction and applying complementarity (possibly multiple times).

Solution: . . . .



End Solution . . . .