Quantum Software

Assignment 5, Michaelmas 2024

Solutions are shown after each question. Note some solutions are marked Sketch. These are intended to be instructions on how to work out the solution yourself, rather than an example of how you should answer this question on an exam.

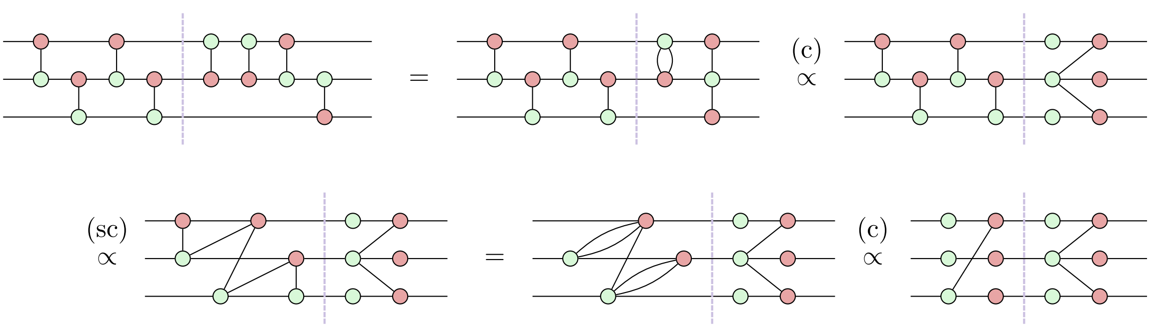
Exercise 1: Write down a CNOT circuit C with at least 3 qubits and at least 8 CNOT gates. Cut the circuit in two halves containing the first 4 CNOT gates and the last 4 CNOT gates. Simplify to parity normal form in two phases. First, form D_1 by simplifying each half individually to parity normal form. Then form D_2 by simplifying all of D_1 the rest of the way to parity normal form. What can you say about:

1. The number of forward-directed paths from each input to each output in C, D_1, and D_2?
2. The relationship between the bi-adjacency matrices of each half of D_1 and the bi-adjacency matrix of D_2?
3. The relationship between facts 1 and 2?

Note there is some ambiguity of what counts as a "forward-directed path", especially for C. This can be resolved as follows: for each diagram, choose a direction for all of the wires such that each Z spider has at most input and each X spider has at most one output. For CNOT gates, this means the wire connecting the two spiders should go from Z to X, not vice-versa.

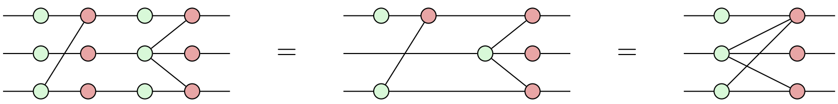
Solution: ....

Here is an example, where we first reduce the two halves to PNF separately:



(1)

Steps are labelled (c) and (sc) for complementarity and strong complementarity, respectively. The identity and spider fusion laws are also being used in most steps above. Then we compute the overall PNF:


(2)

Graphical proof mainly uses (strong) complementarity to eliminate all interior Z/X spiders. As for the questions:

1. The number of paths from a given input to an output is the same in all three diagrams, modulo 2.
2. The parity matrices from the two parts of (1) are:

$$P_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad P_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

   whereas the parity matrix for the overall PNF in (2) is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

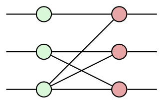
   This is the matrix product $P = P_2 P_1$ of the two parts:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

3. Matrix multiplication in $\mathbb{F}_2$ is the same thing as counting paths, modulo 2. Suppose P_i^k is the number of paths from the i-th qubit at the input to the k-th qubit in the middle of the circuit. Then, suppose Q_k^j is the number of paths from the k-th qubit in the middle to the j-th qubit at the outputs. Each path from the i-th input to the j-th output goes through some qubit in the middle, so I can count out all the paths (modulo 2) from i to j by summing over the middle qubit:
$$\sum_k P_i^k Q_k^j$$
but this is exactly matrix multiplication! Applying the strong complementarity rule preserves the number of paths while removing intermediate steps, so it is like a generalised, "graphical" matrix multiplication.

End Solution . . . .

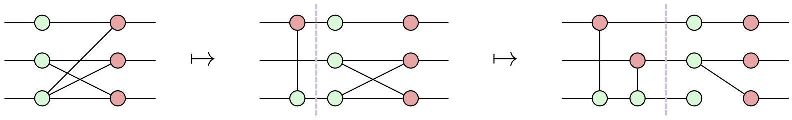
Exercise 2: In the lecture we saw how to reduce a parity normal form diagram to a CNOT circuit if its biadjacency matrix is invertible. Apply the Gaussian elimination procedure to the following diagram which does *not* have an invertible matrix to see what it reduces to:



Use this to argue that the diagram is not unitary.

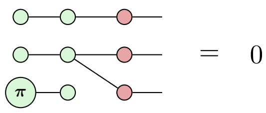
Solution: . . . .

When the matrix is not invertible, one of its columns will be a linear combination of the others, which means at some point in the procedure we'll end up with a zero column:
$$\begin{pmatrix}1&0&1\\0&1&1\\0&1&1\end{pmatrix} \overset{C_3=C_3+C_1}{\longmapsto} \begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix} \overset{C_3=C_3+C_2}{\longmapsto} \begin{pmatrix}1&0&0\\0&1&0\\0&1&0\end{pmatrix}$$
Following this Gaussian elimination graphically, we have:



(3)

The part to the right of the dashed line is clearly not unitary because it sends some non-zero inputs to zero, e.g.



Then, this implies that the whole diagram (3) is not unitary, because it is the composition of something unitary (the CNOT gates) with something non-unitary (the rest). Stated more formally, we can apply this lemma:

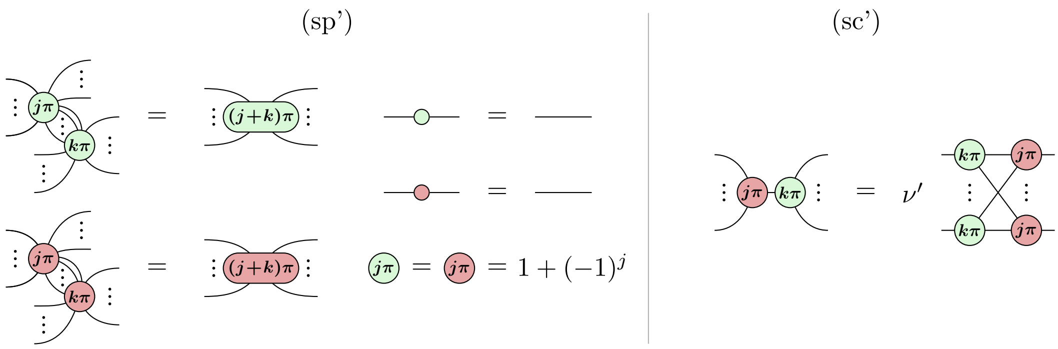
Lemma 1. For a unitary map U, UC is unitary if and only if C is.

Proof. Clearly if C is unitary, then so is UC. Conversely, if UC is unitary, then $C = U^\dagger UC = U^\dagger(UC)$ is a composition of unitaries, hence is itself unitary. □

In particular, if C is not unitary, than neither is UC.

End Solution . . . .

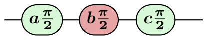
Exercise 3: In *Picturing Quantum Software*, an algorithm is given for reducing a phase-free ZX-diagram to generalised parity form using just the phase-free ZX rules. Consider the following extension to the rules accounting for π phases:



Prove (sc') using the rules of the (full) ZX-calculus. Describe a variation of the algorithm for phase-free diagrams that allows Z and X phases that are integer multiples of π. What do the normal form(s) look like?

Solution: . . . .
Sketch: The (sc') rule can be derived from combining strong complementarity and π-copy. The simplification then proceeds identically to the phase-free version given in the book. The only place a bit of care should be taken is in removing zero-legged spiders. If one of these is a π-spider, terminate with the whole thing going to zero.
End Solution . . . .

Exercise 4 (PQS 5.1): A single-qubit Clifford circuit is constructed out of just Hadamard and S gates. Show that any single-qubit Clifford circuit can be rewritten to the form



for some integers a, b and c. *Hint: We know that a single S or Hadamard can be brought to this form. So you just need to show that when you compose this normal form with an additional S or Hadamard gate that the resulting circuit also be brought to this normal form. You probably will want to make a case distinction on the value of b.*

Solution: . . . .
Sketch: Composing the normal form with S again gives a NF by spider fusion. When we compose a Hadamard we make a case distinction on the value of b. If $b = 0$, then a and c fuse, and we can just decompose the Hadamard into its ZXZ Euler decomposition to get a NF again after spider fusion. We can do something similar when $b = 2$, by moving the X gate out of the way first. If $b = 1$, we realise the normal form can also be written as $Z((c-1)\frac{\pi}{2})HZ((a-1)\frac{\pi}{2})$ by using the Euler decomposition in reverse. The result is then easily proven. The same when $b = 3$.
End Solution . . . .

Exercise 5 (PQS 5.2): Show that the following states are Clifford states. I.e. construct a Clifford circuit C that when applied to $|0\cdots 0\rangle$ gives the desired state.

a) $|1\rangle$

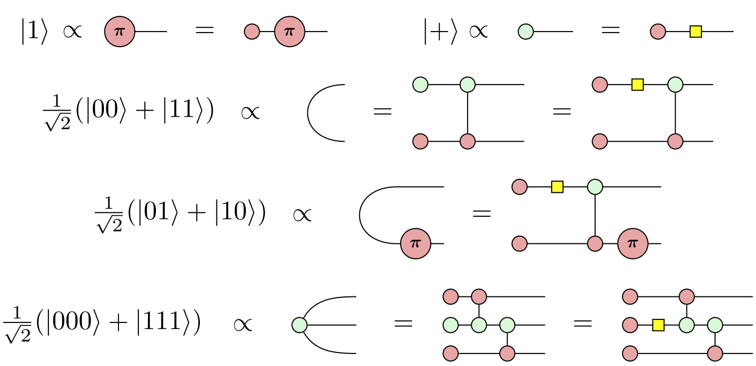
b) $|+\rangle$

c) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

d) $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

e) $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

Solution: . . . .
We can construct each of these states as follows, up to scalar factors, starting from the graphical form of the state and finding its gate decomposition into CNOT, H, and S (where $X = HS^2H$ is also treated as a basic gate):



End Solution . . . .