Picturing Quantum Entanglement ...in MBQC

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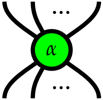
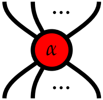
ZX-calculus

- The *ZX-calculus* is a formalism that studies diagrams built from three kinds of generators:



$$:= |0...0\rangle\langle 0...0| + e^{i\alpha}|1...1\rangle\langle 1...1|$$

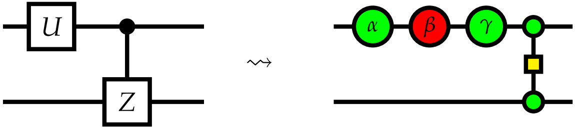


$$:= |+...+\rangle\langle +...+| + e^{i\alpha}|-...-\rangle\langle -...-|$$

$$:= |+\rangle\langle 0| + |-\rangle\langle 1|$$

ZX-calculus in QC

- Admits an encoding of circuits:



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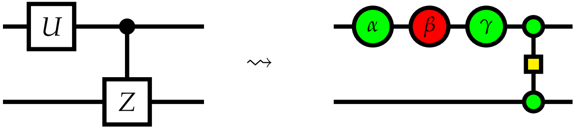
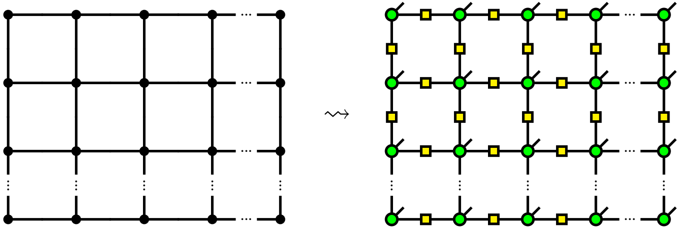


- ...and MBQC:



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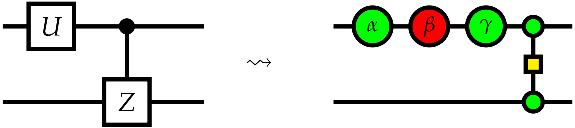

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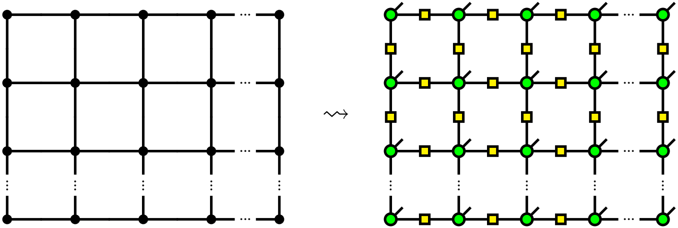
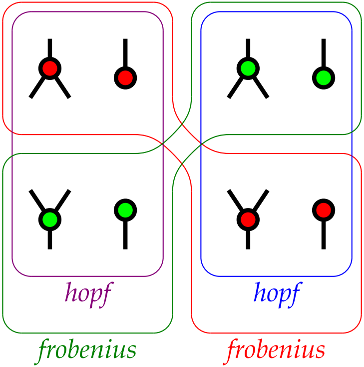


- ...and a means of translating between the two. (⇐)

Algebraic structure

- All of its power comes from its underlying *algebraic structures*:



- ...which have been studied extensively in category theory and representation theory.

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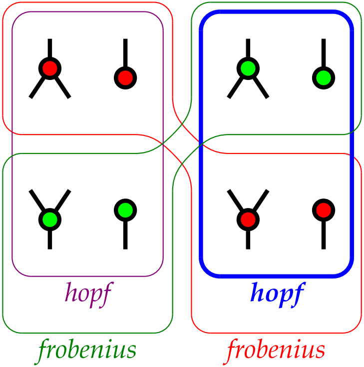

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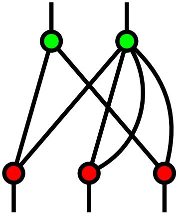
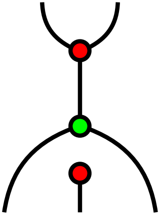
Hopf algebras and $\mathbb{Z}_2$-matrices

- (Commutative, self-inverse) Hopf algebra expressions are totally characterised by their $\mathbb{Z}_2$-path matrices:



$$\leftrightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \leftrightarrow$$



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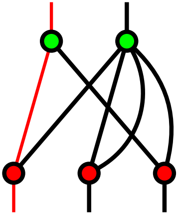
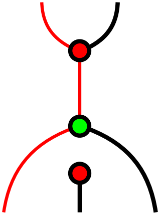


$$\leftrightarrow \quad \begin{pmatrix} \mathbf{1} & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \leftrightarrow$$



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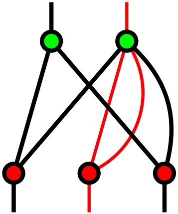
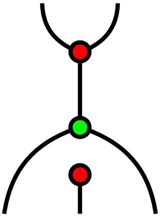


$$\leftrightarrow \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & \textcolor{red}{\mathbf{0}} & 1 \end{pmatrix} \quad \leftrightarrow$$



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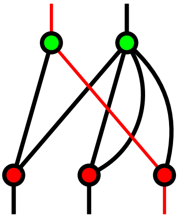
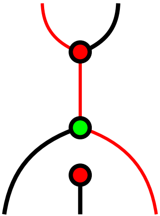


$$\leftrightarrow \begin{pmatrix} 1 & 0 & \color{red}{\mathbf{1}} \\ 1 & 0 & 1 \end{pmatrix} \leftrightarrow$$



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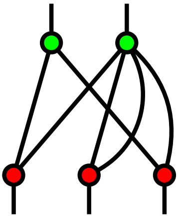

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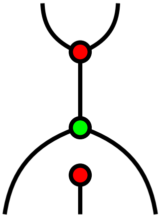
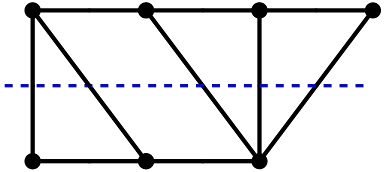


- In category-theoretic terms, this means $\mathrm{Mat}(\mathbb{Z}_2)$ is a PROP for commutative, self-inverse Hopf algebras.

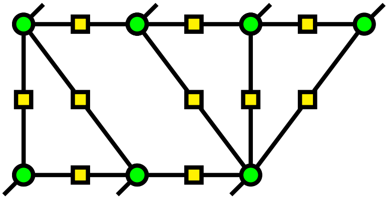
Measuring Entanglement

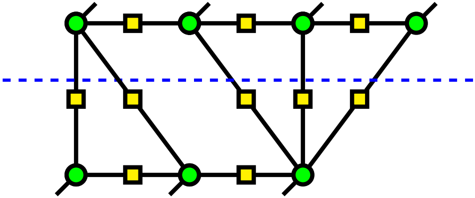
- **Proposition:** The amount of entanglement across any bipartition of a graph state is equal to its cut-rank (i.e. the rank of the associated adjacency matrix over $\mathbb{Z}_2$)[1], e.g.

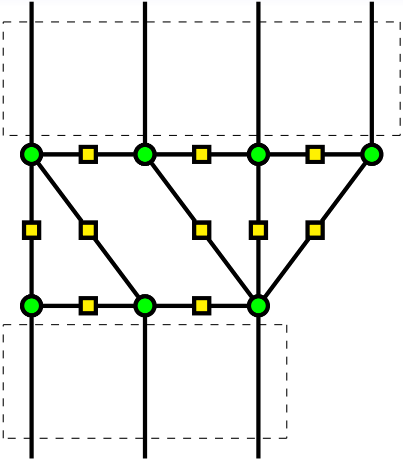


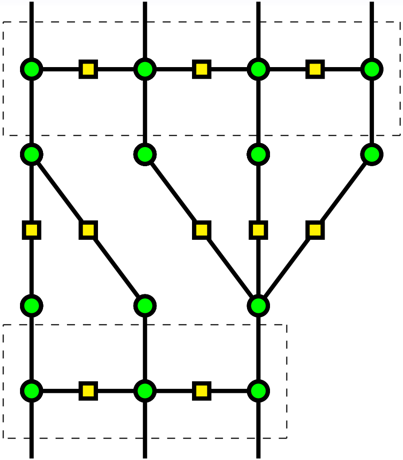
$$\rightsquigarrow \quad \text{rank}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 2 \text{ ebits}$$

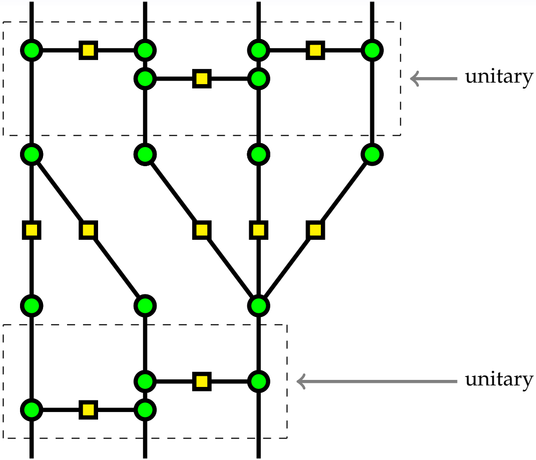
[1] Hein, Eisart, Briegel. arXiv:quant-ph/0602096, Prop. 11










unitary
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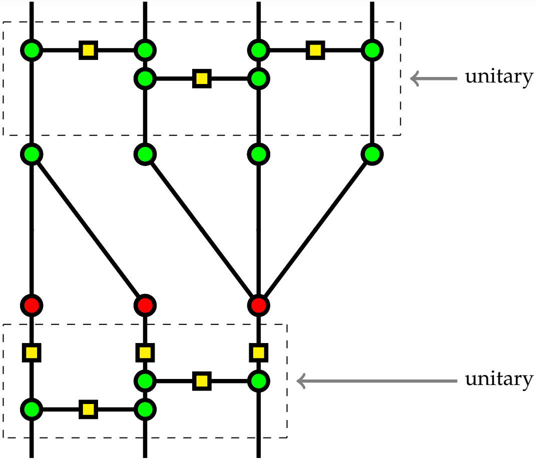
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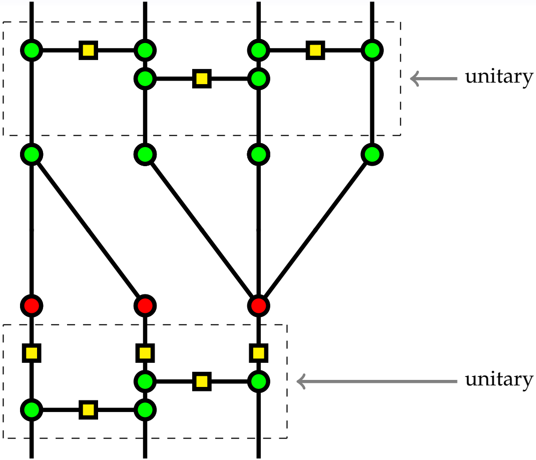
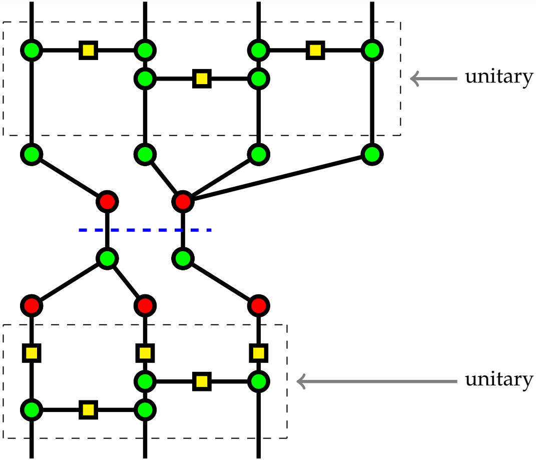


$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 1\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

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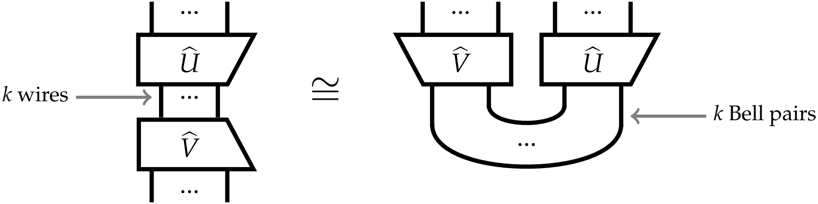
- Writing as a bipartite state:



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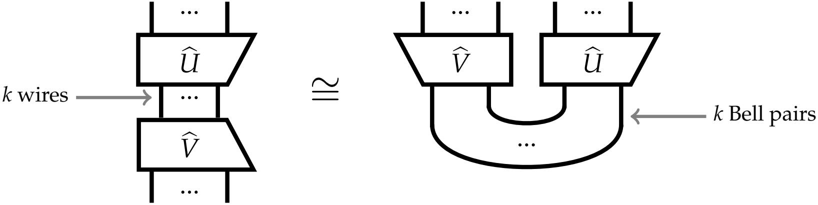
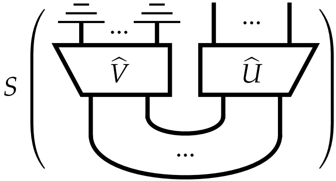


- Computing the entropy of entanglement:



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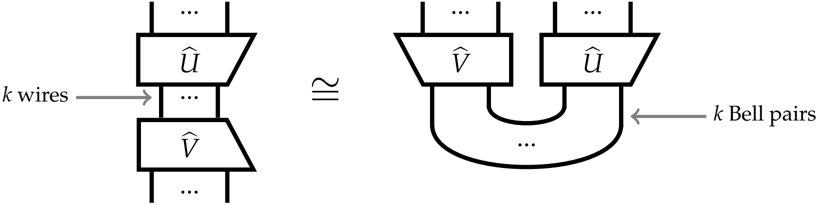

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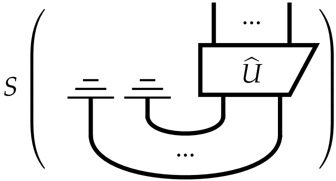

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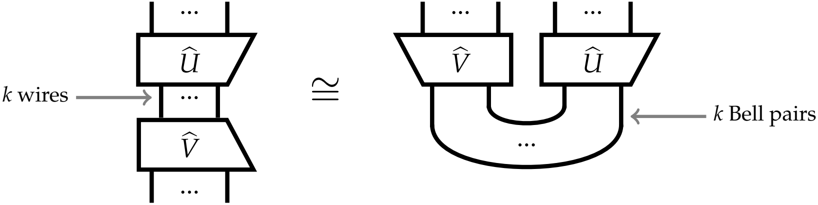

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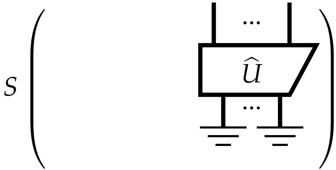

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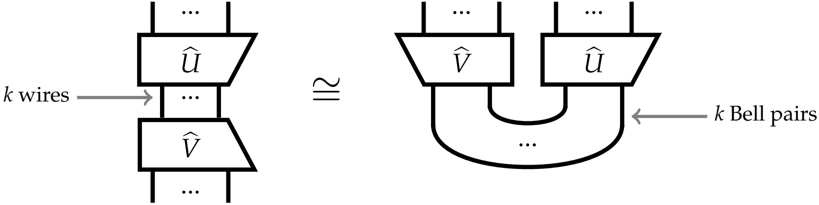

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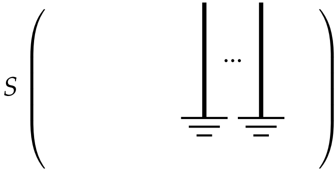

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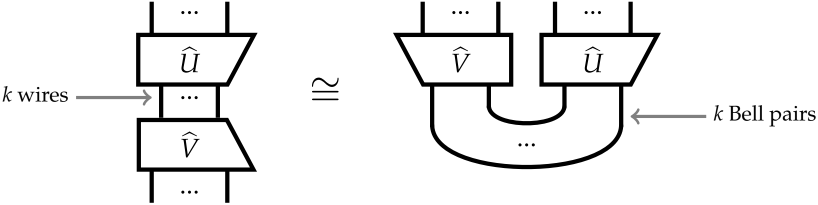

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