

On the complexity of symmetric vs. functional PCSPs

TAMIO-VESA NAKAJIMA, University of Oxford, United Kingdom

STANISLAV ŽIVNÝ, University of Oxford, United Kingdom

The complexity of the *promise constraint satisfaction problem* $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is largely unknown, even for symmetric \mathbf{A} and \mathbf{B} , except for the case when \mathbf{A} and \mathbf{B} are Boolean.

First, we establish a dichotomy for $\text{PCSP}(\mathbf{A}, \mathbf{B})$ where \mathbf{A}, \mathbf{B} are symmetric, \mathbf{B} is *functional* (i.e. any $r - 1$ elements of an r -ary tuple uniquely determines the last one), and (\mathbf{A}, \mathbf{B}) satisfies technical conditions we introduce called *dependency* and *additivity*. This result implies a dichotomy for $\text{PCSP}(\mathbf{A}, \mathbf{B})$ with \mathbf{A}, \mathbf{B} symmetric and \mathbf{B} functional if (i) \mathbf{A} is Boolean, or (ii) \mathbf{A} is a hypergraph of a small uniformity, or (iii) \mathbf{A} has a relation $R^{\mathbf{A}}$ of arity at least 3 such that the hypergraph diameter of $(\mathbf{A}, R^{\mathbf{A}})$ is at most 1.

Second, we show that for $\text{PCSP}(\mathbf{A}, \mathbf{B})$, where \mathbf{A} and \mathbf{B} contain a single relation, \mathbf{A} satisfies a technical condition called *balancedness*, and \mathbf{B} is arbitrary, the combined *basic linear programming* relaxation (BLP) and the *affine integer programming* relaxation (AIP) is no more powerful than the (in general strictly weaker) AIP relaxation. Balanced \mathbf{A} include symmetric \mathbf{A} or, more generally, \mathbf{A} preserved by a transitive permutation group.

CCS Concepts: • **Theory of computation** → **Design and analysis of algorithms; Problems, reductions and completeness.**

Additional Key Words and Phrases: algebraic approach, constraint satisfaction, promise CSP, polymorphisms, minions

ACM Reference Format:

Tamio-Vesa Nakajima and Stanislav Živný. 2024. On the complexity of symmetric vs. functional PCSPs. *ACM Trans. Algor.* 1, 1, Article 1 (January 2024), 29 pages.

1 Introduction

Promise constraint satisfaction problems (PCSPs) are a generalisation of constraint satisfaction problems (CSPs) that allow for capturing many more computational problems [4, 6, 8].

A canonical example of a CSP is the 3-colouring problem: Given a graph G , is it 3-colourable? This can be cast as a CSP. Let \mathbf{K}_k denote a clique on k vertices. Then $\text{CSP}(\mathbf{K}_3)$, the constraint satisfaction problem with the template \mathbf{K}_3 , is the following computational problem (in the decision version): Given a graph G , say Yes if there is a homomorphism from G to \mathbf{K}_3 (indicated by $G \rightarrow \mathbf{K}_3$) and say No otherwise (indicated by $G \not\rightarrow \mathbf{K}_3$). Here a graph homomorphism is an edge preserving map [24]. As graph homomorphisms from G to \mathbf{K}_3 are 3-colourings of G , $\text{CSP}(\mathbf{K}_3)$ is the 3-colouring problem.

Another example of a CSP is 1-in-3-SAT: Given a positive 3-CNF formula, is there an assignment that satisfies exactly one literal in each clause? This is $\text{CSP}(\mathbf{1-in-3})$, where

$$\mathbf{1-in-3} = (\{0, 1\}; \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}).$$

Authors' Contact Information: Tamio-Vesa Nakajima, tamio-vesa.nakajima@cs.ox.ac.uk, University of Oxford, Department of Computer Science, Oxford, United Kingdom; Stanislav Živný, standa.zivny@cs.ox.ac.uk, University of Oxford, Department of Computer Science, Oxford, United Kingdom.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than ACM must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.

© 2024 ACM.

ACM 1549-6333/2024/1-ART1

<https://doi.org/>

Yet another example is NAE-3-SAT: Given a positive 3-CNF formula, is there an assignment that satisfies one or two literals in each clause? This is $\text{CSP}(\text{NAE})$, where

$$\text{NAE} = (\{(0, 1), \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\}\}.$$

A canonical example of a PCSP is the approximate graph colouring problem [22]: Fix $k \leq \ell$. Given a graph G , determine whether G is k -colourable or not even ℓ -colourable. (If neither condition holds then the algorithm can output anything; also, note that if $k = \ell$ this is just k -colouring.) This is the same as the PCSP over cliques; i.e., $\text{PCSP}(\mathbf{K}_k, \mathbf{K}_\ell)$ is the following computational problem (in the decision version): Given a graph G , say YES if $G \rightarrow \mathbf{K}_k$ and say No if $G \not\rightarrow \mathbf{K}_\ell$. In the search version, one is given a k -colourable graph G and the task is to find an ℓ -colouring of G (which necessarily exists by the promise and the fact that $k \leq \ell$).

Another example of a PCSP is $\text{PCSP}(\mathbf{1-in-3}, \text{NAE})$, identified in [8]: Given a satisfiable instance X of $\text{CSP}(\mathbf{1-in-3})$, can one find a solution if X is seen as an instance of $\text{CSP}(\text{NAE})$? I.e., can one find a solution that satisfies one or two literals in each clause given the promise that a solution that satisfies exactly one literal in each clause exists? Although both $\text{CSP}(\mathbf{1-in-3})$ and $\text{CSP}(\text{NAE})$ are NP-complete, Brakensiek and Guruswami showed in [8] that $\text{PCSP}(\mathbf{1-in-3}, \text{NAE})$ is solvable in polynomial time and in particular it is solved by the so-called affine integer programming relaxation (AIP), whose power was characterised in [6].

More generally, one fixes two relational structures A and B with $A \rightarrow B$. The $\text{PCSP}(A, B)$ is then, in the decision version, the computational problem of distinguishing (input) relational structures X with $X \rightarrow A$ from those with $X \not\rightarrow B$. In the search version, $\text{PCSP}(A, B)$ is the problem of finding a homomorphism from an input structure X to B given that one is promised that $X \rightarrow A$. One can think of $\text{PCSP}(A, B)$ as an approximation version of $\text{CSP}(A)$ on satisfiable instances. Another way is to think of $\text{PCSP}(A, B)$ as $\text{CSP}(B)$ with restricted inputs. We refer the reader to [26] for a very recent survey on PCSPs.

For CSPs, a dichotomy conjecture of Feder and Vardi [19] was resolved independently by Bulatov [15] and Zhuk [32] via the so-called algebraic approach [14, 25]: For every fixed finite A , $\text{CSP}(A)$ is either solvable in polynomial time or $\text{CSP}(A)$ is NP-complete.

For PCSPs, even the case of graphs and structures on Boolean domains is widely open; these two were established for CSPs a long time ago [24, 30] and constituted important evidence for conjecturing a dichotomy. Following the important work of Barto et al. [6] on extending the algebraic framework from the realms of CSPs to the world of PCSPs, there have been several recent works on complexity classifications of fragments of PCSPs [2, 5, 8, 9, 12, 13, 20, 23, 27, 28], hardness conditions [6, 7, 12, 31], and power of algorithms [3, 6, 10, 17]. Nevertheless, a classification of more concrete fragments of PCSPs is needed for making progress with the general theory, such as finding hardness and tractability criteria, as well as with resolving longstanding open questions, such as approximate graph colouring.

Brakensiek and Guruswami classified $\text{PCSP}(A, B)$ for all Boolean symmetric structures A and B with disequalities [8]. Fícak, Kozík, Olšák, and Stankiewicz generalised this result by classifying $\text{PCSP}(A, B)$ for all Boolean symmetric structures A and B [20].

Barto, Battistelli, and Berg [5] studied symmetric PCSPs on non-Boolean domains and in particular PCSPs of the form $\text{PCSP}(\mathbf{1-in-3}, B)$, where B contains a single ternary relation over the domain $\{0, 1, \dots, d-1\}$. For $d = 2$, a complete classification $\text{PCSP}(\mathbf{1-in-3}, B)$ is known [8, 20]. For $d = 3$, Barto et al. [5] managed to classify all but one structure B . The remaining open case of “linearly ordered colouring” inspired further investigation in [28]. For $d = 4$, Barto et al. [5] obtained partial results. In particular, for certain structures B they managed to rule out the applicability of the BLP + AIP algorithm from [10]. The significance of BLP + AIP here is that it is the strongest known

algorithm for PCSPs for which a characterisation of its power is known both in terms of a minion¹ and also in terms of polymorphism identities, cf. [10] for details. Furthermore, BLP + AIP solves all currently known tractable Boolean PCSPs [10, 20]. This suggests that those cases are NP-hard (or new algorithmic techniques are needed).

Contributions. We continue the work from [8, 20] and [5] and focus on promise constraint satisfaction problems of the form $\text{PCSP}(\mathbf{A}, \mathbf{B})$, where \mathbf{A} is symmetric and \mathbf{B} is over an arbitrary finite domain.² Since the template \mathbf{A} is symmetric, we can assume without loss of generality that \mathbf{B} is symmetric, as observed in [5] and in [13].³

As our main result, we establish the following result. A structure \mathbf{B} is called *functional* if, for any relation $R^{\mathbf{B}}$ in \mathbf{B} of, say, arity r , and any tuple $x \in R^{\mathbf{B}}$, any $r - 1$ elements of x determine the last element. In detail, $(x_1, \dots, x_{r-1}, y), (x_1, \dots, x_{r-1}, z) \in R^{\mathbf{B}}$ implies $y = z$, and similarly for the other $r - 1$ positions.⁴ The notions of dependency and additivity will be defined in Section 3: for the moment, consider them simply as technical conditions on (\mathbf{A}, \mathbf{B}) . Finite tractability is defined in Section 2 – informally, a template is finitely tractable if it can be solved by treating the instance as though it were an instance of a tractable finite-domain CSP.

Theorem 1. *Let \mathbf{A} be a symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Assume that (\mathbf{A}, \mathbf{B}) is dependent and additive. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

Our main motivation for studying PCSPs with functional structures is the fact (mentioned above) that more complexity classifications of PCSP fragments are needed to make progress with the general theory of PCSPs. Furthermore, functional PCSPs generalise linear equations, an important and fundamental class of CSPs. Finally, the topological methods that proved useful for showing the hardness of certain PCSPs (e.g. $\text{PCSP}(G, K_3)$ for any non-bipartite 3-colourable graph G , where K_3 is the clique [27], or approximate 3-vs.-4 linearly ordered colouring [21]) seem inapplicable to functional PCSPs and thus other methods are required.

Theorem 1 has the following three corollaries. The first corollary applies to structures \mathbf{A} that are Boolean.

Corollary 2. *Let \mathbf{A} be a Boolean symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

The second corollary applies to structures \mathbf{A} consisting of a single relation of a small arity.

Corollary 3. *Let \mathbf{A} be a symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Assume that both \mathbf{A} and \mathbf{B} have exactly one relation of arity at most 4. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

An example of a Boolean symmetric structure \mathbf{A} is **1-in-3**, and more generally **q-in-r**.⁵ The structure **1-in-3** is not only Boolean (and thus is captured by Corollary 2) but also consists of a

¹We will only use polymorphism minions [6] in this paper. More general minions capture not only the power of BLP + AIP [10] but also the power of the CLAP algorithm [17].

²All structures in this article can be assumed to be finite unless they are explicitly stated to be infinite.

³In detail, for any symmetric \mathbf{A} and (not necessarily symmetric) \mathbf{B} with $\mathbf{A} \rightarrow \mathbf{B}$, there is a symmetric \mathbf{B}' with $\mathbf{A} \rightarrow \mathbf{B}'$ such that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ and $\text{PCSP}(\mathbf{A}, \mathbf{B}')$ are polynomial-time equivalent [5, 13]. This \mathbf{B}' is the largest symmetric substructure of \mathbf{B} . Observe that a functional structure has functional substructures, so if \mathbf{B} is functional then \mathbf{B}' remains functional.

⁴Note that for symmetric \mathbf{B} the requirement “for the other $r - 1$ positions” is satisfied automatically.

⁵**q-in-r** is the structure on $\{0, 1\}$ with a single (symmetric) relation of arity r containing all r -tuples with precisely q 1s (and $r - q$ 0s).

single relation of arity 3 (and thus is also captured by Corollary 3). In fact, it satisfies another property, leading to the following generalisation of **1-in-3**, which applies to structures \mathbf{A} with a certain connectivity property. We will need some notation. Let $\text{dist}_R(x, y)$ be the distance between x and y when viewed as vertices in a hypergraph whose edge relation is R ; in particular, $\text{dist}_R(x, x) = 0$ and $\text{dist}_R(x, y) = \infty$ if x and y are in different connected components. Define $\text{diam}(A, R) = \max_{u, v \in A} \text{dist}_R(u, v)$.

Corollary 4. *Let \mathbf{A} be a symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Assume that \mathbf{A} has a relation R^A of arity at least 3 for which $\text{diam}(A, R^A) \leq 1$. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

A hypergraph is called *linear* if no two distinct edges intersect in more than one vertex. We remark that any *linear* r -uniform hypergraph can be seen as a functional symmetric relational structure with one r -ary relation.

Several researchers have informally conjectured that $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ admits a dichotomy. The authors, as well as other researchers, believe that in fact not only is there a dichotomy but also all tractable cases are solvable by AIP (cf. also Remark 39).

Conjecture 5. *For every structure \mathbf{B} , either $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ is solvable in polynomial time by AIP, or $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ is NP-hard.*

Theorem 1 establishes the special case of Conjecture 5 for functional \mathbf{B} . We make further progress towards Conjecture 5 by proving that for any structure \mathbf{A} with a single (*not necessarily Boolean*) symmetric relation, and any (*not necessarily functional*) structure \mathbf{B} for which $\mathbf{A} \rightarrow \mathbf{B}$, BLP + AIP from [10] is no more powerful for $\text{PCSP}(\mathbf{A}, \mathbf{B})$ than AIP from [8], although in general BLP + AIP is strictly stronger than AIP [10], already for (non-promise) CSPs with *two* Boolean symmetric relations, cf. Example 57. In fact, we establish a more general result. We say that a relation R is *balanced* if there exists a matrix M whose columns are tuples of R , where each tuple of R appears as a column (possibly a multiple times), and where the rows of M are permutations of each other. The matrix M below shows that the Boolean **1-in-3** relation is balanced:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 6. *Let \mathbf{A} be any structure with a single relation. If the relation in \mathbf{A} is balanced then, for any \mathbf{B} such that $\mathbf{A} \rightarrow \mathbf{B}$, BLP + AIP solves $\text{PCSP}(\mathbf{A}, \mathbf{B})$ if and only if AIP solves it.*

If the (only) relation in \mathbf{A} is *binary* (i.e., a digraph), the condition of balancedness has a natural combinatorial interpretation: A binary relation is balanced if and only if it is the disjoint union of strongly connected components (cf. Appendix A).

Theorem 6 implies the following corollary. We say that a relation of arity r is *preserved* by a group of permutations of degree r if and only if permuting any tuple of the relation according to any permutation of the group gives another tuple of the relation.

Corollary 7. *Suppose that G is a transitive group of permutations, of order r . Further, suppose that \mathbf{A} is a relational structure with one relation, of arity r , that is preserved by G . Then, for any $\mathbf{A} \rightarrow \mathbf{B}$, BLP + AIP solves $\text{PCSP}(\mathbf{A}, \mathbf{B})$ if and only if AIP does.*

While Corollary 7 is more elegant than Theorem 6, it applies to fewer structures. Indeed, we will show in Remark 60 that there exist balanced relations that are not preserved by any transitive group. Examples of relations that are preserved by some transitive group of permutations G include

symmetric relations (where G is the symmetric group) or cyclic relations (where G contains all cyclic shifts of appropriate degree).

2 Preliminaries

We let $[r] = \{1, \dots, r\}$. We denote by 2^S the powerset of S .

Structures and PCSPs. Promise CSPs have been introduced in [4] and [8]. We follow the notation and terminology of [6].

A (relational) *structure* is a tuple $\mathbf{A} = (A; R_1^A, \dots, R_t^A)$, where $R_i^A \subseteq A^{\text{ar}(R_i)}$ is a relation of arity $\text{ar}(R_i)$ on a set A , called the *domain*. A structure \mathbf{A} is called *Boolean* if $A = \{0, 1\}$ and is called *symmetric* if R_i^A is a symmetric relation for each $i \in [t]$; i.e, if $(x_1, \dots, x_{\text{ar}(R_i)}) \in R_i^A$ then for every permutation π on $[\text{ar}(R_i)]$ we have $(x_{\pi(1)}, \dots, x_{\pi(\text{ar}(R_i))}) \in R_i^A$. A structure \mathbf{A} is called *functional* if $(x_1, \dots, x_{\text{ar}(R_i)-1}, y) \in R_i^A$ and $(x_1, \dots, x_{\text{ar}(R_i)-1}, z) \in R_i^A$ implies $y = z$ for any $i \in [t]$, and that the same hold for all other $r - 1$ positions in the tuple. For any r -ary functional relation $R \subseteq A^r$, we define a partial map also called R from A^{r-1} to A in the following way: for any $x_1, \dots, x_{r-1} \in A$, $R(x_1, \dots, x_{r-1})$ is the unique value y such that $(x_1, \dots, x_{r-1}, y) \in R$, if it exists; $R(x_1, \dots, x_{r-1})$ is undefined if no such value exists.

Consider two structures $\mathbf{A} = (A; R_1^A, \dots, R_t^A)$ and $\mathbf{B} = (B; R_1^B, \dots, R_t^B)$ with t relations, where, for each $i \in [t]$, R_i^A and R_i^B have the same arity. A *homomorphism from \mathbf{A} to \mathbf{B}* is a function $h : A \rightarrow B$ such that, for any $i \in [t]$, for each $x = (x_1, \dots, x_{\text{ar}(R_i)}) \in R_i^A$, we have $h(x) = (h(x_1), \dots, h(x_{\text{ar}(R_i)})) \in R_i^B$. We denote the existence of a homomorphism from \mathbf{A} to \mathbf{B} by $\mathbf{A} \rightarrow \mathbf{B}$.

Let \mathbf{A} and \mathbf{B} be two structures with $\mathbf{A} \rightarrow \mathbf{B}$; we call (\mathbf{A}, \mathbf{B}) a (PCSP) *template*. In the *search* version of the *promise constraint satisfaction problem* (PCSP) with the template (\mathbf{A}, \mathbf{B}) , denoted by $\text{PCSP}(\mathbf{A}, \mathbf{B})$, the task is: Given a structure \mathbf{X} with the promise that $\mathbf{X} \rightarrow \mathbf{A}$, find a homomorphism from \mathbf{X} to \mathbf{B} (which necessarily exists as homomorphisms compose). In the *decision* version of $\text{PCSP}(\mathbf{A}, \mathbf{B})$, the task is: Given a structure \mathbf{X} , output YES if $\mathbf{X} \rightarrow \mathbf{A}$, and output No if $\mathbf{X} \not\rightarrow \mathbf{B}$.⁶

The decision version trivially reduces to the search version. We will use the decision version in this paper.

We will be interested in the complexity of $\text{PCSP}(\mathbf{A}, \mathbf{B})$, in particular for symmetric \mathbf{A} and functional \mathbf{B} . (As discussed in Section 1, the symmetricity of \mathbf{A} means that we can without loss of generality assume symmetricity of \mathbf{B} .)

Operations and polymorphisms. A function $h : A^n \rightarrow B$ is called an *operation* of arity n . A $(2n + 1)$ -ary operation $f : A^{2n+1} \rightarrow B$ is called *2-block-symmetric* if $f(a_1, \dots, a_{2n+1}) = f(a_{\pi(1)}, \dots, a_{\pi(2n+1)})$ for every $a_1, \dots, a_{2n+1} \in A$ and every permutation π on $[2n + 1]$ that preserves parity; i.e, π maps odd values to odd values and even values to even values.

A $(2n + 1)$ -ary operation $f : A^{2n+1} \rightarrow B$ is called *alternating* if it is 2-block-symmetric, and furthermore $f(a_1, \dots, a_{2n-1}, a, a) = f(a_1, \dots, a_{2n-1}, a', a')$ for every $a_1, \dots, a_{2n-1}, a, a' \in A$.

Consider structures \mathbf{A}, \mathbf{B} with t relations with the same arities. We call $h : A^n \rightarrow B$ a *polymorphism* of (\mathbf{A}, \mathbf{B}) if the following holds for any relation $R = R_i$, $i \in [t]$, of arity $r = \text{ar}(R)$. For any $x^1, \dots, x^r \in A^n$, where $x^i = (x_1^i, \dots, x_n^i)$, with $(x_1^i, \dots, x_n^i) \in R^A$ for every $1 \leq i \leq n$, we have $(h(x^1), \dots, h(x^r)) \in R^B$. One can visualise this as an $(r \times n)$ matrix whose rows are the tuples x^1, \dots, x^r . The requirement is that if every column of the matrix is in R^A then the application of h

⁶If neither condition holds then the algorithm can output anything. An equivalent way is to define the problem as follows: Given a relational structure \mathbf{X} such that either $\mathbf{X} \rightarrow \mathbf{A}$ or $\mathbf{X} \not\rightarrow \mathbf{B}$, which is the promise, output YES if $\mathbf{X} \rightarrow \mathbf{A}$ and output No if $\mathbf{X} \not\rightarrow \mathbf{B}$. The equivalence, up to polynomial-time solvability, of the two definitions relies on the fact that polynomials are time-constructible and so a clock can be run.

on the rows of the matrix results in a tuple from $R^{\mathbf{B}}$. We denote by $\text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ the set of n -ary polymorphisms of (\mathbf{A}, \mathbf{B}) and by $\text{Pol}(\mathbf{A}, \mathbf{B})$ the set of all polymorphisms of (\mathbf{A}, \mathbf{B}) .

Relaxations. There are two standard polynomial-time solvable relaxations for PCSPs, the *basic linear programming* relaxation (BLP) and the *affine integer programming* relaxation (AIP) [8]. The AIP solves most tractable PCSPs studied in this paper, with the exception of cases covered in Corollary 3 (cf. also Remark 39). There is also a combination of the two, called BLP + AIP [10], that is provably stronger than both BLP and AIP. We will show that for certain PCSPs, this is not the case (cf. Theorem 6). The precise definitions of the relaxations are not important for us as we will only need the notion of solvability of PCSPs by these relaxations and characterisations of the power of the relaxations; we refer the reader to [6, 8, 10] for details. Let \mathbf{X} be an instance of $\text{PCSP}(\mathbf{A}, \mathbf{B})$. It follows from the definitions of the relaxations that if $\mathbf{X} \rightarrow \mathbf{A}$ then both AIP and BLP + AIP accept [6, 8]. We say that AIP (BLP + AIP, respectively) *solves* $\text{PCSP}(\mathbf{A}, \mathbf{B})$ if, for every \mathbf{X} with $\mathbf{X} \not\rightarrow \mathbf{B}$, AIP (BLP + AIP, respectively) rejects.

The power of AIP and BLP + AIP for PCSPs is characterised by the following results.

Theorem 8 ([6]). *$\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solved by AIP if and only if $\text{Pol}(\mathbf{A}, \mathbf{B})$ contains alternating operations of all odd arities.*

Theorem 9 ([10]). *$\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solved by BLP + AIP if and only if $\text{Pol}(\mathbf{A}, \mathbf{B})$ contains 2-block-symmetric operations of all odd arities.*

We now define the notion of *finite tractability* [2, 6]. We say that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is finitely tractable if $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$ for some finite structure \mathbf{E} and $\text{CSP}(\mathbf{E})$ is tractable. For a group G , we use the standard notation $H \triangleleft G$ to indicate that H is a normal subgroup of G .

Lemma 10. *Suppose $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$, where $E = G$ for some finite Abelian group $(G, +)$, and each relation of \mathbf{E} is either of the form (i) $c + H$ for some $r \in \mathbb{N}$, $c \in G^r$ and $H \triangleleft G^r$, or (ii) empty. Then, $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable.*

PROOF. The following alternating operation is a polymorphism of \mathbf{E}

$$f(x_1, y_1, \dots, y_k, x_{k+1}) = \sum_{i=1}^{k+1} x_i - \sum_{i=1}^k y_i.$$

Consider a relation $R^{\mathbf{E}}$ of \mathbf{E} , of the form $c + H$. Consider a matrix of inputs whose columns are $x_1, y_1, \dots, y_k, x_{k+1} \in R^{\mathbf{E}}$. In other words, $x_i \in c + H$ and $y_i \in c + H$ for each x_i, y_i . Note that the column that results from applying f to the rows of this matrix is just

$$x_1 - y_1 + \dots - y_k + x_{k+1} \in (c + H) - (c + H) + \dots - (c + H) + (c + H) \subseteq c + H$$

Thus f is an alternating polymorphism of \mathbf{E} . It follows that $\text{CSP}(\mathbf{E})$ is solved by AIP, from whence it follows that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is finitely tractable and solved by AIP. \square

Minions. We will use the theory of minions from [6]. Let \mathcal{M} be a set, where each element $f \in \mathcal{M}$ is assigned an arity $\text{ar}(f)$. We write $\mathcal{M}^{(n)} = \{f \in \mathcal{M} \mid \text{ar}(f) = n\}$. Further, let \mathcal{M} be endowed with, for each $\pi : [n] \rightarrow [m]$, a (so-called minor) map $f \mapsto f^\pi : \mathcal{M}^{(n)} \rightarrow \mathcal{M}^{(m)}$ such that, for $\pi : [n] \rightarrow [m]$ and $\sigma : [m] \rightarrow [k]$, and any $f \in \mathcal{M}^{(n)}$ we have $(f^\pi)^\sigma = f^{\sigma \circ \pi} \in \mathcal{M}^{(k)}$, and $f^{\text{id}} = f$. Then, \mathcal{M} is called a *minion*.⁷ We often write $f \xrightarrow{\pi} g$ instead of $g = f^\pi$.

⁷A minion is a functor from the skeleton of the category of finite sets to the category of sets.

Consider two minions \mathcal{M}, \mathcal{N} ; a *minion homomorphism* is a map $\xi : \mathcal{M} \rightarrow \mathcal{N}$ such that, for any $f \in \mathcal{M}^{(n)}$ and $\pi : [n] \rightarrow [m]$, we have that $\xi(f)^\pi = \xi(f^\pi)$.⁸ If such a minion homomorphism exists, we write $\mathcal{M} \rightarrow \mathcal{N}$.

Given an n -ary operation $f : A^n \rightarrow B$ and a map $\pi : [n] \rightarrow [m]$, an m -ary operation $g : A^m \rightarrow B$ is called a *minor* of f given by the map π if

$$g(x_1, \dots, x_m) = f(x_{\pi(1)}, \dots, x_{\pi(m)}).$$

The polymorphisms $\text{Pol}(\mathbf{A}, \mathbf{B})$ thus form a minion, where f^π is given by the minor of f at π .

The main hardness theorem that we will use is the following.⁹

Theorem 11 ([6]). *Fix constants m and C . Take any template (\mathbf{A}, \mathbf{B}) . Suppose $\text{Pol}(\mathbf{A}, \mathbf{B}) = \bigcup_{i=1}^m \mathcal{M}_i$ such that for every $i \in [m]$ there exists a map I_i that takes $f \in \mathcal{M}_i$ to a subset of $[\text{ar}(f)]$ of size at most C such that the following holds: For any $f, g \in \mathcal{M}_i$, $\pi : [\text{ar}(f)] \rightarrow [\text{ar}(g)]$ such that $g = f^\pi$ we have that $I_i(g) \cap \pi(I_i(f)) \neq \emptyset$. Then, $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

3 Additivity and dependency

In this section we will define two new concepts for a template (\mathbf{A}, \mathbf{B}) , additivity and dependency. Let $f \in \text{Pol}(\mathbf{A}, \mathbf{B})$ be a polymorphism of (\mathbf{A}, \mathbf{B}) . Intuitively, additivity constrains the value of f evaluated at two elements and dependency ensures that f is determined by such evaluations. The end goal of this section is to prove the following two theorems.

Theorem 12. *Suppose \mathbf{A} has a symmetric relation $R^{\mathbf{A}}$ of arity at least 3 with $\text{diam}(\mathbf{A}, R^{\mathbf{A}}) \leq 1$. Then, for any functional \mathbf{B} such that $\mathbf{A} \rightarrow \mathbf{B}$, (\mathbf{A}, \mathbf{B}) is additive and dependent.*

Theorem 13. *Suppose \mathbf{A} has a symmetric relation $R^{\mathbf{A}}$ of arity 3 or 4 that is connected when viewed as the edge relation of a hypergraph on vertices A . Then, for any functional \mathbf{B} such that $\mathbf{A} \rightarrow \mathbf{B}$, (\mathbf{A}, \mathbf{B}) is additive and dependent.*

Throughout this section, we will assume implicitly that \mathbf{A}, \mathbf{B} are symmetric.

3.1 Additivity

Consider a polymorphism $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$. For any $(i, j) \in A^2$ (including the case where $i = j$), we can define a function f_{ij} derived from f in the following way: $f_{ij} : 2^{[n]} \rightarrow B$ is a function where

$$f_{ij}(S) = f(x_1, \dots, x_n)$$

where $x_k = j$ if $k \in S$ and $x_k = i$ otherwise. In other words, $f_{ij}(S)$ is f evaluated at the characteristic vector of S , where j indicates membership in S and i indicates non-membership. (Equivalently, let $\pi_S : [n] \rightarrow [2]$ be given by $\pi_S(k) = 2$ if $k \in S$ and $\pi_S(k) = 1$ otherwise. Then, $f_{ij}(S) = f^{\pi_S}(i, j)$.)

We define $f^P : 2^{[n]} \rightarrow B^{A^2}$ to be the function¹⁰

$$f^P(S)(i, j) = f_{ij}(S).$$

We will be interested in templates (\mathbf{A}, \mathbf{B}) that have the following property.

Definition 14. Consider a template (\mathbf{A}, \mathbf{B}) with \mathbf{A} symmetric and \mathbf{B} functional. We say that (\mathbf{A}, \mathbf{B}) is *additive* if there exists an operator $+ : B^{A^2} \times B^{A^2} \rightarrow B^{A^2}$ such that, for any $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ and disjoint $S, T \subseteq [n]$ we have

$$f^P(S) + f^P(T) = f^P(S \cup T).$$

⁸Minion homomorphisms are natural transformations.

⁹In [6], this is Theorem 5.21 together with Lemma 5.11, as detailed after the proof of Theorem 5.21 therein.

¹⁰The superscript “ P ” indicates the word “pair”.

Lemma 15. *If (A, B) is additive, there exists an operator $- : B^{A^2} \times B^{A^2} \rightarrow B^{A^2}$ such that, for any $f \in \text{Pol}^{(n)}(A, B)$ and $S \subseteq T \subseteq [n]$, we have*

$$f^p(T) - f^p(S) = f^p(T \setminus S).$$

PROOF. While $f^p(S)$ has been written as a function from A^2 to B above, we can also see it as an $|A| \times |A|$ matrix of elements of B . Thus we can take the transpose of this matrix, denoted by the superscript T below. Observe that $(f^p(S))^T(i, j) = f_{ji}(S) = f_{ij}(\bar{S}) = f^p(\bar{S})(i, j)$, where \bar{S} denotes the complement of S . In other words, $(f^p(S))^T = f^p(\bar{S})$.

Set $x - y = (x^T + y)^T$. Then for $S \subseteq T \subseteq [n]$,

$$f^p(T) - f^p(S) = (f^p(T))^T + f^p(S))^T = f^p(\overline{T \cup S}) = f^p(T \setminus S). \quad \square$$

Lemma 16. *Suppose (A, B) is additive. Consider a polymorphism $f \in \text{Pol}^{(n)}(A, B)$. Consider any family of disjoint sets $\mathcal{A} \subseteq 2^{[n]}$, containing at least $|B|^{|A|^2}$ sets. Then some nonempty subset $\mathcal{B} \subseteq \mathcal{A}$ exists such that $f^p(\cup \mathcal{B}) = f^p(\emptyset)$.*

The approach used to prove this is analogous to the following well known exercise (first set out by Vázsonyi and Sved, according to Erdős [1]): Prove that any sequence of n integers has a subsequence whose sum is divisible by n .

PROOF. \mathcal{A} contains at least $|B|^{|A|^2} \geq |\text{range}(f^p)|$ different sets. Let $A_1, \dots, A_{|\text{range}(f^p)|}$ be some of these sets. Define $B_i = \bigcup_{j \leq i} A_j$ for $0 \leq i \leq |\text{range}(f^p)|$; note that $B_0 = \emptyset$. By the pigeonhole principle there exists $0 \leq i < j \leq |\text{range}(f^p)|$ such that $f^p(B_i) = f^p(B_j)$. Then, using Lemma 15, $f^p(B_j \setminus B_i) = f^p(B_j) - f^p(B_i) = f^p(B_i) - f^p(B_i) = f^p(B_i \setminus B_i) = f^p(\emptyset)$. Thus $\mathcal{B} = \{A_{i+1}, \dots, A_j\}$ is the required family of sets. \square

Lemma 17. *Suppose (A, B) is additive. Consider a polymorphism $f \in \text{Pol}^{(n)}(A, B)$. Consider any $S \subseteq [n]$. There exists $T \subseteq S$ of size at most $|B|^{|A|^2}$ such that $f^p(S) = f^p(T)$.*

PROOF. Suppose this is not the case, and suppose that S is a minimal counterexample (with respect to inclusion) to this claim. Clearly $|S| > |B|^{|A|^2}$, or else taking $T = S$ shows that S is no counterexample at all. Thus, apply Lemma 16 to the family $\{\{x\} \mid x \in S\}$ to find that some nonempty subset $U \subseteq S$ exists such that $f^p(U) = f^p(\emptyset)$. But now, take $S' = S \setminus U \subseteq S$, and note that $f^p(S') = f^p(S \setminus U) = f^p(S) - f^p(U) = f^p(S) - f^p(\emptyset) = f^p(S \setminus \emptyset) = f^p(S)$. By the minimality of S , S' has a subset T of size at most $|B|^{|A|^2}$ such that $f^p(T) = f^p(S') = f^p(S)$, which contradicts the fact that S is a counterexample. \square

3.2 Dependency

In the sequel, we will use “tuple builder” notation. In other words, for an indexing set I , we write

$$(f(i) \mid i \in I),$$

to denote the tuple, indexed by I , with elements $f(i)$ for every $i \in I$. For instance,

$$(x^2 \mid x \in [5]) = (1, 4, 9, 16, 25).$$

For a more complicated example, we have the tuple

$$(f(x) \mid x \in [5], f : [5] \rightarrow [6]),$$

where the elements are $f(x)$, indexed by $x \in [5]$, and by functions $f : [5] \rightarrow [6]$.

This notation is justified since an n -ary tuple is just a function from $[n]$ to some base set; our notation $(f(i) \mid i \in I)$ “builds” the tuple equivalent to the function f .

Definition 18. We call (A, B) *dependent* if there exists a map h such that, for every $f \in \text{Pol}^{(n)}(A, B)$, we have that

$$f(x_1, \dots, x_n) = h(f(\alpha(x_1), \dots, \alpha(x_n)) \mid \alpha : A \rightarrow S, S \in A^{\leq 2}),$$

where $A^{\leq 2}$ is the set of nonempty subsets of A of size at most 2.

Informally, (A, B) is dependent if, for every f , we can deduce the value of $f(x_1, \dots, x_n)$ from the values $f(\alpha(x_1), \dots, \alpha(x_n))$ for α a function that maps A to a subset of A of size at most 2. One can think of α as “flattening” the values of x_1, \dots, x_n to only two values. This deduction is encoded by h , and must be independent of the choice of f .

Any polymorphism $f \in \text{Pol}^{(n)}(A, B)$ is just a function $f : A^n \rightarrow B$. Without loss of generality, identify A with the set $[a]$ from now on. Note that a tuple from $A^n = [a]^n$ can be seen as a partition of $[n]$ into a parts S_1, \dots, S_a : S_i is the set of coordinates in the tuple set to i . We will thus denote by $a^{[n]}$ both the set tuples and the set of such partitions. Thus we can, for example, evaluate f at a partition S_1, \dots, S_a of $[n]$ and get $f(S_1, \dots, S_a)$.

Definition 19. For any polymorphism $f \in \text{Pol}^{(n)}(A, B)$, we define $f^\star : a^{[n]} \rightarrow (B^{A^2})^a$ by

$$f^\star(S_1, \dots, S_a) = (f^p(S_1), \dots, f^p(S_a)).$$

The usefulness of all the concepts introduced so far comes from the following fact.

Lemma 20. *Suppose (A, B) is dependent and additive. Then there exists a function $h : (B^{A^2})^a \rightarrow B$ such that for any polymorphism $f \in \text{Pol}^{(n)}(A, B)$, we have $f = h \circ f^\star$.*

In the proof we will freely mix the two notations for functions of form $[a]^n \rightarrow B$. If the input to such functions is written with capital letters it uses the “families of disjoint sets” notation, and if it is written with lower case letters it uses the tuple notation.

PROOF. We claim that for any $\alpha : A \rightarrow S, S \in A^{\leq 2}$ we can deduce $f(\alpha(x_1), \dots, \alpha(x_n))$ from the values of $f^\star(x_1, \dots, x_n)$ in a manner independent of f . If this is the case, then by running this deduction for each α , there exists some map h' such that for every f and x_1, \dots, x_n we have

$$h'(f^\star(x_1, \dots, x_n)) = (f(\alpha(x_1), \dots, \alpha(x_n)) \mid \alpha : A \rightarrow S, S \in A^{\leq 2}).$$

Hence by the dependency of (A, B) there exists a map h'' such that $f = h'' \circ h' \circ f^\star$, which implies our conclusion.

First suppose $S = \{s\}$. Then we want to find $f(s, \dots, s)$. But this value is just $f_{ss}(X)$ for any X , and thus is included in every $f^p(S_i)$ that is within $f^\star(x_1, \dots, x_n)$.

Now suppose $S = \{b, c\}$, and that $A = B \cup C$ such that α maps B to b and C to c . In this case we observe that $(\alpha(x_1), \dots, \alpha(x_n))$ contains a c at all places where (x_1, \dots, x_n) contained an element in C , and contains a b elsewhere. Suppose S_1, \dots, S_a are sets such that S_i contains all the indices where (x_1, \dots, x_n) is equal to i . Thus, $f(\alpha(x_1), \dots, \alpha(x_n)) = f_{bc}(\bigcup_{i \in C} S_i)$. This value is an element of the tuple $f^p(\bigcup_{i \in C} S_i)$, so it is sufficient to show that we can deduce this latter value from $f^\star(x_1, \dots, x_n) = f^\star(S_1, \dots, S_a)$. But

$$f^p\left(\bigcup_{i \in C} S_i\right) = \sum_{i \in C} f^p(S_i).$$

All of the elements within this sum are contained within $f^\star(S_1, \dots, S_a) = (f^p(S_1), \dots, f^p(S_a))$. Thus we can deduce the value of $f(\alpha(x_1), \dots, \alpha(x_n))$ uniquely from $f^\star(x_1, \dots, x_n)$, without reference to f , as required. \square

3.3 Formal system

We will now describe a system of formal proofs that will help us reason about additivity and dependency. First we define the semantics of this system.

Definition 21. Suppose $S \subseteq A^n$ is a set of n -ary tuples, and $t \in A^n$ is an n -tuple. We write $S \vDash_{\mathbf{A}, \mathbf{B}} t$ (omitting \mathbf{A}, \mathbf{B} if there is no chance for confusion) if there exists a function $h : B^S \rightarrow B$ (where we interpret B^S as being a tuple indexed by S), such that, for any polymorphism $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$, we have that

$$f(t) = h(f(u) \mid u \in S).$$

In other words, we write $S \vDash t$ if the value of $f(t)$ is uniquely determined by $f(u)$ for $u \in S$, in a manner independent of f .

Definition 22. For any \mathbf{A} , we write $\Gamma_{\mathbf{A}} = \{(r, s, s) \mid s, r \in A\} \cup \{(s, r, s) \mid s, r \in A\}$ for the set of triples from A^3 with the last two elements equal or with the first and last elements equal. Write $\Delta_{\mathbf{A}}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in A, |\{x_1, \dots, x_n\}| \leq 2\}$ i.e. $\Delta_{\mathbf{A}}^n$ contains all the n -ary tuples that contain at most two distinct elements.

Lemma 23. (\mathbf{A}, \mathbf{B}) is additive if for every $p, q \in A$ we have

$$\Gamma_{\mathbf{A}} \vDash_{\mathbf{A}, \mathbf{B}} (p, p, q).$$

PROOF. For $p, q \in A$, suppose h_{qp} is the function that witnesses $\Gamma_{\mathbf{A}} \vDash_{\mathbf{A}, \mathbf{B}} (p, p, q)$. Define, for $X, Y \in B^{A^2}$,

$$(X + Y)(q, p) = h_{qp}(t_{xyz} \mid (x, y, z) \in \Gamma_{\mathbf{A}}),$$

where $t_{sss} = X(s, s)$, $t_{rss} = X(s, r)$ and $t_{srs} = Y(s, r)$, for $s, r \in A$. (This defines t_{xyz} for all $(x, y, z) \in \Gamma_{\mathbf{A}}$.)

We claim that this shows the additivity of (\mathbf{A}, \mathbf{B}) . Consider any polymorphism $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ and disjoint $S, T \subseteq [n]$. Write $\pi : [n] \rightarrow [3]$ for the function that takes S to 1, T to 2, and $[n] \setminus (S \cup T)$ to 3. By the definition of h_{qp} and that of taking minors, we have

$$(f^p(S \cup T))(q, p) = f_{qp}(S \cup T) = f^\pi(p, p, q) = h_{qp}(f^\pi(x, y, z) \mid (x, y, z) \in \Gamma_{\mathbf{A}}).$$

Now, let $t_{xyz} = f^\pi(x, y, z)$. Observe that $t_{sss} = f^\pi(s, s, s) = f^p(S)(s, s)$. Also, $t_{rss} = f^\pi(r, s, s) = f^p(S)(s, r)$, and $t_{srs} = f^\pi(s, r, s) = f^p(T)(s, r)$. Thus we deduce that

$$(f^p(S \cup T))(q, p) = h_{qp}(t_{xyz} \mid (x, y, z) \in \Gamma_{\mathbf{A}}) = (f^p(S) + f^p(T))(q, p).$$

Thus,

$$f^p(S \cup T) = f^p(S) + f^p(T),$$

as required by Definition 14. \square

Lemma 24. Assuming $A = [a]$, (\mathbf{A}, \mathbf{B}) is dependent if we have

$$\Delta_{\mathbf{A}}^a \vDash_{\mathbf{A}, \mathbf{B}} (1, \dots, a).$$

PROOF. Suppose h witnesses that $\Delta_{\mathbf{A}}^a \vDash_{\mathbf{A}, \mathbf{B}} (1, \dots, a)$. Note that every tuple in $\Delta_{\mathbf{A}}^a$ can be seen as a function from $[a]$ to some $S \subseteq A$ for $0 < |S| \leq 2$. In other words h takes a tuple of elements from B , indexed by functions $\alpha : A \rightarrow S, S \in A^{\leq 2}$. Equivalently, we see that for any a -ary polymorphism f ,

$$f(1, \dots, a) = h(f(\alpha(1), \dots, \alpha(a)) \mid \alpha : A \rightarrow S, S \in A^{\leq 2}).$$

The type of this function (not coincidentally) is exactly the same as the function that ought to witness the dependency of (\mathbf{A}, \mathbf{B}) ; and indeed, we claim that it does in fact witness this.

In other words, we must show that, for any $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ and $x_1, \dots, x_n \in A$ we have that

$$f(x_1, \dots, x_n) = h(f(\alpha(x_1), \dots, \alpha(x_n)) \mid A \rightarrow S, S \in A^{\leq 2}).$$

Observe that for any x_1, \dots, x_n , there exists a function $\pi : [n] \rightarrow [a]$ such that $f(x_1, \dots, x_n) = f^\pi(1, \dots, a)$; namely $\pi(i) = x_i$. Furthermore f^π is an a -ary polymorphism; thus

$$\begin{aligned} f(x_1, \dots, x_n) &= f^\pi(1, \dots, a) = h(f^\pi(\alpha(1), \dots, \alpha(a)) \mid \alpha : A \rightarrow S, S \in A^{\leq 2}) \\ &= h(f(\alpha(x_1), \dots, \alpha(x_n)) \mid A \rightarrow S, S \in A^{\leq 2}), \end{aligned}$$

as required. \square

We now move on to a syntactic description of the formal proof system.

Definition 25. Fix some $n \in \mathbb{N}$. Fix also some relation R^A of A . For sets of n -ary tuples $S \subseteq A^n$ and $t \in A^n$, we define $S \vdash_{A, R^A} t$ as the minimal relation that satisfies the following.

- (1) If $S \subseteq T$ and $S \vdash t$ then $T \vdash t$.
- (2) $t \vdash t$.
- (3) If $S \vdash t$ and $t, T \vdash t'$ then $S, T \vdash t'$.
- (4) If there exists a matrix with n columns and r rows, whose rows are t_1, \dots, t_r , and whose columns are tuples of R^A , then $t_2, \dots, t_r \vdash t_1$.

We omit A, R^A if they are obvious from context.

Remark 26. Suppose R^A has arity r . Any judgement of the form $S \vdash_{A, R^A} t$ must have a finite proof using the rules above by minimality. From this proof, we can create a *proof-tree* where the vertices are n -ary tuples of A , the root is t , the leaves belong to S , and every non-leaf has $r - 1$ children such that if t_1 is the non-leaf and t_2, \dots, t_r are its children, then the matrix whose rows are t_1, \dots, t_r has as its columns only tuples of R^A . To create this tree, proceed inductively on the proof, from the conclusion backwards. The first and second rule do not modify the proof tree. The third rule corresponds to recursively constructing a subtree. The final rule is the only one that adds new vertices to the proof tree.

Lemma 27. If $S \vdash_{A, R^A} t$ then $S \vDash_{A, B} t$ for any symmetric A and functional B .

PROOF. By minimality, it is sufficient to show that $\vDash_{A, B}$ satisfies all the rules satisfied by \vdash_{A, R^A} , which we do now rule-by-rule. For the following assume always that all tuples are of arity n .

- (1) Suppose $S \subseteq T$ and $S \vDash t$. Thus there exists a function $h : A^S \rightarrow A$ such that, for any polymorphism $f \in \text{Pol}^{(n)}(A, B)$, we have

$$f(t) = h(f(u) \mid u \in S).$$

Now, we define the function $h' : A^T \rightarrow A$ as follows:

$$h'(x_u \mid u \in T) = h(x_u \mid u \in S).$$

In other words, h' ignores all inputs in $T \setminus S$ and otherwise acts like h . We claim that h' witnesses that $T \vDash t$: for any polymorphism $f \in \text{Pol}^{(n)}(A, B)$, we have

$$f(t) = h(f(u) \mid u \in S) = h'(f(u) \mid u \in T).$$

- (2) For any tuple t , we have $t \vDash t$; indeed, the identity function $\text{id}_B : B \rightarrow B$ witnesses this fact.
- (3) Suppose $S \vDash t$ and $t, T \vDash u$, as witnessed by $h : A^S \rightarrow A$ and $h' : B^{T \setminus S} \rightarrow B$. If $t \in T$ then $T \vDash u$ and we have by the first rule that $S, T \vDash u$ as required; thus suppose $t \notin T$. We will thus interpret h' as having signature $h' : B \rightarrow B^T \rightarrow B$. With this interpretation, by definition, for any polymorphism $f \in \text{Pol}^{(n)}(A, B)$, we have

$$\begin{aligned} f(t) &= h(f(v) \mid v \in S) \\ f(u) &= h'(f(t))(f(v) \mid v \in T). \end{aligned}$$

Now define $h'' : B^{S \cup T} \rightarrow B$ in the following way:

$$h''(x_v \mid v \in S \cup T) = h'(h(x_v \mid v \in S))(x_v \mid v \in T).$$

We find that, for any $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$,

$$\begin{aligned} f(u) = h'(f(t))(f(v) \mid v \in T) &= h'(h(f(v) \mid v \in S))(f(v) \mid v \in T) \\ &= h''(f(v) \mid v \in S \cup T). \end{aligned}$$

Thus we find that $S, T \vDash u$, as required.

- (4) Suppose R^A has arity r , and suppose there exists a matrix with n columns and r rows, whose rows are t_1, \dots, t_r , and whose columns are tuples of R^A . Let $h : B^{\{t_2, \dots, t_r\}} \rightarrow B$ be defined as follows, interpreting an element of $B^{\{t_2, \dots, t_r\}}$ as an $(r-1)$ -ary tuple in the natural way:

$$h(x_2, \dots, x_r) = R^B(x_2, \dots, x_r).$$

Now, we claim that h witnesses that $t_2, \dots, t_r \vDash t_1$. Indeed, for any n -ary polymorphism $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$, we have that $(f(t_1), \dots, f(t_r)) \in R^B$, and thus

$$f(t_1) = R^B(f(t_2), \dots, f(t_r)) = h(f(t_2), \dots, f(t_r)) = h(f(u) \mid u \in \{t_2, \dots, t_r\}).$$

Thus we conclude that $\vDash_{\mathbf{A}, \mathbf{B}}$ satisfies the rules of $\vdash_{\mathbf{A}, R^A}$, which implies our conclusion. \square

For a relation R , we define its support $\text{supp}(R)$ as the elements that appear in at least one tuple of R . Formally,

$$\text{supp}(R) = \bigcup_{i=1}^{\text{ar}(R)} \{a_i \mid (a_1, \dots, a_{\text{ar}(R)}) \in R\}.$$

Lemma 28. *Suppose that $S \vdash_{\mathbf{A}, R^A} t$ and $x \in \text{supp}(R^A)$. Then $S \times \text{supp}(R^A) \vdash_{\mathbf{A}, R^A} (t, x)$.¹¹*

PROOF. We show this by induction on the formal proof that proves that $S \vdash_{\mathbf{A}, R^A} t$. Based on the last step in the proof used to prove this fact, we have the following cases.

- (1) Suppose $S' \subseteq S$ and $S' \vdash t$. By the inductive hypothesis, we know that $S' \times \text{supp}(R^A) \vdash (t, x)$. As $S' \times \text{supp}(R^A) \subseteq S \times \text{supp}(R^A)$, then $S \times \text{supp}(R^A) \vdash (t, x)$ as required.
- (2) Suppose $S = \{t\}$. In this case the result is immediate, as $S \times \text{supp}(R^A) \supseteq \{(t, u)\} \vdash (t, u)$.
- (3) Suppose $S = X \cup Y$, $X \vdash s$ and $s, Y \vdash t$. By the inductive hypothesis, for $y \in \text{supp}(R^A)$, $X \times \text{supp}(R^A) \vdash (s, y)$ and $\{(s, y) \mid y \in \text{supp}(R^A)\}, Y \times \text{supp}(R^A) \vdash (t, x)$. We can therefore deduce, by this rule applied $|\text{supp}(R^A)|$ times, that $S \times \text{supp}(R^A) = (X \cup Y) \times \text{supp}(R^A) \vdash (t, x)$.
- (4) Suppose that $S = \{t_2, \dots, t_r\}$ and there exists a matrix whose rows are t, t_2, \dots, t_r and all of whose columns are elements of R^A . Suppose $(x, x_2, \dots, x_r) \in R^A$. Then by applying this rule we find that $(t_2, x), \dots, (t_r, x_r) \vdash (t, x)$, which implies that $S \times \text{supp}(R^A) \vdash (t, x)$.

Thus our conclusion follows. \square

Any tuple $x \in A^n$ can be seen as a function from $[n]$ to A . Thus for any function $\pi : [m] \rightarrow [n]$, we let $x \circ \pi \in A^m$ be the tuple which, at position i , has value $x_{\pi(i)}$. For a set of tuples $S \in A^n$, we let $S \circ \pi = \{x \circ \pi \mid x \in S\}$.

Lemma 29. *Suppose $S \subseteq A^n$, $t \in A^n$ and $\pi : [m] \rightarrow [n]$. If $S \vdash_{\mathbf{A}, R^A} t$ then $S \circ \pi \vdash_{\mathbf{A}, R^A} t \circ \pi$.*

¹¹Here, if $t = (t_1, \dots, t_r)$ then $(t, x) = (t_1, \dots, t_r, x)$, and likewise $S \times \text{supp}(R^A) = \{(s_1, \dots, s_r, x) \mid (s_1, \dots, s_r) \in S, x \in \text{supp}(R^A)\}$.

PROOF. First, consider any matrix M whose rows are r_1, \dots, r_n and whose columns are c_1, \dots, c_m . If we create a matrix M' whose rows are $r_1 \circ \sigma, \dots, r_n \circ \sigma$ for some $\sigma : [k] \rightarrow [m]$, we observe that the columns of this new matrix are $c_{\sigma(1)}, \dots, c_{\sigma(k)}$. Thus, the columns of M' are a subset of the columns of M .

Consider the proof tree that proves that $S \vdash_{A, R^A} t$. If a vertex is labeled by a tuple u , transform it into a vertex labeled by $u \circ \pi$. By the previous observation, this remains a valid proof tree, whose leaves belong to $S \circ \pi$ and whose root is $t \circ \pi$. Thus $S \circ \pi \vdash t \circ \pi$. \square

3.4 Super-connectedness

Rather than dealing directly with additivity and dependency, we will instead deal with a notion that implies them.

Definition 30. We say that A is *super-connected* if A has a relation R^A such that, for every $x, y, z \in A$ (perhaps even with $|\{x, y, z\}| < 3$), we have

$$\Gamma_A \vdash_{A, R^A} (x, y, z).$$

Super-connectedness is a very useful concept: it implies both additivity and dependency.

Lemma 31. *If A is super-connected then (A, B) is additive for functional B where $A \rightarrow B$.*

PROOF. An immediate consequence of Lemma 23 and Lemma 27. \square

Lemma 32. *If A is super-connected then (A, B) is dependent for functional B where $A \rightarrow B$.*

We first prove two simple propositions. Assume for them that A is super-connected, witnessed by relation R^A , and that $|A| > 1$.

Proposition 33. $\text{supp}(R^A) = A$.

PROOF. Suppose for contradiction that this is not the case. Let $x \in A \setminus \text{supp}(R^A)$, and take $y \in A$ such that $x \neq y$. Now,

$$\Gamma_A \vdash_{A, R^A} (x, x, y).$$

Consider any proof tree concluding in this judgement. The root vertex cannot have any children, since no matrix containing (x, x, y) can have its columns belong to R^A , as $x \notin \text{supp}(R^A)$. Thus (x, x, y) is a leaf vertex in the tree, and $(x, x, y) \in \Gamma_A$. This is not possible, as $x \neq y$. \square

Proposition 34. *For any $x_1, \dots, x_n \in A$, we have that $\Delta_A^n \vdash (x_1, \dots, x_n)$.*

PROOF. We prove this fact by induction. We do induction over the lexicographical ordering on $\{(n, d) \in \mathbb{N}^2 \mid n \geq d\}$. For any $(n, d) \in \mathbb{N}^2$ with $n \geq d$ we prove that $\Delta_A^n \vdash (x_1, \dots, x_n)$ whenever $|\{x_1, \dots, x_n\}| \leq d$. As our base case, note that the result is immediate when $n \leq 2$, as $(x_1, \dots, x_n) \in \Delta_A^n$ in this case. Thus suppose $n \geq 3$.

First suppose that x_1, \dots, x_n contains a duplicated pair of values, say $x_{n-1} = x_n$. Now, define $\pi : [n-1] \rightarrow [n]$, where $\pi(i) = i$. By the inductive hypothesis, we know that $\Delta_A^{n-1} \vdash (x_1, \dots, x_{n-1})$. Thus, by Lemma 29, $\Delta_A^{n-1} \circ \pi \vdash (x_1, \dots, x_{n-1}) \circ \pi = (x_1, \dots, x_{n-1}, x_{n-1}) = (x_1, \dots, x_n)$. Furthermore, every tuple of $\Delta_A^{n-1} \circ \pi$ contains at most 2 values, so it belongs to Δ_A^n . Thus in this case $\Delta_A^n \vdash (x_1, \dots, x_n)$.

Now suppose that x_1, \dots, x_n are all distinct. Since A is super-connected, we find that $\Gamma_A \vdash (x_1, x_2, x_3)$. Thus by Lemma 28, we find that $\Gamma_A \times \text{supp}(R^A)^{n-3} \vdash (x_1, \dots, x_n)$. By Proposition 33, $\text{supp}(R^A) = A$, so $\Gamma_A \times A^{n-3} \vdash (x_1, \dots, x_n)$. But, every tuple $t \in \Gamma_A \times A^{n-3}$ contains at most $n-1$ distinct values, so we can apply the inductive hypothesis to them, and find that $\Delta_A^n \vdash t$. Thus $\Delta_A^n \vdash (x_1, \dots, x_n)$, as required. This completes the proof. \square

PROOF OF LEMMA 32. $\Delta_A^a \vdash (1, \dots, a)$ by Proposition 34. Thus, by Lemma 27 we have that $\Delta_A^a \vDash_{A,B} (1, \dots, a)$, and therefore dependency follows by Lemma 24. \square

Lemma 35. *Suppose A has a symmetric relation R^A of arity at least 3 with $\text{diam}(A, R^A) \leq 1$. Then A is super-connected.*

PROOF. Take $x, y, z \in A$; we must show that $\Gamma_A \vdash (x, y, z)$. We have several cases.

$x = z$ or $y = z$. In this case $(x, y, z) \in \Gamma_A$, so there is nothing left to be proved.

$x = y \neq z$. In this case $\text{dist}_{R^A}(x, z) = 1$, so there exists an edge $(x, z, a_3, \dots, a_r) \in R^A$. Consider the matrices

$$M = \begin{pmatrix} x & x & z \\ z & a_3 & x \\ a_3 & z & a_3 \\ a_4 & a_4 & a_4 \\ \vdots & \vdots & \vdots \\ a_r & a_r & a_r \end{pmatrix}, \quad N = \begin{pmatrix} z & a_3 & x \\ x & z & z \\ a_3 & x & a_3 \\ a_4 & a_4 & a_4 \\ \vdots & \vdots & \vdots \\ a_r & a_r & a_r \end{pmatrix}.$$

Thus we deduce that $(z, a_3, x), \Gamma_A \vdash (x, x, z)$ and $\Gamma_A \vdash (z, a_3, x)$, and thus $\Gamma_A \vdash (x, x, z) = (x, y, z)$.

$x \neq y, y \neq z, x \neq z$. In this case $\text{dist}_{R^A}(x, z) = \text{dist}_{R^A}(x, y) = 1$, so there must exist edges $(x, z, a_3, \dots, a_r), (x, y, b_3, \dots, b_r) \in R^A$. Consider the following matrix

$$\begin{pmatrix} x & y & z \\ z & x & x \\ a_3 & b_3 & a_3 \\ \vdots & \vdots & \vdots \\ a_r & b_r & a_r \end{pmatrix}.$$

Thus we find that $(z, x, x), (a_3, b_3, a_3), \dots, (a_r, b_r, a_r) \vdash (x, y, z)$, whence $\Gamma_A \vdash (x, y, z)$.

Thus our conclusion follows in all cases. \square

Lemma 36. *If A has a connected symmetric relation R^A of arity 3 then A is super-connected.*

PROOF. We see (A, R^A) as a connected 3-uniform hypergraph. For $x, y \in A$ recall that $\text{dist}(x, y)$ is the distance between x and y in this hypergraph. We show that $\Gamma_A \vdash_{A, R^A} (x, y, z)$ for all $x, y, z \in A$ by lexicographic induction on $\min(\text{dist}(x, z), \text{dist}(y, z))$ (i.e. we first order by the minimum, and in case of equality the maximum). In all the cases that follow assume $\text{dist}(x, z) \leq \text{dist}(y, z)$.

$\text{dist}(x, z) = 0$. In this case $(x, y, z) = (x, y, x) \in \Gamma_A$ so there is nothing left to prove.

$\text{dist}(x, z) = \text{dist}(y, z) = 1$. In this case, there exists edges $(x, a, z), (y, b, z)$. Now consider the following matrices:

$$M = \begin{pmatrix} x & y & z \\ z & z & x \\ a & b & a \end{pmatrix}, \quad N = \begin{pmatrix} z & z & x \\ a & x & z \\ x & a & a \end{pmatrix}, \quad P = \begin{pmatrix} a & x & z \\ x & z & x \\ z & a & a \end{pmatrix}.$$

Thus we conclude that $(a, b, a), (z, z, x) \vdash (x, y, z)$, $(a, x, z), (x, a, a) \vdash (z, z, x)$ and also $(x, z, x), (z, a, a) \vdash (a, x, z)$, from where the conclusion follows.

$\text{dist}(y, z) > \text{dist}(x, z) = 1$. In this case, there exist edges (x, a, z) and (y, b, y') with $\text{dist}(y', z) = \text{dist}(y, z) - 1$. Furthermore, as $\text{dist}(y', z) > 0$, there exists an edge (y', c, d) such that

$$\text{dist}(c, z), \text{dist}(d, z) \leq \text{dist}(y', z) \leq \text{dist}(y, z) - 1,$$

Now consider matrices

$$M = \begin{pmatrix} x & y & z \\ a & b & a \\ z & y' & x \end{pmatrix}, \quad N = \begin{pmatrix} z & y' & x \\ a & c & a \\ x & d & z \end{pmatrix}.$$

Hence $(z, y', x), \Gamma_A \vdash (x, y, z)$ and $(x, d, z), \Gamma_A \vdash (z, y', x)$. Since $\text{dist}(d, z) < \text{dist}(y, z)$ we can apply the inductive hypothesis to (x, d, z) , so $\Gamma_A \vdash (x, y, z)$ as required.

dist $(y, z) \geq \text{dist}(x, z) > 1$. In this case, there exist edges (x, a, x') , (z, b, z') such that it is true that $\text{dist}(a, z'), \text{dist}(b, x') < \text{dist}(x, z)$ — take edges from x, z towards the other one. (Such edges only exist when $\text{dist}(x, z) > 1$ as assumed.) Furthermore, by connectedness, an edge (y, c, d) exists. Thus consider the matrix

$$\begin{pmatrix} x & y & z \\ a & c & z' \\ x' & d & b \end{pmatrix}.$$

Hence $(a, c, z'), (x, d, b) \vdash (x, y, z)$. By induction $\Gamma_A \vdash (a, c, z')$ and $\Gamma_A \vdash (x', d, b)$, so we have $\Gamma_A \vdash (x, y, z)$.

Thus we find by induction that \mathbf{A} is super-connected. \square

Lemma 37. *If \mathbf{A} has a connected symmetric relation R^A of arity 4 then \mathbf{A} is super-connected.*

PROOF. We see (A, R^A) as a connected 4-uniform hypergraph. We show that $\Gamma_A \vdash_{A, R^A} (x, y, z)$ for all $x, y, z \in A$ by lexicographic induction on $\min(\text{dist}(x, z), \text{dist}(y, z))$. In all the cases that follow assume $\text{dist}(x, z) \leq \text{dist}(y, z)$.

dist $(x, z) = 0$. In this case $(x, y, z) = (x, y, x) \in \Gamma_A$ so there is nothing left to prove.

dist $(x, z) = \text{dist}(y, z) = 1$. In this case, there exist edges (x, a, a', z) , (y, b, b', z) . Now consider the following matrices:

$$M = \begin{pmatrix} x & y & z \\ z & z & x \\ a & b & a \\ a' & b' & a' \end{pmatrix}, \quad N = \begin{pmatrix} z & z & x \\ a & x & z \\ x & a & a \\ a' & a' & a' \end{pmatrix}, \quad P = \begin{pmatrix} a & x & z \\ x & z & x \\ z & a & a \\ a' & a' & a' \end{pmatrix}.$$

Hence, $\Gamma_A \vdash (x, y, z)$.

dist $(y, z) > \text{dist}(x, z) = 1$. In this case, there exist edges (x, a, a', z) and (y, b, b', y') such that $\text{dist}(y', z) = \text{dist}(y, z) - 1$. Furthermore, as $\text{dist}(y', z) > 0$, there exists an edge (y', c, d, e) such that

$$\text{dist}(c, z), \text{dist}(d, z), \text{dist}(e, z) \leq \text{dist}(y', z) \leq \text{dist}(y, z) - 1,$$

Now consider matrices

$$M = \begin{pmatrix} x & y & z \\ a & b & a \\ a' & b' & a' \\ z & y' & x \end{pmatrix}, \quad N = \begin{pmatrix} z & y' & x \\ a & c & a \\ a' & d & a' \\ x & e & z \end{pmatrix}.$$

Hence $(z, y', x), \Gamma_A \vdash (x, y, z)$ and $(x, d, z), \Gamma_A \vdash (z, y', x)$. Since $\text{dist}(d, z) < \text{dist}(y, z)$ we can apply the inductive hypothesis to (x, d, z) , so $\Gamma_A \vdash (x, y, z)$ as required.

dist $(y, z) \geq \text{dist}(x, z) > 1$. In this case, there exist edges (x, a, a', x') , (z, b, b', z') such that it is true that $\text{dist}(a, z'), \text{dist}(a', z'), \text{dist}(b, x'), \text{dist}(b', x') < \text{dist}(x, z)$ — take edges from

x, z towards the other one. Furthermore, as $\text{dist}(y, z) > 2$ there exist edges (y, c, c', y') and (y', d, d', y'') , with $\text{dist}(y'', z) = \text{dist}(y, z) - 2$. Thus consider the matrices

$$M = \begin{pmatrix} x & y & z \\ a' & c & z' \\ x' & c' & b' \\ a & y' & b \end{pmatrix}, \quad N = \begin{pmatrix} a & y' & b \\ a' & d & z' \\ x' & d' & b' \\ x & y'' & z \end{pmatrix}.$$

Note that $\text{dist}(a', z'), \text{dist}(x', b') < \text{dist}(x, z)$, and furthermore $\text{dist}(y'', z) < \text{dist}(y, z)$, so by induction $\Gamma_A \vdash (x, y, z)$.

Thus we find by induction that \mathbf{A} is super-connected. \square

With all of this, we are finally ready to prove Theorem 12 and Theorem 13.

PROOF OF THEOREM 12 AND THEOREM 13. The structures from Theorem 12 are super-connected by Lemma 35. The structures from Theorem 13 are super-connected by Lemma 36 and Lemma 37. This is sufficient for additivity by Lemma 31 and for dependency by Lemma 32. \square

4 Dichotomy

In this section we will prove our main result.

Theorem 1. *Let \mathbf{A} be a symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Assume that (\mathbf{A}, \mathbf{B}) is dependent and additive. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

Before proving Theorem 1, we will prove three interesting corollaries.

Corollary 2. *Let \mathbf{A} be a Boolean symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

For the proof of Corollary 2, we will need a simple lemma.

Lemma 38. *Suppose \mathbf{A} is Boolean and symmetric, \mathbf{B} is functional, and every relation of \mathbf{A} is either binary or contains only constant tuples. Then, $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable.*

PROOF. Consider any $h : \mathbf{A} \rightarrow \mathbf{B}$. Suppose $h(0) = h(1)$. Then every relation in \mathbf{B} contains a constant tuple of the form $(h(0), \dots, h(0))$; in this case, $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is trivially solved by AIP and is finitely tractable. Thus suppose $h(0) \neq h(1)$. Any empty relation in \mathbf{A} can be removed (together with the corresponding relation in \mathbf{B}) as it does not affect $\text{Pol}(\mathbf{A}, \mathbf{B})$ and the complexity of $\text{PCSP}(\mathbf{A}, \mathbf{B})$.

Since \mathbf{B} is functional, the binary relations of \mathbf{A} that do not contain only constant tuples must be the binary disequality. To see why, consider any relation $R^{\mathbf{A}}$ in \mathbf{A} that contains the tuple $(0, 1)$. $R^{\mathbf{A}}$ cannot contain $(0, 0)$ or $(1, 1)$, since the corresponding relation $R^{\mathbf{B}}$ in \mathbf{B} contains $(h(0), h(1))$ and if it contained $(h(0), h(0))$ or $(h(1), h(1))$ it would not be functional. Since \mathbf{A} is symmetric, $R^{\mathbf{A}}$ also contains the tuple $(1, 0)$. Thus $R^{\mathbf{A}} = \{(0, 1), (1, 0)\}$ is the disequality relation. It follows that every relation in \mathbf{A} is either a binary disequality, or consists only of constant tuples. In this case, $\text{CSP}(\mathbf{A})$ is solved by AIP and thus $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solved by AIP and is finitely tractable. \square

PROOF OF COROLLARY 2. If \mathbf{A} contains only binary relations or relations that contain only constant tuples, the conclusion follows by Lemma 38. Otherwise, \mathbf{A} has a relation $R^{\mathbf{A}}$ of arity at least 3 with a non-constant tuple i.e. for which $\text{diam}(A, R^{\mathbf{A}}) = 1$, and the conclusion follows from Theorem 1 together with super-connectivity of such structures (cf. Lemma 35): super-connectedness implies additivity (cf. Lemma 31) and dependency (cf. Lemma 32). \square

Corollary 3. *Let \mathbf{A} be a symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Assume that both \mathbf{A} and \mathbf{B} have exactly one relation of arity at most 4. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

PROOF. First, observe that the conclusion holds trivially if $R^{\mathbf{B}}$ has arity 1 as in this case $\text{CSP}(\mathbf{B})$ is trivially tractable. Second, suppose $R^{\mathbf{B}}$ has arity 2. In this case we claim that $\text{CSP}(\mathbf{B})$ is solvable in polynomial time. Indeed, an easy modification of the algorithm for 2-colouring will solve this problem. Thus $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is tractable in this case.

Now, suppose that $R^{\mathbf{B}}$ has arity 3 or 4. The conclusion holds for connected \mathbf{A} by the super-connectedness of structures \mathbf{A} that have a relation of arity 3 or 4 that is connected. Thus suppose that \mathbf{A} is not connected, and that its connected components are $\mathbf{A}_1, \dots, \mathbf{A}_k$ i.e. $\mathbf{A} = \mathbf{A}_1 + \dots + \mathbf{A}_k$. If $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$ is NP-hard for some $i \in [k]$ then $\text{PCSP}(\mathbf{A}, \mathbf{B})$ will also be NP-hard (since there is a trivial reduction from $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$ to $\text{PCSP}(\mathbf{A}, \mathbf{B})$, as $\mathbf{A}_i \rightarrow \mathbf{A}$). Thus suppose $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$ is solvable in polynomial time for all $i \in [k]$. Consider any input hypergraph $\mathbf{X} = \mathbf{X}_1 + \dots + \mathbf{X}_n$, where $\mathbf{X}_1, \dots, \mathbf{X}_n$ are connected. Since the homomorphic image of a connected hypergraph is connected, if $\mathbf{X}_i \rightarrow \mathbf{A}$ then $\mathbf{X}_i \rightarrow \mathbf{A}_j$ for some $j \in [k]$. Thus, for the decision version of $\text{PCSP}(\mathbf{A}, \mathbf{B})$, it is sufficient to see if, for all $i \in [n]$ there exists some $j \in [k]$ such that \mathbf{X}_i is a YES-instance of $\text{PCSP}(\mathbf{A}_j, \mathbf{B})$. If so then \mathbf{X} is a YES-instance overall. For the search version something similar happens: see if we can produce a homomorphism $\mathbf{X}_i \rightarrow \mathbf{B}$ for each \mathbf{X}_i by running the algorithm for $\text{PCSP}(\mathbf{A}_j, \mathbf{B})$ for each \mathbf{A}_j , and combine these homomorphisms to find a homomorphism $\mathbf{A} \rightarrow \mathbf{B}$. \square

Remark 39. Unlike in all other results in this paper, solvability by AIP is not proved in Corollary 3, only polynomial-time solvability. As far as we know, it well may be that the PCSPs considered in Corollary 3 are not solved by AIP (or even BLP + AIP). The way that tractability is deduced in the proof of Corollary 3 is as follows: If $\mathbf{A}_1, \dots, \mathbf{A}_k$ are connected and $\text{PCSP}(\mathbf{A}_i, \mathbf{B})$ is tractable, then $\text{PCSP}(\mathbf{A}_1 + \dots + \mathbf{A}_k, \mathbf{B})$ is tractable. We give a concrete example of a template with a single symmetric relation of arity 6 that shows that this reduction *does not* preserve solvability by AIP, or even by BLP + AIP. Note however that Corollary 3 only applies to relations of arity at most 4, so the corollary might still be strengthened to show solvability by AIP.

Let $\mathbb{Z}' = \{x' \mid x \in \mathbb{Z}\}$ be a disjoint copy of \mathbb{Z} . Then, let $\mathbf{B} = \mathbf{A}_1 + \mathbf{A}_2$, where

$$\begin{aligned} \mathbf{A}_1 &= (\{0, 1\}; \{(x_1, \dots, x_6) \mid x_1 + \dots + x_6 \equiv 1 \pmod{2}\}), \\ \mathbf{A}_2 &= (\{0', 1', 2'\}; \{(x_1, \dots, x_6) \mid x_1 + \dots + x_6 \equiv 2' \pmod{3'}\}). \end{aligned}$$

Note that $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}$ are all functional and symmetric. Furthermore, $\text{PCSP}(\mathbf{A}_1, \mathbf{B})$ and $\text{PCSP}(\mathbf{A}_2, \mathbf{B})$ are solved by AIP by Theorem 8: a $(2k + 1)$ -ary alternating polymorphism is

$$(x_1, y_1, \dots, x_k, y_k, x_{k+1}) \mapsto x_1 - y_1 + \dots + x_k - y_k + x_{k+1} \pmod{m},$$

where $m = 2$ for \mathbf{A}_1 and $m = 3'$ for \mathbf{A}_2 . We will now show that BLP + AIP fails to solve $\text{PCSP}(\mathbf{B}, \mathbf{B}) = \text{CSP}(\mathbf{B})$; to see why, suppose that it does solve it. Thus, by Theorem 9, $\text{Pol}(\mathbf{B}, \mathbf{B})$ should contain a 2-block symmetric polymorphism of arity 5, say $f \in \text{Pol}^{(5)}(\mathbf{B}, \mathbf{B})$. For the ease of notation, suppose that the first block of symmetry of f contains the first 3 inputs, and the second block of symmetry contains the last 2 inputs (rather than the blocks being based on parity). Now consider the following matrix

$$\left(\begin{array}{ccc|cc} 1' & 0' & 0' & 1 & 0 \\ 0' & 1' & 0' & 0 & 1 \\ 0' & 0' & 1' & 1 & 0 \\ 1' & 0' & 0' & 0 & 1 \\ 0' & 1' & 0' & 1 & 0 \\ 0' & 0' & 1' & 0 & 1 \end{array} \right)$$

Every column of this matrix is an element of $R^{\mathbf{B}}$; thus f applied to every row gives a tuple of $R^{\mathbf{B}}$. But, due to block-symmetry, the image of every row through f is the same! This contradicts the lack of constant tuples in \mathbf{B} .

The non-solvability of $\text{PCSP}(\mathbf{B}, \mathbf{B})$ by AIP is relevant in view of Conjecture 5 – it shows that the “AIP being a universal algorithm for $\text{PCSP}(\mathbf{1-in-3}, -)$ ” part of Conjecture 5 cannot be extended to *arbitrary* symmetric templates. We believe that it might hold true for any *connected* \mathbf{A} with one symmetric relation.

Corollary 4. *Let \mathbf{A} be a symmetric structure and \mathbf{B} be a functional structure such that $\mathbf{A} \rightarrow \mathbf{B}$. Assume that \mathbf{A} has a relation $R^{\mathbf{A}}$ of arity at least 3 for which $\text{diam}(A, R^{\mathbf{A}}) \leq 1$. Then, either $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time by AIP and is finitely tractable, or $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

PROOF. Such structures are super-connected by Lemma 35, which implies additivity and dependency of (\mathbf{A}, \mathbf{B}) by Lemma 31 and Lemma 32, respectively. \square

We will now move on to a proof of Theorem 1. Suppose (\mathbf{A}, \mathbf{B}) is dependent and additive. Suppose generally that $A = [a]$.

Definition 40. Consider a polymorphism $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$. We call it k -degenerate if there exist $x_1, \dots, x_k \in \text{range}(f^p)$ such that for any $S_1, \dots, S_k \subseteq [n]$ for which $f^p(S_i) = x_i$ we have that not all S_i are disjoint. Note that no polymorphism can be 1-degenerate as a single set is a disjoint family.

For any polymorphism $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$, we call a set $S \subseteq [n]$ a *hard set* if, for any $T \supseteq S$, we have $f^p(T) \neq f^p(\emptyset)$.¹²

We will prove Theorem 1 using the following two cases. For the following, define $N_d = \max(1 + |B|^{a^2} a^{2r_{\max}}, 3)$ and $N_h = |B|^{a^2}$, where r_{\max} is the maximum arity of any relation in \mathbf{A} .

Theorem 41. *If $\text{Pol}(\mathbf{A}, \mathbf{B})$ contains a polymorphism that is not k -degenerate, for any $k \leq N_d$, and that has no hard sets of size at most N_h , then $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solved by AIP and is finitely tractable.*

Theorem 42. *If every polymorphism within $\text{Pol}(\mathbf{A}, \mathbf{B})$ is k -degenerate for some $k \leq N_d$, or has a hard set of size at most N_h , then $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard.*

These two theorems will be proved in their own sections later.

PROOF OF THEOREM 1. A result of Theorem 41 and Theorem 42. \square

Remark 43. The following turns out to be an equivalent characterisation of the solvability of $\text{PCSP}(\mathbf{A}, \mathbf{B})$ for symmetric \mathbf{A} and functional \mathbf{B} . Suppose \mathbf{A} has domain A and relations $R_1^{\mathbf{A}}, \dots, R_k^{\mathbf{A}}$. For any positive integer m , define a structure \mathbf{A}_m in the following way. The domain of \mathbf{A}_m is the free \mathbb{Z}_m -module of functions \mathbb{Z}_m^A . We will write elements of this module as formal sums of the form $\sum_{a \in A} x_i \bar{a}$, where $x_i \in \mathbb{Z}_m$ and $\{\bar{a} \mid a \in A\}$ is a basis for \mathbb{Z}_m^A . Extend the map $x \mapsto \bar{x}$ to tuples of functions, in the following way: $(x_1, \dots, x_n) = (\bar{x}_1, \dots, \bar{x}_n)$. To define the relation $R_k^{\mathbf{A}_m}$, consider the set of tuples of functions $S = \{\bar{t} \mid t \in R_k^{\mathbf{A}}\}$. Tuples of functions also form a free \mathbb{Z}_m -module; thus take $R_k^{\mathbf{A}_m}$ to be the minimal affine space containing S . Equivalently, $R_k^{\mathbf{A}_m}$ is the set of tuples that are equivalent to some $t \in R_k^{\mathbf{A}}$ modulo $\{t - t' \mid t, t' \in R_k^{\mathbf{A}}\}$, if the set of tuple of functions from \mathbb{Z}_m^A is seen as merely an Abelian group.

With the relational structure \mathbf{A}_m thus defined, and noting that $\mathbf{A} \rightarrow \mathbf{A}_m$ always via the homomorphism $x \mapsto \bar{x}$, we note that our result is equivalent to the following: for symmetric \mathbf{A} and functional \mathbf{B} where (\mathbf{A}, \mathbf{B}) is additive and dependent, $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solvable in polynomial time if

¹²These two notions are similar to those of *unbounded antichains* and *fixing sets* in [20]. The notion of hard-set is similar to the notion of an $f(\emptyset)$ -avoiding set from [18, 31].

and only if $\mathbf{A} \rightarrow \mathbf{A}_m \rightarrow \mathbf{B}$ for some positive integer m . That this condition is sufficient is clear, as $\text{CSP}(\mathbf{A}_m)$ is solved by AIP as per Lemma 10. Necessity follows from our proof of Theorem 41 and Theorem 42.

We note also that \mathbf{A}_m can also be described as a free structure [6]. Namely, define \mathcal{Z}_m to be a minion such that

$$\mathcal{Z}_m^{(n)} = \{(x_1, \dots, x_n) \mid 0 \leq x_1, \dots, x_n < m, \sum_{i=1}^n x_i \equiv 1 \pmod{m}\},$$

and let minoring be defined as follows. For $x = (x_1, \dots, x_n) \in \mathcal{Z}_m^{(n)}$, and for $\pi : [n] \rightarrow [n']$, define $y = x^\pi$ by $y_i = \sum_{\pi(j)=i} x_j \pmod{m}$. With this in mind, it can be seen that $\mathbf{A}_m = \mathbf{F}_{\mathcal{Z}_m}(\mathbf{A})$.

4.1 Proof of Theorem 41

In this section we assume that $\text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ has a polymorphism f of arity n that is not k -degenerate for k at most N_d , and has no hard sets of size at most N_h . Given this, we will prove that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solved by AIP and is finitely tractable.

Definition 44. Define $0 = f^P(\emptyset)$ and $1 = f^P([n])$.

Lemma 45. $(\text{range}(f^P), +, 0)$ forms a group.¹³

PROOF. We prove this in a few parts.

Closure, well-definedness. Consider $x, y \in \text{range}(f^P)$. As f is not 2-degenerate, there exist disjoint S, T such that $f^P(S) = x, f^P(T) = y$. Thus $x + y = f^P(S) + f^P(T) = f^P(S \cup T) \in \text{range}(f^P)$, so $+$ is closed and well-defined.

Associativity. Consider any $x, y, z \in \text{range}(f^P)$. Since f is not 3-degenerate, there exist disjoint $S, T, U \subseteq [n]$ such that $f^P(S) = x, f^P(T) = y, f^P(U) = z$. Thus,

$$x + (y + z) = x + f^P(T \cup U) = f^P(S \cup (T \cup U)) = f^P((S \cup T) \cup U) = f^P(S \cup T) + z = (x + y) + z.$$

Identity element. Consider any $x \in \text{range}(f^P)$. Suppose $f^P(S) = x$ for some $S \subseteq [n]$. Thus, $x + 0 = f^P(S \cup \emptyset) = f^P(S) = x$.

Inverses. Consider any $x \in \text{range}(f^P)$. Suppose that $f^P(S) = x$; by Lemma 17, some $T \subseteq S$ exists with size at most $|B|^{a^2}$ such that $f^P(T) = f^P(S) = x$. Since f has no hard sets of size at most $N_h = |B|^{a^2}$, T is not a hard set, and thus some $U \supseteq T$ exists such that $f^P(U) = 0$. Thus $x + f^P(U \setminus T) = f^P(T) + f^P(U \setminus T) = f^P(U) = 0$, so x has an inverse.

Thus we conclude that $(\text{range}(f^P), +, 0)$ is a group. \square

Definition 46. Let G be the Abelian subgroup of $(\text{range}(f^P), +, 0)$ generated by $1 = f^P([n])$. Let m be the order of 1 in G . Thus $G \cong \mathbb{Z}_m$. Note that $m \leq |\text{range}(f^P)| \leq |B|^{a^2}$. We will identify \mathbb{Z}_m with G (e.g. we allow ourselves to write $1 + 1 = 2$, provided $m \geq 3$, where $1, 2 \in \text{range}(f^P)$).

Define the Abelian group $(H, +) = G^a$. We will identify H with \mathbb{Z}_m^a . We will also define 0 to be the 0 element in H as well as G .

For any $i \in [a]$, define $\bar{i} \in H$ as the unit vector that has a 1 at position i . For some tuple $(x_1, \dots, x_r) \in [a]^r$, define $\overline{(x_1, \dots, x_r)} = (\overline{x_1}, \dots, \overline{x_r}) \in H^r$. Define 0 to be the zero vector in H^r as well.¹⁴

¹³This group happens to be Abelian, but this is not needed for the proof.

¹⁴We can see the elements of H as frequency vectors modulo m . Indeed, for $x_1, \dots, x_n \in [a]$, $\overline{x_1} + \dots + \overline{x_n}$ counts the number of appearances of $1, 2, \dots, a$ modulo m among x_1, \dots, x_n . In line with this, the elements of H^r can be seen as tuples of r frequency vectors. Under this view, for $t_1, \dots, t_n \in [a]^r$, the sum $\overline{t_1} + \dots + \overline{t_n}$ is a tuple of r frequency vectors, where the i -th frequency vector counts the frequencies of the elements of $[a]$ among the i -th elements of the tuples t_1, \dots, t_n , modulo m .

For any relation R^A of \mathbf{A} of arity r , define $M(R^A)$ to be the subgroup of H^r generated by $\bar{p} - \bar{q}$ for $p, q \in R^A$. Since H^r is Abelian, $M(R^A)$ is a normal subgroup.

Lemma 47. *Fix some relation R^A of \mathbf{A} ; suppose it has arity r . Let t be some tuple of R^A . Define $M = M(R^A)$. Consider any $(a_1, \dots, a_r) \in H^r$ such that $(a_1, \dots, a_r) \equiv \bar{t} \pmod{M}$. There exists a matrix (x_{ij}) with $N \leq N_d$ columns and r rows, where $N \equiv 1 \pmod{m}$, with elements in $[a]$, such that each column is a tuple of R^A , and, for each row i , we have*

$$\sum_{j=1}^N \bar{x}_{ij} = a_i.$$

PROOF. Note that every element in H^r has order that divides m (since $H^r \cong (G^a)^r \cong (\mathbb{Z}_m^a)^r$). Thus, since $(a_1, \dots, a_r) \equiv \bar{t} \pmod{M}$, and since M is generated by $\bar{p} - \bar{q}$ for $p, q \in R^A$, it follows that there exist coefficients $c_{pq} \in \{0, \dots, m-1\}$ for $p, q \in R^A$ such that

$$(a_1, \dots, a_r) = \bar{t} + \sum_{p,q \in R^A} c_{pq}(\bar{p} - \bar{q}) = \bar{t} + \sum_{p,q \in R^A} c_{pq}\bar{p} + (m - c_{pq})\bar{q}. \quad (1)$$

This indicates the matrix we will use: let (x_{ij}) be a matrix whose first column is t , and, for each $p, q \in R^A$, has c_{pq} columns equal to p and $m - c_{pq}$ columns equal to q . Clearly we use $N = 1 + m|R^A|^2 \leq 1 + |B|^{a^2} a^{2r_{\max}} \leq N_d$ columns, and $N \equiv 1 \pmod{m}$. To see why $\sum_{j=1}^N \bar{x}_{ij} = a_i$ for each i , note that this condition is equivalent to $(a_1, \dots, a_r) = \sum_{j=1}^N \bar{c}_j$, where c_1, \dots, c_N are the columns of the matrix. But this is precisely Equation (1). Thus we have created the required matrix. \square

Lemma 48. *Suppose $f = h \circ f^*$. For every $N \leq N_d$ such that $N \equiv 1 \pmod{m}$, the function $h_N : A^N \rightarrow B$ defined by*

$$h_N(x_1, \dots, x_N) = h\left(\sum_{i=1}^N \bar{x}_i\right)$$

is a polymorphism of (\mathbf{A}, \mathbf{B}) .

PROOF. By assumption, f is not N -degenerate. Thus there exist disjoint subsets S_1, \dots, S_N of $[n]$ where $f^p(S_1) = \dots = f^p(S_N) = f^p([n]) = 1$. Let $T = [n] \setminus (S_1 \cup \dots \cup S_N)$. Note that S_1, \dots, S_N, T form a partition of $[n]$. Furthermore,

$$1 = f^p([n]) = f^p(S_1) + \dots + f^p(S_N) + f^p(T) = N + f^p(T) = 1 + f^p(T).$$

The last equation holds as $N \equiv 1 \pmod{m}$, and addition is done in $G \cong \mathbb{Z}_m$. Thus $f^p(T) = 0$.

Let $\pi : [n] \rightarrow [N+1]$ be a function that takes $x \in S_i$ to i and $x \in T$ to $N+1$. Consider the polymorphism f^π . Since $f = h \circ f^*$, by the definition of f^* we can see that

$$f^\pi(U_1, \dots, U_a) = h(f^p(\pi^{-1}(U_1)), \dots, f^p(\pi^{-1}(U_a))).$$

Now consider $f^p(\pi^{-1}(U))$. Note that $\pi^{-1}(U) = T_U \cup \bigcup_{i \in U \cap [N]} S_i$, where $T_U = T$ if $N+1 \in U$, and $T_U = \emptyset$ otherwise. Thus

$$\begin{aligned} f^p(\pi^{-1}(U)) &= f^p\left(T_U \cup \bigcup_{i \in U \cap [N]} S_i\right) = f^p(T_U) + \sum_{i \in U \cap [N]} f^p(S_i) \\ &= 0 + \sum_{i \in U \cap [N]} 1 = |U \cap [N]| \pmod{m}, \end{aligned}$$

where $|U \cap [N]| \bmod m$ is taken as an element of $\mathbb{Z}_m \cong G$. In other words,

$$f^\pi(U_1, \dots, U_a) = h(|U_1 \cap [N]| \bmod m, \dots, |U_a \cap [N]| \bmod m).$$

Suppose now that U_1, \dots, U_a are the set family representation of the input vector x_1, \dots, x_{N+1} (i.e. $x_i = j$ if and only if $i \in U_j$) and consider the sum $\sum_{i=1}^N \bar{x}_i$. The j -th coordinate of this sum is the number of j 's that appear in x_1, \dots, x_N , modulo m , i.e. $|U_j \cap [N]| \bmod m$. Thus we see that the equation above is equivalent to

$$f^\pi(x_1, \dots, x_{N+1}) = h\left(\sum_{i=1}^N \bar{x}_i\right).$$

This polymorphism ignores x_{N+1} , so we find that the function

$$(x_1, \dots, x_N) \mapsto h\left(\sum_{i=1}^N \bar{x}_i\right)$$

is also a polymorphism. But this is just h_N , which is thus a polymorphism as required. \square

We can now prove the main theorem in this subsection.

PROOF OF THEOREM 41. We will show that (\mathbf{A}, \mathbf{B}) admits a homomorphic sandwich $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$, where \mathbf{E} is a relational structure whose domain is H , and where each relation will be of the form (i) $c + M$ for some $c \in H^r$ and $M \triangleleft H^r$, or (ii) empty. By Lemma 10 this implies our desired conclusion. The homomorphism $\mathbf{A} \rightarrow \mathbf{E}$ will be given by the map $g(x) = \bar{x}$. The homomorphism $\mathbf{E} \rightarrow \mathbf{B}$ will be given by any function h for which $f = h \circ f^*$. (Recall that such a function exists by Lemma 20.) We will construct \mathbf{E} relation by relation, showing along the way that g and h are in fact homomorphisms.

Consider some relation $R^{\mathbf{A}}$ of \mathbf{A} , of arity r , that corresponds to a relation $R^{\mathbf{B}}$ of \mathbf{B} , and $R^{\mathbf{E}}$ in \mathbf{E} . If $R^{\mathbf{A}}$ is empty then we can simply set $R^{\mathbf{E}}$ to be empty, and then g and h map tuples of $R^{\mathbf{A}}$ to tuples of $R^{\mathbf{E}}$, and then to tuples of $R^{\mathbf{B}}$ vacuously. Thus, suppose $t = (t_1, \dots, t_r)$ is some tuple of $R^{\mathbf{A}}$, and let $M = M(R^{\mathbf{A}})$. Then we set $R^{\mathbf{E}} = \bar{t} + M$; in other words, a tuple $x \in H^r$ will belong to this relation if and only if $x \equiv \bar{t} \bmod M$.

We first show that g maps $R^{\mathbf{A}}$ to $R^{\mathbf{E}} = \bar{t} + M$. Indeed, consider any tuple $x \in R^{\mathbf{A}}$. We know that $g(x) = \bar{x}$ by definition. Thus, $g(x) = \bar{x} = \bar{t} + (\bar{x} - \bar{t}) \in \bar{t} + M$. Thus g maps $R^{\mathbf{A}}$ to $\bar{t} + M$.

We now show that h maps $R^{\mathbf{E}} = \bar{t} + M$ to $R^{\mathbf{B}}$. Consider any tuple $(a_1, \dots, a_r) \in \bar{t} + M$. By Lemma 47 there exists some matrix $X = (x_{ij})$ with $N \leq N_d$ columns and r rows, where $N \equiv 1 \bmod m$, such that each column is an element of $R^{\mathbf{A}}$, and for each $i \in [r]$ we have

$$\sum_{j=1}^N \bar{x}_{ij} = a_i.$$

Furthermore, by Lemma 48, the function $h_N : A^N \rightarrow B$ given by

$$h_N(x_1, \dots, x_N) = h\left(\sum_{i=1}^N \bar{x}_i\right)$$

is a polymorphism. Note now that

$$\begin{aligned} (h(a_1), \dots, h(a_r)) &= \left(h\left(\sum_{j=1}^N \bar{x}_{1j}\right), \dots, h\left(\sum_{j=1}^N \bar{x}_{rj}\right) \right) \\ &= (h_N(x_{11}, \dots, x_{1N}), \dots, h_N(x_{r1}, \dots, x_{rN})) \in R^{\mathbf{B}}. \end{aligned}$$

The last inclusion holds since the tuple in question is the result of applying h_N , a polymorphism of (\mathbf{A}, \mathbf{B}) , to the rows of a matrix whose columns are elements in R^A .

Thus we note that $\mathbf{A} \rightarrow \mathbf{E} \rightarrow \mathbf{B}$ for some structure \mathbf{E} that satisfies the conditions in Lemma 10. In conclusion, $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is solved by AIP and is finitely tractable. \square

4.2 Proof of Theorem 42

In this section we will prove that $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard if each polymorphism $f \in \text{Pol}(\mathbf{A}, \mathbf{B})$ is k -degenerate for some k at most N_d , or has a hard set of size at most N_h .

Lemma 49. *If $f \in \text{Pol}(\mathbf{A}, \mathbf{B})$ then f cannot have more than $|B|^{a^2}$ disjoint hard sets.*

PROOF. Equivalently we show that any family \mathcal{G} of more than $|B|^{a^2}$ disjoint sets contains a non-hard set. Apply Lemma 16 to \mathcal{G} to find a nonempty subfamily $\{G_1, \dots\}$ such that $f^p(\bigcup_i G_i) = f^p(\emptyset)$. Thus $G_i \in \mathcal{G}$ is not a hard set. \square

Lemma 50. *Suppose $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ and $\pi : [n] \rightarrow [m]$. Then $(f^\pi)^p = f^p \circ \pi^{-1}$.*

PROOF. Note that $f_{ij}(S) = f(T_1, \dots, T_a)$ where $T_j = S$, $T_i = [n] \setminus S$, and all the other inputs are \emptyset . Now, $(f^\pi)_{ij}(S) = f^\pi(T_1, \dots, T_a) = f(\pi^{-1}(T_1), \dots, \pi^{-1}(T_a)) = f_{ij}(\pi^{-1}(S))$, and so $(f^\pi)_{ij} = f_{ij} \circ \pi^{-1}$. Our conclusion follows by applying this fact for each $i, j \in A$. \square

Lemma 51. *Suppose $f \in \text{Pol}^{(n)}(\mathbf{A}, \mathbf{B})$ and $\pi : [n] \rightarrow [m]$. If S is a hard set of f then $\pi(S)$ is a hard set of f^π .*

PROOF. We prove this by contrapositive. Suppose $\pi(S)$ is not a hard set of f^π . Then some $T \supseteq \pi(S)$ exists such that $(f^\pi)^p(T) = (f^\pi)^p(\emptyset)$. So $(f^p)(\pi^{-1}(T)) = (f^\pi)^p(T) = (f^\pi)^p(\emptyset) = (f^p)(\pi^{-1}(\emptyset)) = f^p(\emptyset)$. Thus $f^p(\pi^{-1}(T)) = f^p(\emptyset)$, and S is not a hard set, as $S \subseteq \pi^{-1}(T)$. \square

Let \mathcal{M}_h denote the subset of $\text{Pol}(\mathbf{A}, \mathbf{B})$ whose polymorphisms have hard sets of size at most N_h . Let $\mathcal{M}_{x_1, \dots, x_k}$ denote the subset of $\text{Pol}(\mathbf{A}, \mathbf{B})$ whose polymorphisms are k -degenerate, yet not $(k-1)$ -degenerate, where $x_1, \dots, x_k \in B^{A^2}$ are witnesses to this degeneracy. By assumption, and as no polymorphism is 1-degenerate,

$$\text{Pol}(\mathbf{A}, \mathbf{B}) = \mathcal{M}_h \cup \bigcup_{k=2}^{N_d} \bigcup_{x_1 \in \text{range}(f^p)} \dots \bigcup_{x_k \in \text{range}(f^p)} \mathcal{M}_{x_1, \dots, x_k}. \quad (2)$$

Lemma 52. *There exists some assignment I that takes $f \in \mathcal{M}_h^{(n)}$ to a subset of $[n]$ of size at most $|B|^{2a^2}$ such that, whenever $g \in \mathcal{M}_h^{(m)}$ and $g = f^\pi$ for some $\pi : [n] \rightarrow [m]$, we have that $\pi(I(f)) \cap I(g) \neq \emptyset$.*

PROOF. To construct $I(f)$, let S_1, \dots be a maximal sequence of disjoint hard sets of f of size at most $|B|^{a^2}$, constructed greedily, and then set $I(f)$ to be the union of these sets. Since there can be at most $|B|^{a^2}$ disjoint hard sets by Lemma 49, it follows that $|I(f)| \leq |B|^{2a^2}$.

Consider now any $f, g \in \mathcal{M}_h$ such that $g = f^\pi$. Note that $I(f)$ contains within it a hard set S of size at most $|B|^{a^2}$. Thus $\pi(I(f)) \supseteq \pi(S)$, which is a hard set of size at most $|B|^{a^2}$ by Lemma 51, and thus must intersect $I(g)$ by maximality. It follows that $\pi(I(f)) \cap I(g) \neq \emptyset$. \square

Lemma 53. *For $k \geq 2$, $x_1, \dots, x_k \in \text{range}(f^p)$, there exists some assignment I that takes $f \in \mathcal{M}_{x_1, \dots, x_k}^{(n)}$ to a subset of $[n]$ of size at most $k|B|^{a^2}$ such that, whenever $g \in \mathcal{M}_{x_1, \dots, x_k}^{(m)}$ and $g = f^\pi$ for some $\pi : [n] \rightarrow [m]$ we have that $\pi(I(f)) \cap I(g) \neq \emptyset$.*

PROOF. To construct $I(f)$, take S_1, \dots, S_{k-1} to be disjoint sets such that $f^p(S_i) = x_i$, and take T to be any set such that $f(T) = x_k$. Such sets exist since f is not $(k-1)$ -degenerate, and we can take all of these sets to be of size at most $|B|^{a^2}$, by replacing them with the subsets given by Lemma 17. Let $I(f)$ be the union of S_1, \dots, S_{k-1}, T . Note that $|I(f)| \leq k|B|^{a^2}$.

Consider now any $f, g \in \mathcal{M}_{x_1, \dots, x_k}$ such that $g = f^\pi$. Note that $I(f)$ contains within it disjoint sets S_1, \dots, S_{k-1} such that $f^p(S_i) = x_i$, and $I(g)$ contains within it a set T such that $g^p(T) = x_k$. Now, $f^p(\pi^{-1}(T)) = (f^\pi)^p(T) = g^p(T) = x_k$, and thus by the k -degeneracy of f and the disjointness of S_1, \dots, S_{k-1} it follows that $\pi^{-1}(T)$ and S_1, \dots, S_{k-1} must intersect. It follows that $\pi(I(f)) \cap I(g) \neq \emptyset$, as required. \square

PROOF OF THEOREM 42. We see in (2) that $\text{Pol}(\mathbf{A}, \mathbf{B})$ is the union of $m = 1 + \sum_{k=2}^{N_d} (|B|^{a^2})^k$ sets, each of which has an assignment I that satisfies the condition of Theorem 11 if we take $C = \max(N_d |B|^{a^2}, |B|^{2a^2})$. Thus $\text{PCSP}(\mathbf{A}, \mathbf{B})$ is NP-hard. \square

5 BLP+AIP = AIP when A has one balanced relation

In this section we prove Theorem 6 and Corollary 7. Recall that we call a relation R balanced if there exists a matrix M whose columns are tuples of R , where each tuple of R appears as a column (possibly a multiple times), and where the rows of M are permutations of each other.

Theorem 6. *Let \mathbf{A} be any structure with a single relation. If the relation in \mathbf{A} is balanced then, for any \mathbf{B} such that $\mathbf{A} \rightarrow \mathbf{B}$, BLP + AIP solves $\text{PCSP}(\mathbf{A}, \mathbf{B})$ if and only if AIP solves it.*

Suppose that $A = [a]$, and the relation of \mathbf{A} is $R = R^A$. Furthermore suppose that each element in $[a]$ appears in R (otherwise these elements can just be eliminated from A). Suppose $A \neq \emptyset, R \neq \emptyset$ (otherwise the conclusion is trivially true). We name the columns of the matrix that witness the balancedness of R as $t_1, \dots, t_N \in R$.

For any $i \in [a]$, let \bar{i} be a unit vector in \mathbb{Z}^a ; i.e., it has a 1 at position i . For any tuple $(a_1, \dots, a_r) \in A^r$, let $(\bar{a}_1, \dots, \bar{a}_r) = (\bar{a}_1, \dots, \bar{a}_r) \in (\mathbb{Z}^a)^r$. Let $\bar{R} = \{\bar{t} \mid t \in R\} \subseteq (\mathbb{Z}^a)^r$. (We see the elements of \mathbb{Z}^a as frequency vectors, and the elements of $(\mathbb{Z}^a)^r$ as tuples of frequency vectors.) Endow all of these with additive structure. For any sets of vectors A, B , we let $A + B = \{x + y \mid x \in A, y \in B\}$ and $A - B = \{x - y \mid x \in A, y \in B\}$. For any $k \in \mathbb{Z}$, we denote by $k\bar{R}$ the set of sums of k vectors from \bar{R} .

Lemma 54. $(k+1)\bar{R} - k\bar{R} + k \sum_i \bar{t}_i \subseteq (kN+1)\bar{R}$.

PROOF. If $x \in (k+1)\bar{R} - k\bar{R} + k \sum_i \bar{t}_i$, it can be written as a sum of $k+1$ vectors from \bar{R} , minus k vectors from \bar{R} , plus k copies of each vector \bar{t}_i . Since each tuple of R appears among t_1, \dots, t_N , the last kN vectors in the sum above include at least k copies of each vector in \bar{R} . By removing the k subtracted vectors from the k copies of each vector from \bar{R} , we find that x can be written as a sum of $k+1 - k + kN = kN+1$ vectors from \bar{R} , i.e. $x \in (kN+1)\bar{R}$. \square

For any $k \in \mathbb{Z}$, we define $S_k \subseteq \mathbb{Z}^a$ to be the set of sequences of integers that sum up to k , with non-negative coordinates.

Lemma 55. *If (\mathbf{A}, \mathbf{B}) has a 2-block-symmetric polymorphism f of arity $2k+1$ then there exists a function $g: S_k \times S_{k+1} \rightarrow B$ such that $(g(x_1, y_1), \dots, g(x_r, y_r)) \in R^B$ for all $(x_1, \dots, x_r) \in k\bar{R}, (y_1, \dots, y_r) \in (k+1)\bar{R}$.*

PROOF. To compute $g(x, y)$, create two sequences of elements in $[a]$, of lengths k and $k+1$, whose frequencies correspond to x and y respectively (i.e. the sequence for $x = (x_1, \dots, x_a)$ has x_i appearances of i), and interleave these to create a sequence a_1, \dots, a_{2k+1} . Then $g(x, y) = f(a_1, \dots, a_{2k+1})$. To see why this function satisfies the required condition, suppose $(x_1, \dots, x_r) \in k\bar{R}$

and $(y_1, \dots, y_r) \in (k+1)\overline{R}$. Thus we can, by definition, create matrices M and N , with k and $k+1$ columns respectively, and r rows, where each column is an element of R , and each row i has frequencies corresponding to x_i and y_i respectively. Interleave the columns of these matrices to create a matrix A . Apply f to the rows of A . We find that the image of row i of A is $g(x_i, y_i)$ by the symmetry of f ; furthermore, the images of the rows of A must form a tuple of R^B , since f is a polymorphism. This is the desired conclusion. \square

Lemma 56. *Assume that there exists a function $f : (S_{k+1}-S_k) \rightarrow B$ for which $(f(x_1), \dots, f(x_r)) \in R^B$ for any $x_1, \dots, x_r \in S_{k+1}-S_k$ with $(x_1, \dots, x_r) \in (k+1)\overline{R} - k\overline{R}$. Then, $\text{PCSP}(\mathbf{A}, \mathbf{B})$ has an alternating polymorphism of arity $2k+1$.*

PROOF. If such a function exists, then

$$g(x_1, \dots, x_{2k+1}) = f(\overline{x_1} + \overline{x_3} + \dots + \overline{x_{2k+1}} - \overline{x_2} - \overline{x_4} - \dots - \overline{x_{2k}})$$

is the required polymorphism. \square

PROOF OF THEOREM 6. By Theorem 8, AIP solves $\text{PCSP}(\mathbf{A}, \mathbf{B})$ if and only if $\text{Pol}(\mathbf{A}, \mathbf{B})$ contains alternating operations of all odd arities. By Theorem 9, BLP + AIP solves $\text{PCSP}(\mathbf{A}, \mathbf{B})$ if and only if $\text{Pol}(\mathbf{A}, \mathbf{B})$ contains 2-block-symmetric operations of all odd arities. As any alternating operation is 2-block-symmetric, it follows that any PCSP solved by AIP is also solved by BLP + AIP.¹⁵ It suffices to show that 2-block-symmetric operations in $\text{Pol}(\mathbf{A}, \mathbf{B})$ imply alternating operations.

Fix some natural number k ; we will now show that there exists an alternating operation in $\text{Pol}(\mathbf{A}, \mathbf{B})$ of arity $2k+1$. Since $\text{Pol}(\mathbf{A}, \mathbf{B})$ contains a 2-block-symmetric operation of arity $2kN+1$, let $f : S_{kN} \times S_{kN+1} \rightarrow B$ be the function given by Lemma 55. We will construct the function required by Lemma 56 in order to prove the existence of an alternating polymorphism.

Consider the vector $v = \sum_i \overline{t_i} \in (\mathbb{Z}^a)^r$. We claim that v is a constant vector. To see why this is the case, observe that one way to compute $\sum_i \overline{t_i}$ is to make t_1, \dots, t_N into the columns of a matrix, and then to compute the frequencies of each element of $[a]$ in each row. Element i of v is a tuple, where component j is the number of appearances of j in row i in this matrix. But, since t_1, \dots, t_N witness the balancedness of R , these frequencies are equal for each row. Thus v is indeed a constant vector; suppose that $v = (c, \dots, c)$ for some $c \in S_N$. Note that each element in $[a]$ appears in some tuple of R by assumption, and each tuple of R appears in the sum $\sum_i \overline{t_i}$. Thus each coordinate in $c \in \mathbb{Z}^a$ is at least 1.

The function we are interested in is $g : (S_{k+1}-S_k) \rightarrow B$, where $g(x) = f(kc, x + kc)$. First note that these inputs are legal inputs for the function f . To see why, note first that $c \in S_N$ and thus $kc \in S_{kN}$. Second, consider $x + kc$. As $x \in S_{k+1}-S_k$, the components in x sum up to 1. Since the components in kc sum up to kN , it follows that the components in $x + kc$ sum up to $1 + kN$ as required. Furthermore, all the components of $x + kc$ are non-negative: each component of x is at least $-k$, whereas each component of c is at least 1, and thus each component of kc is at least k . Thus $x + kc \in S_{kN+1}$.

Why does g satisfy the conditions from Lemma 56? Consider any $x_1, \dots, x_r \in S_{k+1}-S_k$ such that $(x_1, \dots, x_r) \in (k+1)\overline{R} - k\overline{R}$. Note that

$$(kc, \dots, kc) = k(c, \dots, c) = k \sum_i \overline{t_i} \in kN\overline{R},$$

$$(x_1 + kc, \dots, x_r + kc) = (x_1, \dots, x_r) + k(c, \dots, c) \in (k+1)\overline{R} - k\overline{R} + k \sum_i \overline{t_i} \subseteq (kN+1)\overline{R},$$

¹⁵This also directly follows from the definitions of the AIP and BLP + AIP algorithms [10].

due to Lemma 54. Thus, since f satisfies the conditions in Lemma 55,

$$(g(x_1), \dots, g(x_r)) = (f(kc, x_1 + kc), \dots, f(kc, x_r + kc)) \in R^B.$$

Thus $(g(x_1), \dots, g(x_r)) \in R^B$, as required. \square

Theorem 6 does not generalise to structures with multiple relations (even just two), as the following examples show.

Example 57. Consider a Boolean symmetric template \mathbf{A} that has two balanced (and in fact even symmetric) relations, namely $R^{\mathbf{A}} = \{(0)\}$ and $Q^{\mathbf{A}} = \{(0, 1), (1, 0), (1, 1)\}$, which are unary and binary, respectively. Then $\text{CSP}(\mathbf{A})$ is solved by BLP + AIP, and indeed by BLP, since the symmetric operation $\max(x_1, \dots, x_n)$ is a polymorphism for any n [6], but *not* by AIP. This is because \mathbf{A} fails to have any alternating non-unary polymorphisms, even of arity 3: suppose $f(x, y, z)$ is such a polymorphism. Then $f(1, 1, 0) = f(0, 0, 0) = f(0, 1, 1)$ as f is alternating; and $f(0, 0, 0) = 0$ due to $R^{\mathbf{A}}$. However, due to $Q^{\mathbf{A}}$, $f(1, 1, 0)$ and $f(0, 1, 1)$ cannot both be 0. This contradiction implies our conclusion.

One cannot simply remove the balancedness condition from Theorem 6, as the following example shows.

Example 58. Let \mathbf{A} be a Boolean template with relation $S^{\mathbf{A}} = \{(0, 0, 1), (0, 1, 0), (0, 1, 1)\}$. Note that $S^{\mathbf{A}}$ is *not* balanced. Then $\text{CSP}(\mathbf{A})$ is solved by BLP + AIP, and indeed by BLP, since the symmetric operation $\max(x_1, \dots, x_n)$ is a polymorphism for any n [6], but not by AIP. \mathbf{A} fails to have any alternating polymorphism, even of arity 3, for exactly the same reason as the problem from Example 57. (The identities that would result from $R^{\mathbf{A}}$ in that example now result from the first component in each tuple in $S^{\mathbf{A}}$, and the identities that would result from $Q^{\mathbf{A}}$ in that example now result from the last two components in each tuple in $S^{\mathbf{A}}$.)

On the other hand, there are templates that are unbalanced for which AIP and BLP + AIP have equivalent power, as the following example shows.

Example 59. Consider a Boolean template \mathbf{A} that has one relation $P^{\mathbf{A}} = \{(0, 1)\}$. Then $\text{CSP}(\mathbf{A})$ is solved by AIP and by BLP + AIP, since the alternating operation $x_1 + \dots + x_n \pmod 2$ is a polymorphism of \mathbf{A} for every odd n . This is in spite of the fact that $P^{\mathbf{A}}$ is unbalanced.

We now prove Corollary 7.

Corollary 7. *Suppose that G is a transitive group of permutations, of order r . Further, suppose that \mathbf{A} is a relational structure with one relation, of arity r , that is preserved by G . Then, for any $\mathbf{A} \rightarrow \mathbf{B}$, BLP + AIP solves $\text{PCSP}(\mathbf{A}, \mathbf{B})$ if and only if AIP does.*

PROOF. Let R be the relation of \mathbf{A} , of arity r . It is sufficient to show that R is balanced. Let M be a matrix whose columns are the tuples of R . Suppose that the rows of M are r_1, \dots, r_n . We show that row i is a permutation of row j , for arbitrary $i, j \in [n]$.

Represent the elements of G as permutation matrices. Let $\pi \in G$ be a permutation (matrix) that sends i to j (it exists by transitivity). Consider πM . Note that no two columns of πM can be equal, since then two columns of $\pi^{-1}\pi M = M$ would be equal, which is false. Furthermore each column of πM is a tuple of R , and thus a column of M , since R is preserved by π . Thus we see that πM can be seen as M but with its columns permuted. In other words, for some permutation matrix σ , we have $\pi M = M\sigma^T$.

Now, let us look at row j in $\pi M = M\sigma^T$. In πM this is r_i (since π sends i to j). In $M\sigma^T$ this is $r_j\sigma^T$. Thus $r_i = r_j\sigma^T$, i.e. row i of M and row j of M are permutations of each other. We conclude that R is balanced, as required. \square

Corollary 7 applies to fewer structures than Theorem 6, as shown in the next example.

Remark 60. Consider any digraph \mathbf{A} with edge relation $E^{\mathbf{A}}$ that is strongly connected but not symmetric. Then $E^{\mathbf{A}}$ is balanced (cf. Appendix A). On the other hand, the unique transitive permutation group with degree 2 (i.e. the group containing the identity permutation and the permutation swapping two elements) does not preserve $E^{\mathbf{A}}$.

6 Conclusion

Our first result classifies certain $\text{PCSP}(\mathbf{A}, \mathbf{B})$, where \mathbf{A} and \mathbf{B} are symmetric and \mathbf{B} is functional, into being either NP-hard or solvable in polynomial time. This is the first step towards the following more general problem.

Problem 61. Classify the complexity of $\text{PCSP}(\mathbf{A}, \mathbf{B})$ for functional \mathbf{B} .

Looking more specifically at the case $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$, we note that our proof of Theorem 1 implies that, for functional \mathbf{B} , we have that $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ is tractable if and only if $\text{Eqn}_{m,1} \rightarrow \mathbf{B}$ for some $m \leq |\mathbf{B}|$, where $\text{Eqn}_{m,1}$ is a relational structure over $\{0, \dots, m-1\}$ with one ternary relation defined by $x+y+z \equiv 1 \pmod{m}$. By using the Chinese remainder theorem, $\text{Eqn}_{m,1} = \text{Eqn}_{3^p,1} \times \text{Eqn}_{q,1}$, where q is coprime to 3. Since this latter template contains a constant tuple (namely (x, x, x) where x is the inverse of 3 modulo q), we find that, for functional \mathbf{B} , $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ is tractable if and only if $\text{Eqn}_{3^p,1} \rightarrow \mathbf{B}$.

Looking at non-functional templates $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ that are tractable, all the examples the authors are aware of are either tractable for the same reason as a functional template is (i.e. $\text{Eqn}_{3^p,1} \rightarrow \mathbf{B}$), or because they include the not-all-equal predicate (i.e. $\text{NAE} \rightarrow \mathbf{B}$). Thus, we pose the following problem.

Problem 62. Is $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ tractable if and only if $\text{Eqn}_{3^p,1} \times \text{NAE} \rightarrow \mathbf{B}$ for some p ?

Problem 62 has a link with the problem of determining the complexity of $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_k^+)$, where \mathbf{C}_k^+ is a ternary symmetric template on domain $[k]$ which contains tuples of the form $(1, 1, 2), \dots, (k-1, k-1, k), (k, k, 1)$, as well as all tuples of three distinct elements (rainbow tuples). Such templates are called *cyclic*, with the cycle being $1 \rightarrow \dots \rightarrow k \rightarrow 1$.

The link is the following: $\text{Eqn}_{3^p,1} \times \text{NAE}$ is a template containing one cycle of length 2×3^p , together with certain rainbow tuples – in other words, $\text{Eqn}_{3^p,1} \times \text{NAE} \rightarrow \mathbf{C}_{2 \times 3^p}^+$. Likewise, $\text{Eqn}_{3^p,1}$ has a cycle of length 3^p and some rainbow tuples, i.e. $\text{Eqn}_{3^p,1} \rightarrow \mathbf{C}_{3^p}^+$. That $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_k^+)$ is tractable whenever $k = 3^p$ or $k = 2 \times 3^p$ was first observed in [11]. If Problem 62 were answered in the affirmative then we would have that $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_k^+)$ is tractable if and only if $k = 3^p$ or $k = 2 \times 3^p$. In particular, this would mean that $\text{PCSP}(\mathbf{1-in-3}, \mathbf{C}_4^+)$ is NP-hard, as conjectured in [5].¹⁶

Answering Problem 62 in the affirmative would resolve Conjecture 5, i.e., $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ would be tractable (via AIP) if and only if $\text{Eqn}_{3^p,1} \times \text{NAE} \rightarrow \mathbf{B}$. Perhaps determining whether this equivalence is true might be easier than resolving Conjecture 5; thus we pose the following problem.

Problem 63. Is $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ solved by AIP if and only if $\text{Eqn}_{3^p,1} \times \text{NAE} \rightarrow \mathbf{B}$ for some p ?

There already exists such a characterisation for the power of AIP using an infinite structure [6]. In particular, if we let \mathbf{Z} be an infinite structure whose domain is \mathbb{Z} , and with a tuple (x, y, z) in the relation if and only if $x + y + z = 1$, then $\text{PCSP}(\mathbf{1-in-3}, \mathbf{B})$ is solved by AIP if and only if $\mathbf{Z} \rightarrow \mathbf{B}$. We are interested in a finite template of this kind.

¹⁶Our structure \mathbf{C}_4^+ is called $\tilde{\mathbf{C}}_4^+$ in [5].

Turning from problems to algorithms, our second result shows us that, for certain problems of the form $\text{PCSP}(\mathbf{A}, \mathbf{B})$ where \mathbf{B} need not be functional, and \mathbf{A}, \mathbf{B} have one relation, AIP and $\text{BLP} + \text{AIP}$ have the same power. A natural question is for which other templates is it true?

Problem 64. For which templates (\mathbf{A}, \mathbf{B}) do AIP and $\text{BLP} + \text{AIP}$ have the same power?

Equivalently [6, 10], for which templates (\mathbf{A}, \mathbf{B}) does the existence of 2-block symmetric operations of all odd arities in $\text{Pol}(\mathbf{A}, \mathbf{B})$ imply the existence of alternating operations of all odd arities in $\text{Pol}(\mathbf{A}, \mathbf{B})$?

We remark that the recent work [16] does not answer any problem from this section, and the results from [16] are consistent with positive answers to Problem 62 and Problem 63.

Acknowledgments

We would like to thank the anonymous referees of both the conference [29] and this full version of the paper. This work was supported by UKRI EP/X024431/1 and a Clarendon Fund Scholarship.

References

- [1] Martin Aigner and Günter M. Ziegler. 2009. *Proofs from THE BOOK* (4th ed.). Springer Publishing Company, Incorporated, Germany.
- [2] Kristina Asimi and Libor Barto. 2021. Finitely Tractable Promise Constraint Satisfaction Problems. In *Proc. 46th International Symposium on Mathematical Foundations of Computer Science (MFCS'21) (LIPIcs, Vol. 202)*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 11:1–11:16. <https://doi.org/10.4230/LIPIcs.MFCS.2021.11>
- [3] Albert Atserias and Víctor Dalmau. 2022. Promise Constraint Satisfaction and Width. In *Proc. 2022 ACM-SIAM Symposium on Discrete Algorithms (SODA'22)*. SIAM, USA, 1129–1153. <https://doi.org/10.1137/1.9781611977073.48> arXiv:2107.05886
- [4] Per Austrin, Venkatesan Guruswami, and Johan Håstad. 2017. $(2+\epsilon)$ -Sat Is NP-hard. *SIAM J. Comput.* 46, 5 (2017), 1554–1573. <https://doi.org/10.1137/15M1006507>
- [5] Libor Barto, Diego Battistelli, and Kevin M. Berg. 2021. Symmetric Promise Constraint Satisfaction Problems: Beyond the Boolean Case. In *Proc. 38th International Symposium on Theoretical Aspects of Computer Science (STACS'21) (LIPIcs, Vol. 187)*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 10:1–10:16. <https://doi.org/10.4230/LIPIcs.STACS.2021.10> arXiv:2010.04623
- [6] Libor Barto, Jakub Bulín, Andrei A. Krokhin, and Jakub Opršal. 2021. Algebraic approach to promise constraint satisfaction. *J. ACM* 68, 4 (2021), 28:1–28:66. <https://doi.org/10.1145/3457606> arXiv:1811.00970
- [7] Libor Barto and Marcin Kozik. 2022. Combinatorial Gap Theorem and Reductions between Promise CSPs. In *Proc. 2022 ACM-SIAM Symposium on Discrete Algorithms (SODA'22)*. SIAM, USA, 1204–1220. <https://doi.org/10.1137/1.9781611977073.50> arXiv:2107.09423
- [8] Joshua Brakensiek and Venkatesan Guruswami. 2021. Promise Constraint Satisfaction: Algebraic Structure and a Symmetric Boolean Dichotomy. *SIAM J. Comput.* 50, 6 (2021), 1663–1700. <https://doi.org/10.1137/19M128212X> arXiv:1704.01937
- [9] Joshua Brakensiek and Venkatesan Guruswami. 2021. The Quest for Strong Inapproximability Results with Perfect Completeness. *ACM Trans. Algorithms* 17, 3 (2021), 27:1–27:35. <https://doi.org/10.1145/3459668>
- [10] Joshua Brakensiek, Venkatesan Guruswami, Marcin Wrochna, and Stanislav Živný. 2020. The power of the combined basic LP and affine relaxation for promise CSPs. *SIAM J. Comput.* 49 (2020), 1232–1248. Issue 6. <https://doi.org/10.1137/20M1312745> arXiv:1907.04383
- [11] Alex Brandts. 2022. *Promise Constraint Satisfaction Problems*. Ph.D. Dissertation. University of Oxford. <https://ora.ox.ac.uk/objects/uuid:5efeff56-d5c8-42e8-9bd2-95243a1367ad>
- [12] Alex Brandts, Marcin Wrochna, and Stanislav Živný. 2021. The complexity of promise SAT on non-Boolean domains. *ACM Trans. Comput. Theory* 13, 4 (2021), 26:1–26:20. <https://doi.org/10.1145/3470867> arXiv:1911.09065
- [13] Alex Brandts and Stanislav Živný. 2022. Beyond PCSP(1-in-3,NAE). *Inf. Comput.* 289, Part A (2022), 104954. <https://doi.org/10.1016/j.ic.2022.104954> arXiv:2104.12800
- [14] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin. 2005. Classifying the Complexity of Constraints using Finite Algebras. *SIAM J. Comput.* 34, 3 (2005), 720–742. <https://doi.org/10.1137/S0097539700376676>
- [15] Andrei A. Bulatov. 2017. A Dichotomy Theorem for Nonuniform CSPs. In *Proc. 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS'17)*. IEEE, USA, 319–330. <https://doi.org/10.1109/FOCS.2017.37> arXiv:1703.03021

- [16] Lorenzo Ciardo, Marcin Kozik, Andrei Krokhin, Tamio-Vesa Nakajima, and Stanislav Živný. 2024. 1-in-3 vs. Not-All-Equal: Dichotomy of a broken promise. arXiv:2302.03456v2
- [17] Lorenzo Ciardo and Stanislav Živný. 2023. CLAP: A New Algorithm for Promise CSPs. *SIAM J. Comput.* 52, 1 (2023), 1–37. <https://doi.org/10.1137/22m1476435> arXiv:2107.05018
- [18] Irit Dinur, Oded Regev, and Clifford Smyth. 2005. The Hardness of 3-Uniform Hypergraph Coloring. *Comb.* 25, 5 (2005), 519–535. <https://doi.org/10.1007/s00493-005-0032-4>
- [19] Tomás Feder and Moshe Y. Vardi. 1998. The Computational Structure of Monotone Monadic SNP and Constraint Satisfaction: A Study through Datalog and Group Theory. *SIAM J. Comput.* 28, 1 (1998), 57–104. <https://doi.org/10.1137/S0097539794266766>
- [20] Miron Ficak, Marcin Kozik, Miroslav Olšák, and Szymon Stankiewicz. 2019. Dichotomy for Symmetric Boolean PCSPs. In *Proc. 46th International Colloquium on Automata, Languages, and Programming (ICALP'19)*, Vol. 132. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 57:1–57:12. <https://doi.org/10.4230/LIPIcs.ICALP.2019.57> arXiv:1904.12424
- [21] Marek Filakovský, Tamio-Vesa Nakajima, Jakub Opršal, Gianluca Tasinato, and Uli Wagner. 2024. Hardness of Linearly Ordered 4-Colouring of 3-Colourable 3-Uniform Hypergraphs. In *Proc. 41st International Symposium on Theoretical Aspects of Computer Science (STACS'24) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 289)*. 34:1–34:19. <https://doi.org/10.4230/LIPIcs.STACS.2024.34> arXiv:2312.12981
- [22] M. R. Garey and D. S. Johnson. 1976. The Complexity of Near-Optimal Graph Coloring. *J. ACM* 23, 1 (1976), 43–49. <https://doi.org/10.1145/321921.321926>
- [23] Venkatesan Guruswami and Sai Sandeep. 2020. d-To-1 Hardness of Coloring 3-Colorable Graphs with $O(1)$ Colors. In *Proc. 47th International Colloquium on Automata, Languages, and Programming (ICALP'20) (LIPIcs, Vol. 168)*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 62:1–62:12. <https://doi.org/10.4230/LIPIcs.ICALP.2020.62>
- [24] Pavol Hell and Jaroslav Nešetřil. 1990. On the Complexity of H -coloring. *J. Comb. Theory, Ser. B* 48, 1 (1990), 92–110. [https://doi.org/10.1016/0095-8956\(90\)90132-J](https://doi.org/10.1016/0095-8956(90)90132-J)
- [25] Peter G. Jeavons. 1998. On the Algebraic Structure of Combinatorial Problems. *Theor. Comput. Sci.* 200, 1-2 (1998), 185–204. [https://doi.org/10.1016/S0304-3975\(97\)00230-2](https://doi.org/10.1016/S0304-3975(97)00230-2)
- [26] Andrei Krokhin and Jakub Opršal. 2022. An Invitation to the Promise Constraint Satisfaction Problem. *ACM SIGLOG News* 9, 3 (2022), 30–59. <https://doi.org/10.1145/3559736.3559740>
- [27] Andrei A. Krokhin, Jakub Opršal, Marcin Wrochna, and Stanislav Živný. 2023. Topology and adjunction in promise constraint satisfaction. *SIAM J. Comput.* 52, 1 (2023), 38–79. <https://doi.org/10.1137/20m1378223> arXiv:2003.11351
- [28] Tamio-Vesa Nakajima and Stanislav Živný. 2022. Linearly Ordered Colourings of Hypergraphs. *ACM Trans. Comput. Theory* 14, 3-4 (2022), 12:1–12:19. <https://doi.org/10.1145/3570909> arXiv:2204.05628
- [29] Tamio-Vesa Nakajima and Stanislav Živný. 2023. Boolean symmetric vs. functional PCSP dichotomy. In *Proc. 38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'23)*. IEEE, 1–12. <https://doi.org/10.1109/LICS56636.2023.10175746>
- [30] Thomas Schaefer. 1978. The complexity of satisfiability problems. In *Proc. 10th Annual ACM Symposium on the Theory of Computing (STOC'78)*. ACM, USA, 216–226. <https://doi.org/10.1145/800133.804350>
- [31] Marcin Wrochna. 2022. A note on hardness of promise hypergraph colouring. arXiv:2205.14719
- [32] Dmitriy Zhuk. 2020. A Proof of the CSP Dichotomy Conjecture. *J. ACM* 67, 5 (2020), 30:1–30:78. <https://doi.org/10.1145/3402029> arXiv:1704.01914

A Deferred proof

Lemma 65. *A digraph A is balanced if and only if it is a disjoint union of strongly connected components.*

PROOF. If the relation of A is balanced, then there exists a collection of tours of A containing every edge at least once. These exist since the multi-digraph formed by the columns of the matrix witnessing the balancedness of the relation of A is Eulerian. Thus, considering the tour that contains some edge (u, v) , we see that there exists a walk from v to u . It follows that every weakly connected component of A is strongly connected, or equivalently A is the disjoint union of strongly connected components.

If A is the disjoint union of strongly connected components, then we create a matrix that witnesses the balancedness of the relation of A . Consider any edge (u, v) of A . Take the cycle formed by (u, v) together with the path from v to u . Add all of these edges to the matrix as columns. The resulting

matrix contains each edge at least once; furthermore every vertex appears the same number of times in the first and the second row (since the edges form cycles). Thus A is balanced. \square