Equivariant Subspaces of Orbit-Finite-Dimensional Vector Spaces

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Abstract

Weighted orbit-finite automata provide a linear generalisation to register automata where the alphabet $\mathbb{A}$ is only required to be finite modulo symmetries. To adapt Schützenberger's algorithm for zeroness checking to the orbit-finite setting, we study chains $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_l$ of equivariant subspaces in $\operatorname{Lin} \mathbb{A}^{(k)}$ — the complex vector space whose basis comprises k-tuples of distinct elements of $\mathbb{A}$ — which are closed under both linear combinations and permutations. The existence of a maximum length l guarantees termination of the algorithm, and furthermore gives an upper bound on its complexity.

For the equality atoms $\mathbb{A} = (\mathbb{N}, =)$, we have $\operatorname{Lin} \mathbb{A}^{(k)} = \bigcup_n \operatorname{Lin} [n]^{(k)}$ where each $\operatorname{Lin} [n]^{(k)}$ is an S_n-representation with well understood substructures. Through rudimentary calculations, we find all the equivariant subspaces of $\operatorname{Lin} \mathbb{A}$ and $\operatorname{Lin} \mathbb{A}^{(2)}$, in turn determining $\operatorname{length}(\operatorname{Lin} \mathbb{A})$ and $\operatorname{length}(\operatorname{Lin} \mathbb{A}^{(2)})$. Also of interest is the existence of infinitely many such subspaces in $\operatorname{Lin} \mathbb{A}^{(2)}$ (even though there are only finitely many isomorphism classes). More generally, we deduce from the stabilisation of the sequence $\left(\operatorname{length}(\operatorname{Lin} [n]^{(k)})\right)_{n \geq 2k}$ that $\operatorname{length}(\operatorname{Lin} \mathbb{A}^{(k)}) \leq \operatorname{length}(\operatorname{Lin} [2k]^{(k)})$ for all $k \geq 0$; the latter coincides with the recursively defined OEIS sequence `A005425`(k), which refines a previous upper bound obtained by Bojańczyk, Klin, and Moerman. Finally, we give evidence that our bound is tight.

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Contents

1 Introduction

Finite-state automata have proven to be a ubiquitous model of computation with numerous generalisations. The formalism is by itself simplistic: we consider an abstract machine with a finite set of states Q whose *configuration* evolves as an input word over a finite alphabet Σ is processed one letter at a time; after all letters are read, an output is then produced from the final configuration. For a deterministic finite automaton (DFA; see, e.g., [RS59, Definition 1]) with initial state $q_0 \in Q$, transition function

$$\delta : Q \times \Sigma \to Q,$$

and final states $Q_F \subseteq Q$, an input word $a_1 a_2 \dots a_n \in \Sigma^*$ gives rise to a unique *run* $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n$: the states $q_1, \dots, q_n \in Q$ are given by $\delta(q_{i-1}, a_i) = q_i$ for $1 \leq i \leq n$; the output is just a boolean value determined by whether $q_n \in Q_F$. Since a configuration is simply a state in Q, we say that the DFA has *state space* Q. A non-deterministic finite automaton (NFA; see, e.g., [RS59, Definitions 9–10]), in comparison, comes with a transition function

$$\delta : Q \times \Sigma \to \mathcal{P}(Q)$$

so that an input word may have zero or more than one run. The output then indicates whether any of the runs finishes in a final state. Correspondingly, the state space becomes the power set $\mathcal{P}(Q)$ of Q. A weighted automaton [Sch61, §I.6] takes a step further:

$$\delta : Q \times \Sigma \to \mathrm{Lin}(Q)$$

assigns to each $(q, a) \in Q \times \Sigma$ a linear combination of the states in Q over a field $\mathbb{F}$. Given a run $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n$, its weight is given by $I(q_0) \cdot c_1 \cdots c_n \cdot F(q_n)$ where $I : Q \to \mathbb{F}$ is the initial weight, $c_i \in \mathbb{F}$ is the coefficient of q_i in the vector $\delta(q_{i-1}, a_i)$ for $1 \le i \le n$, and $F : Q \to \mathbb{F}$ is the final weight. The output associated with a word $a_1 a_2 \dots a_n \in \Sigma^*$ is then obtained by summing the weights across all such runs. As the state space here is the free $\mathbb{F}$-vector space $\mathrm{Lin}(Q)$ on the set of states Q, standard results from linear algebra come in handy. Schützenberger's algorithm for equivalence checking* relies heavily on the dimension theory of finite-dimensional vector spaces in particular.

But these finite-state automata face an intrinsic limitation: the alphabet is finite (and typically very small), severely impeding applications in XML and verification amongst other concerns. As a recourse, Kaminski and Francez introduced finite-memory automata in [KF94, Definition 1] to handle infinite alphabets by allowing an NFA to store previously processed letters in a bounded number of *registers*. Input letters can only be compared for equality; when $\Sigma = \mathbb{N}$, if $q \xrightarrow{1} q' \xrightarrow{2} q''$ is a run, then so must be $q \xrightarrow{m} q' \xrightarrow{n} q''$ given any distinct $m, n \in \mathbb{N}$ for example. This is enforced by syntactic guards on the transition function. Inspired by the work of Gabbay and Pitts [GP02], Bojańczyk et al. describe an equivalent model in [BKL14, Definition 3.1 and Theorem 6.4] availing of sets with group actions. Here Q and Σ need only be *orbit-finite*: for instance, $\{q_n \mid n \in \mathbb{N}\} \cup \{q'_n \mid n \in \mathbb{N}\} \cup \{q''_n \mid n \in \mathbb{N}\}$ contains infinitely many elements but only 3 orbits under the bijective renamings of $\mathbb{N}$; that the graph of

$$\delta : Q \times \Sigma \to \mathcal{P}(Q)$$

is *equivariant* — i.e., the set is closed under group actions — then semantically ensures that letters are only tested for equality. We refer readers to [Boj19, §1.1] for a more gentle introduction.

Weighted orbit-finite automata [BKM21, Definition III.1] unify these two lines

*For a modern treatment, see [DK21, §8]; we also offer a detailed account in Section 2.3.

of generalisation with an equivariant transition function

$$\delta : Q \times \Sigma \to \mathrm{Lin}(Q).$$

The state space $\mathrm{Lin}(Q)$ is *orbit-finite-dimensional*: it has an orbit-finite basis Q. As a vector space with actions from the automorphism group G of $\mathbb{N}$, its substructures are *equivariant subspaces* — subsets that are closed under both linear combinations and G-actions; mathematically, one speaks of a *G-representation* or a *module over the group ring* $\mathbb{F}[G]$. To show that Schützenberger's algorithm can be adapted to the orbit-finite setting, a *length* theory of orbit-finite-dimensional vector spaces needs to be developed analogously to the dimension theory of finite-dimensional vector spaces for weighted finite automata. Bojańczyk et al. mention this algebraic connection before Lemma 4.2 in their work but do not pursue it further.

In this paper, we chase the representation-theoretic aspect fully. The archetypical orbit-finite set we study is $\mathbb{N}^{(k)}$, the k-tuples of distinct natural numbers endowed with the natural actions from the bijections of $\mathbb{N}$. As portended by the stability phenomena investigated in [BVO14], [SS15], and [OZ21], we find that the structure theory of $\mathrm{Lin}(\mathbb{N}^{(k)})$ is closely related to that of $\mathrm{Lin}(\{1, \dots, n\}^{(k)})$ for large n's; this allows us to draw upon classical results for finite symmetric groups, notably from [Sag01, §2–3] and [FH04, §4]. Our contributions are as follows.

- In Chapter 2, we provide a self-contained exposition of weighted orbit-finite automata and Schützenberger's equivalence algorithm.
- We lay out an elementary argument leading up to the internal direct sum (3.21) which appears to be original, even though explicit projection maps onto isomorphic copies have been determined in [CM11, §5].
- We establish the full structure theory of $\mathrm{Lin}\,\mathbb{N}^{(2)}$ as depicted by Figure B, which in particular exposes an infinite family of distinct equivariant subspaces.
- In Corollary 4.4 we exhibit the [OEIS] sequence `A005425`(k) as an upper bound for $\mathrm{length}(\mathbb{N}^{(k)})$, improving [BKM21, Lemma IV.9] by a large margin.

2 Atoms and automata

2.1 Atoms

The key motif in the computation theory with atoms[†] consists of replacing finite structures with infinite but highly symmetric ones. We begin by recalling the underlying formalism; we will closely follow a combination of [Boj19, §3] and [BKL14, §4].

2.1.1 $\mathrm{Aut}(\mathbb{A})$-sets

Let $\mathbb{A}$ be a countable relational structure in the model-theoretical sense; that is, $\mathbb{A} = (A, \{R_i\}_{i \in I})$ where the universe A is countable, and $R_i \subseteq A^{k_i}$ are k_i-ary relations. It is easy to see that the set of structure-preserving bijections

$$\mathrm{Aut}(\mathbb{A}) \coloneqq \{\pi : A \to A \text{ bijective} \mid \forall i : \overline{a} \in R_i \iff \pi(\overline{a}) \in R_i\}$$

forms a group under function composition — the identity function id_A lies in $\mathrm{Aut}(\mathbb{A})$, and for $\pi, \pi' \in \mathrm{Aut}(\mathbb{A})$ we have $\pi \circ \pi' \in \mathrm{Aut}(\mathbb{A})$ as well as $\pi^{-1} \in \mathrm{Aut}(\mathbb{A})$. We call $\mathrm{Aut}(\mathbb{A})$ the *automorphism group* of $\mathbb{A}$. When the context is clear, by abuse of notation we will use $\mathbb{A}$ and A interchangeably.

Example 2.1. We will focus exclusively on

- the *infinite equality atoms* $(\mathbb{N}, =)$, and

[†]The nomenclature stems from the Fraenkel–Mostowski permutation model of set theory with atoms in the 1930s. Under the guise of nominal sets, the theory has recently been rediscovered as an elegant model of name binding [GP02].

- the *finite equality atoms* $([n], =)$, where we write $[n]$ for $\{1, \ldots, n\}$.

Here $\mathrm{Aut}(\mathbb{N})$ is the full symmetric group of $\mathbb{N}$, whereas $\mathrm{Aut}([n])$ is the finite symmetric group S_n.

Two remarks are in order. First, the natural numbers regrettably start at 1 in this work. Second, finite symmetry groups are unusual in the nominal literature; we only intend to use $([n], =)$ as an auxiliary tool to study vector spaces with the infinite equality atoms later. ■

Recall that an $\mathrm{Aut}(\mathbb{A})$-*set* is a set X equipped with a group homomorphism $\pi \mapsto (\pi \cdot -)$ between $\mathrm{Aut}(\mathbb{A})$ and the bijections of X. By uncurrying, we see this amounts to a *group action* $- \cdot - : \mathrm{Aut}(\mathbb{A}) \times X \to X$ satisfying $(\pi \circ \pi') \cdot x = \pi \cdot (\pi' \cdot x)$ and $\mathrm{id}_{\mathbb{A}} \cdot x = x$ for all $\pi, \pi' \in \mathrm{Aut}(\mathbb{A})$ and $x \in X$.

$\mathbb{A}$ is canonically an $\mathrm{Aut}(\mathbb{A})$-set with $\pi \cdot a \coloneqq \pi(a)$. On the other hand, any set Z constructed without using $\mathbb{A}$ can also be viewed as an $\mathrm{Aut}(\mathbb{A})$-set if we endow it with the trivial action $\pi \cdot z \coloneqq z$ for all π and z. Now let X, Y be two $\mathrm{Aut}(\mathbb{A})$-sets. We can check that the following are also $\mathrm{Aut}(\mathbb{A})$-sets naturally:

- the cartesian product $X \times Y$ where $\pi \cdot (x, y) \coloneqq (\pi \cdot x, \pi \cdot y)$;

  in particular, it follows that $\mathbb{A}^k$ is an $\mathrm{Aut}(\mathbb{A})$-set with the diagonal action;
- the disjoint union $X \sqcup Y$ where $\pi \cdot -$ is defined by case analysis;
- the set ${}^X Y$ of all functions from X to Y, where given $f : X \to Y$ we define $\pi \cdot f : X \to Y$ pointwise via

  $$(\pi \cdot f)(x) \coloneqq \pi \cdot f(\pi^{-1} \cdot x)$$

  so that $\pi \cdot x$ is mapped to $\pi \cdot f(x)$ for all $x \in X$;
- the power set $\mathcal{P}(X)$ where

  $$\pi \cdot P \coloneqq \{\pi \cdot x \mid x \in P\}$$

  for all $P \subseteq X$.

2.1.2 Equivariance

Definition 2.2. Let X be an $\mathrm{Aut}(\mathbb{A})$-set. We say that $x \in X$ is *supported by* $S \subseteq \mathbb{A}$ if $\pi \cdot x = x$ whenever $\pi \in \mathrm{Aut}(\mathbb{A})$ satisfies $\pi|_S = \mathrm{id}_S$. In particular, we say that x is *equivariant* if it is supported by $\emptyset$.

Remark. By definition, a subset $P \subseteq X$ is equivariant iff $\pi \cdot x \in P$ for every $\pi \in \mathrm{Aut}(\mathbb{A})$ and $x \in P$. In that case, any action $\pi \cdot - : X \to X$ restricts to $(\pi \cdot -)|_P : P \to P$, and thus P itself is an $\mathrm{Aut}(\mathbb{A})$-set. For instance, we see that

$$\mathbb{A}^{(k)} \coloneqq \{(a_1, \ldots, a_k) \in \mathbb{A}^k \mid a_i \neq a_j \text{ whenever } i \neq j\}$$

is an equivariant subset of $\mathbb{A}^k$ for $k \geq 0$ with $\mathbb{A}^{(0)} = \{()\}$, and thus inherits the structure of an $\mathrm{Aut}(\mathbb{A})$-set with the diagonal action $\pi \cdot (a_1, \ldots, a_k) = (\pi \cdot a_1, \ldots, \pi \cdot a_k)$.

Also, a function $f : X \to Y$ is equivariant iff $f(\pi \cdot x) = \pi \cdot f(x)$ for every $\pi \in \mathrm{Aut}(\mathbb{A})$ and $x \in X$. ■

Example 2.3. If X and Y are $\mathrm{Aut}(\mathbb{A})$-sets with trivial actions, then any function $f : X \to Y$ is equivariant: we have $f(\pi \cdot x) = f(x) = \pi \cdot f(x)$. ■

2.1.3 Orbit-finiteness

The notion of $\mathrm{Aut}(\mathbb{A})$-sets is rather abstract. In contrast, one would expect a model of computation to have components which admit concrete descriptions that are amenable to algorithms. We will therefore restrict our attention to $\mathrm{Aut}(\mathbb{A})$-sets with the properties that we define next.

Definition 2.4. An $\mathrm{Aut}(\mathbb{A})$-set X is *nominal* if every $x \in X$ is supported by some $S \subseteq \mathbb{A}$ which is finite.

Remark. Clearly $\mathbb{A}$ is nominal: each $a \in \mathbb{A}$ is supported by the singleton $\{a\}$. In fact, it is straightforward to show that nominal sets are closed under cartesian products, equivariant subsets, and disjoint unions. It follows that $\mathbb{A}^k$, $\mathbb{A}^{(k)}$, and $\mathbb{A}^* = \sqcup_{k \geq 0} \mathbb{A}^k$

are all nominal. Also, if X is an $\mathrm{Aut}(\mathbb{A})$-set with a trivial action then X is obviously nominal, as each $x \in X$ is by definition equivariant.

Nonetheless, $\mathcal{P}(\mathbb{A})$ is not nominal when we work with the equality atoms $\mathbb{A} = (\mathbb{N}, =)$. To see this, consider the odd numbers $O \coloneqq \{1, 3, 5, \dots\} \subseteq \mathbb{A}$, and suppose to the contrary that we had a finite support $S \subseteq \mathbb{A}$. As both O and $\mathbb{A} \setminus O$ are infinite, clearly we can find $x \in O \setminus S$ and $y \in (\mathbb{A} \setminus O) \setminus S$. Now the transposition $(x\ y)$ acts trivially on S, yet $y \in (x\ y) \cdot O$ so $(x\ y) \cdot O \neq O$. ■

Definition 2.5. A nominal set X is *orbit-finite* if we can find $x_1, \dots, x_n \in X$ with

$$X = \mathrm{Aut}(\mathbb{A}) \cdot x_1 \sqcup \cdots \sqcup \mathrm{Aut}(\mathbb{A}) \cdot x_n,$$

where $\mathrm{Aut}(\mathbb{A}) \cdot x_i \coloneqq \{\pi \cdot x_i \mid \pi \in \mathrm{Aut}(\mathbb{A})\}$ is called the *orbit* of x_i.

Example 2.6. If X is an $\mathrm{Aut}(\mathbb{A})$-set with a trivial action, then $\mathrm{Aut}(\mathbb{A}) \cdot x = \{x\}$ for $x \in X$. Hence the notions of orbit-finiteness and finiteness coincide for such X. ■

The upshot is that orbit-finite nominal sets can be represented by finite means; checking membership, equality and forming products, disjoint unions, etc. are all computable. We refer the interested readers to [BKL14] and [Boj19, §4.1] for the theory and to [BBKL12] for a concrete implementation. Here, we will be content with knowing that orbit-finite sets can be used as inputs to algorithms and continue to work at an abstract level.

Proposition 2.7 (Equality atoms are oligomorphic [Boj19, Definition 3.9]). *Suppose $\mathbb{A} = (\mathbb{N}, =)$ or $\mathbb{A} = ([n], =)$ with $n \geq k$. Then $\mathbb{A}$, $\mathbb{A}^k$, and $\mathbb{A}^{(k)}$ are all orbit-finite, but $\mathbb{A}^*$ is not.*

Proof. We shall show $\mathbb{A}^{(k)} = \mathrm{Aut}(\mathbb{A}) \cdot (1, \dots, k)$. For $k = 0$, this is trivial: both sides are the singleton $\{()\}$. Thus let $(a_1, \dots, a_k) \in \mathbb{A}^{(k)}$ be arbitrary for $k \geq 1$, and put $t \coloneqq \max_i a_i$ so that $a_i \in [t]$ for all i. Then $k \leq t$, and $f : i \mapsto a_i$ defines an injection $[k] \to [t]$. But the finite sets $[t] \setminus [k]$ and $[t] \setminus f([k])$ have the same number of elements, so f extends to a bijection $[t] \to [t]$. Finally, by setting $x \mapsto x$ for $x \in \mathbb{A} \setminus [t]$ we obtain an element of $\mathrm{Aut}(\mathbb{A})$ whose action sends $(1, \ldots, k)$ to $(a_1, \ldots, a_k)$. Therefore $\mathbb{A}^{(k)}$ has a single orbit, and so does $\mathbb{A}$ since $\mathbb{A} = \mathbb{A}^{(1)}$.

Now consider $(a_1, \ldots, a_k) \in \mathbb{A}^k$. By above, we see that its orbit is determined by its equality type, i.e., the partition $[k]/{\sim}$ where $i \sim j$ iff $a_i = a_j$. Furthermore, if there are t equivalence classes, then by removing repetitions we obtain an equivariant bijection $\mathrm{Aut}(\mathbb{A}){\cdot}(a_1, \ldots, a_k) \simeq \mathbb{A}^{(t)}$. Overall, the orbits give an equivariant bijection

$$\mathbb{A}^k \simeq \bigsqcup_{t=0}^{k} \bigsqcup_{i=1}^{S(k,t)} \mathbb{A}^{(t)} \tag{2.8}$$

where $S(k, t)$ is the second Stirling number counting the number of ways to partition $[k]$ into t equivalence classes. In particular $\mathbb{A}^k$ has $\sum_{t=0}^{k} S(k, t)$ orbits, whereas $\mathbb{A}^*$ contains all these $\mathbb{A}^k$'s in a disjoint union and thus has infinitely many orbits. □

It turns out that, akin to (2.8), any orbit-finite set is related to the $\mathbb{A}^{(t)}$'s.

Lemma 2.9 (Variant of [Boj19, Lemma 3.20]). *Let X be an orbit-finite set. Then there exists an equivariant surjection $f : \sqcup_{i=1}^{k} \mathbb{A}^{(t_i)} \to X$.*

Proof. Suppose that $X = \sqcup_{i=1}^{k} \mathrm{Aut}(\mathbb{A}) \cdot x_i$. As X is nominal, x_i is supported by some finite set $S_i \subseteq \mathbb{A}$ whose elements we can enumerate as $(a_1, \ldots, a_{t_i}) \in \mathbb{A}^{(t_i)}$. Now observe that

$$\{(\pi \cdot (a_1, \ldots, a_{t_i}), \pi \cdot x_i) \mid \pi \in \mathrm{Aut}(\mathbb{A})\}$$

is the graph of an equivariant surjection $f_i : \mathbb{A}^{(t_i)} \to \mathrm{Aut}(\mathbb{A}) \cdot x_i$. Indeed, if we have $\pi \cdot (a_1, \ldots, a_{t_i}) = \pi' \cdot (a_1, \ldots, a_{t_i})$ then $\pi^{-1} \circ \pi'$ is the identity on $S_i = \{a_1, \ldots, a_{t_i}\}$, so $x_i = (\pi^{-1} \circ \pi') \cdot x_i$ by assumption and thus $\pi \cdot x_i = \pi' \cdot x_i$; it is then easy to see that the function f_i is equivariant and surjective. By combining all the f_i's, we can thus define an equivariant surjection $f : \sqcup_{i=1}^{k} \mathbb{A}^{(t_i)} \to X$ by cases. □

We shall expand our arsenal of orbit-finite sets.

Corollary 2.10 (Closure properties [Boj19, Lemma 3.24]). *With $\mathbb{A} = (\mathbb{N}, =)$, orbit-finite sets are closed under equivariant subsets, disjoint unions, and cartesian products.*

Proof. Recall that these constructions preserve nominal sets as we observed after Definition 2.4, so we only need to count the number of orbits.

Suppose that $X = \bigsqcup_{i \in I} \mathrm{Aut}(\mathbb{A}) \cdot x_i$ is orbit-finite, and let $P \subseteq X$ be equivariant. Then each $p \in P$ lies in the orbit of some x_{i_p}, so we have $P \subseteq \bigsqcup_{i \in I'} \mathrm{Aut}(\mathbb{A}) \cdot x_i$ where $I' \coloneqq \{i_p \mid p \in P\} \subseteq I$ is necessarily finite. But as the reverse containment also holds by the equivariance of P, we conclude that P is orbit-finite.

Now let $Y = \bigsqcup_{j \in J} \mathrm{Aut}(\mathbb{A}) \cdot y_j$ also be orbit-finite. Clearly $X \sqcup Y$ is orbit-finite: it has orbits $\mathrm{Aut}(\mathbb{A}) \cdot x_i, \mathrm{Aut}(\mathbb{A}) \cdot y_j$. But $X \times Y = \sqcup_{i,j} (\mathrm{Aut}(\mathbb{A}) \cdot x_i) \times (\mathrm{Aut}(\mathbb{A}) \cdot y_j)$, so it suffices to prove that $(\mathrm{Aut}(\mathbb{A}) \cdot x_i) \times (\mathrm{Aut}(\mathbb{A}) \cdot y_j)$ is orbit-finite for each i, j. Consider the equivariant surjections $f : \mathbb{A}^{(m)} \to \mathrm{Aut}(\mathbb{A}) \cdot x_i$ and $g : \mathbb{A}^{(n)} \to \mathrm{Aut}(\mathbb{A}) \cdot y_j$ that we constructed in the proof of Lemma 2.9. Define

$$
\begin{aligned}
h : \mathbb{A}^{(m)} \times \mathbb{A}^{(n)} &\to (\mathrm{Aut}(\mathbb{A}) \cdot x_i) \times (\mathrm{Aut}(\mathbb{A}) \cdot y_j) \\
(a, b) &\mapsto (f(a), g(b))
\end{aligned}
$$

which is easily seen to be an equivariant surjection. Since $\mathbb{A}^{(m)} \times \mathbb{A}^{(n)}$ is an equivariant subset of $\mathbb{A}^{m+n}$ which is orbit-finite by Proposition 2.7, by above $\mathbb{A}^{(m)} \times \mathbb{A}^{(n)}$ too is orbit-finite; say $\mathbb{A}^{(m)} \times \mathbb{A}^{(n)} = \bigsqcup_{k \in K} \mathrm{Aut}(\mathbb{A}) \cdot z_k$. Finally note that $\mathrm{Aut}(\mathbb{A}) \cdot h(z_k) = h(\mathrm{Aut}(\mathbb{A}) \cdot z_k)$ as h is equivariant, and thus by its surjectivity we have

$$
\begin{aligned}
(\mathrm{Aut}(\mathbb{A}) \cdot x_i) \times (\mathrm{Aut}(\mathbb{A}) \cdot y_j) &= h\Big(\bigsqcup_{k \in K} \mathrm{Aut}(\mathbb{A}) \cdot z_k\Big) \\
&= \bigcup_{k \in K} h(\mathrm{Aut}(\mathbb{A}) \cdot z_k) = \bigcup_{k \in K} \mathrm{Aut}(\mathbb{A}) \cdot h(z_k).
\end{aligned}
$$

This establishes the desired orbit-finiteness after we remove repeated orbits. □

2.2 Weighted automata

We are now ready to introduce the weighted orbit-finite automata with atoms from [BKM21].

2.2.1 Definition

Consider henceforth the infinite equality atoms $\mathbb{A} = (\mathbb{N}, =)$, and view the field of complex numbers $\mathbb{C}$ as a nominal $\mathrm{Aut}(\mathbb{A})$-set endowed with the trivial action.

Definition 2.11. A *weighted orbit-finite automaton* $\mathcal{A} = (Q, \Sigma, I, \delta, F)$ consists of

- an $\mathrm{Aut}(\mathbb{A})$-set of states Q,
- an $\mathrm{Aut}(\mathbb{A})$-set Σ (called the *alphabet*),
- initial weights $I : Q \to \mathbb{C}$,
- transition weights $\delta : Q \times \Sigma \times Q \to \mathbb{C}$, and
- final weights $F : Q \to \mathbb{C}$

where Q, Σ are orbit-finite and I, δ, F are equivariant.Moreover, we require I and $\delta(q, a, -)$ to be zero on all but finitely many states — this is called the *non-guessing condition* — so that given any $a_1 \dots a_n \in \Sigma^*$ the sum of weights

$$L_{\mathcal{A}}(a_1 \dots a_n) \coloneqq \sum_{(q_0, q_1, \dots, q_n) \in Q^{n+1}} I(q_0) \Big(\prod_{1 \le i \le n} \delta(q_{i-1}, a_i, q_i) \Big) F(q_n) \tag{2.12}$$

across all runs $q_0 \xrightarrow{a_1} q_1 \cdots \xrightarrow{a_n} q_n$ is well-defined. We say that $\mathcal{A}$ recognises the *weighted language* $L_{\mathcal{A}} : \Sigma^* \to \mathbb{C}$.

Remark. Any equivariant function $f : X \to \mathbb{C}$ is constant on each orbit in X: we have $f(\pi \cdot x) = \pi \cdot f(x) = f(x)$ for $x \in X, \pi \in \mathrm{Aut}(\mathbb{A})$. Therefore f amounts to finitely many values of $\mathbb{C}$ provided that X is moreover orbit-finite, and it can certainly be represented by finite means.

As Q and Σ are orbit-finite, so is $Q \times \Sigma \times Q$ by Corollary 2.10. It follows that I, δ, F are all determined by their values on the finitely many orbits of their respective domains.

Consider the case $Q, \Sigma = \mathbb{A}$. We see by (2.8) that $Q \times \Sigma \times Q = \mathbb{A}^3$ has 5 orbits, but due to the non-guessing condition δ cannot take a non-zero value at (a, a, b) or (a, b, c) where $a, b, c \in \mathbb{A}$ are distinct. The remaining 3 orbits are generated by (a, a, a), (a, b, a), and (a, b, b) respectively. Operationally, the transition (a_0, a_1, a_2)

can be interpreted as follows: "the automaton begins with a_0 stored in a 'register'; when it reads the letter a_1, it updates the register with a_2." Then the equivariance of δ ensures that the letters a_i are only compared for equality, and the non-guessing condition prohibits the automaton from using a necessarily divined letter $a_2 \in \mathbb{A} \setminus \{a_0, a_1\}$. ■

Example 2.13. In view of Examples 2.3 and 2.6, if Q and Σ are Aut($\mathbb{A}$)-sets with trivial actions then we recover the classical definition of a weighted finite automaton [DK21, Definitions 2.1 and 2.3] over the field $\mathbb{C}$. In this light, if $\mathcal{N} = (Q, \Sigma, I, \delta, F)$ is an NFA, then by putting

$$I'(q) = \begin{cases} 1 & \text{if } q \in I, \\ 0 & \text{otherwise,} \end{cases} \quad \delta'(p, a, q) = \begin{cases} 1 & \text{if } q \in \delta(p, a), \\ 0 & \text{otherwise,} \end{cases} \quad F'(q) = \begin{cases} 1 & \text{if } q \in F, \\ 0 & \text{otherwise} \end{cases}$$

we get a weighted automaton $\mathcal{A} = (Q, \Sigma, I', \delta', F')$ whose language $L_{\mathcal{A}}(a_1 \dots a_n)$ counts the number of accepting runs $q_0 \xrightarrow{a_1} q_1 \cdots \xrightarrow{a_n} q_n$ in $\mathcal{N}$ — observe that each such run is assigned a weight of 1 in $\mathcal{A}$. ■

2.2.2 Equivalence and zeroness

Recall that the equivalence problem for DFAs can be reduced to the emptiness problem: $\mathcal{A}_1$ and $\mathcal{A}_2$ recognise the same language iff $(L_{\mathcal{A}_1} \setminus L_{\mathcal{A}_2}) \cup (L_{\mathcal{A}_2} \setminus L_{\mathcal{A}_1})$ is empty, and we can build a new DFA from $\mathcal{A}_1, \mathcal{A}_2$ to recognise this latter language. An analogue holds for weighted orbit-finite automata: $\mathcal{A}_1$ and $\mathcal{A}_2$ recognise the same language iff $L_{\mathcal{A}_1} - L_{\mathcal{A}_2}$ is the zero function $\Sigma^* \to \mathbb{C}$. To recognise this latter language, we can construct a new weighted automaton from $\mathcal{A}_i = (Q_i, \Sigma, I_i, \delta_i, F_i), i \in \{1, 2\}$ as follows. Let $\mathcal{A}_\Delta \coloneqq (Q_1 \sqcup Q_2, \Sigma, I, \delta, F)$, where

$$I(q) \coloneqq \begin{cases} I_1(q) & \text{if } q \in Q_1, \\ I_2(q) & \text{if } q \in Q_2, \end{cases} \quad F(q) \coloneqq \begin{cases} F_1(q) & \text{if } q \in Q_1, \\ -F_2(q) & \text{if } q \in Q_2, \end{cases}$$

$$\delta(p,a,q) \coloneqq \begin{cases} \delta_1(p,a,q) & \text{if } p,q \in Q_1, \\ \delta_2(p,a,q) & \text{if } p,q \in Q_2, \\ 0 & \text{otherwise.} \end{cases}$$

We then see from (2.12) that $L_{\mathcal{A}_\Delta}(w) = L_{\mathcal{A}_1}(w) - L_{\mathcal{A}_2}(w)$ for all $w \in \Sigma^*$ as desired.

2.2.3 The state space $\operatorname{Lin} Q$

Before proceeding to address the zeroness problem, we take a detour to introduce an algebraic counterpart to the operational definition that we gave above. Given $q \in Q$, write $\mathbb{C}\underline{q}$ for the 1-dimensional vector space $\{c\underline{q} \mid c \in \mathbb{C}\}$ so that we can tell $\underline{q} \in \operatorname{Lin} Q$ apart from $q \in Q$; we then define $\operatorname{Lin} Q$ to be the direct sum $\bigoplus_{q \in Q} \mathbb{C}\underline{q}$. A typical element of $\operatorname{Lin} Q$ is a formal sum $\sum_{q \in Q} c_q \underline{q}$ where $c_q \in \mathbb{C}$ is zero for all but finitely many states $q \in Q$. Notice that permutations from $\operatorname{Aut}(\mathbb{A})$ act naturally on $\operatorname{Lin} Q$ via

$$\pi \cdot (\sum_{q \in Q} c_q \underline{q}) \coloneqq \sum_{q \in Q} c_q \underline{\pi \cdot q} \tag{2.14}$$

in a way that is compatible with the vector space structure of $\operatorname{Lin} Q$. We capture this structure as follows.

Definition 2.15. The *group ring* $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$ is the vector space $\operatorname{Lin}(\operatorname{Aut} \mathbb{A})$ equipped with a ring structure by extending the multiplication $\underline{\pi} * \underline{\pi'} \coloneqq \underline{\pi \circ \pi'}$ bilinearly. A $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-*module* M is an abelian group M together with a binary operation $- \cdot - : \mathbb{C}[\operatorname{Aut} \mathbb{A}] \times M \to M$ satisfying

$$\phi \cdot (m + m') = \phi \cdot m + \phi \cdot m'$$

$$(\phi + \phi') \cdot m = \phi \cdot m + \phi' \cdot m$$

$$(\phi * \phi') \cdot m = \phi \cdot (\phi' \cdot m)$$

$$\underline{\mathrm{id}_{\mathbb{A}}} \cdot m = m$$

for all $\phi, \phi' \in \mathbb{C}[\operatorname{Aut} \mathbb{A}]$ and $m, m' \in M$.

A $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-*submodule* is a subset $N \subseteq M$ with $0, n+n', \phi\cdot n \in N$ for all $n, n' \in N$ and $\phi \in \mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$, so that N is itself a $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-module with the restricted actions.

A $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-*module homomorphism* between two $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-modules M, M' is a function $f : M \to M'$ such that $f(m+m') = f(m) + f(m')$ and $f(\phi \cdot m) = \phi \cdot f(m)$ for all $m, m' \in M$ and $\phi \in \mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$.

One may wish to read up on the relevant module theory in [EH18, §1.1.2 and Chapter 2], but all we really need is the following.

Example 2.16. Let Q be an $\mathrm{Aut}(\mathbb{A})$-set. Define $- \cdot - : \mathbb{C}[\mathrm{Aut}\,\mathbb{A}] \times \mathrm{Lin}\,Q \to \mathrm{Lin}\,Q$ by extending the group action $- \cdot - : \mathrm{Aut}(\mathbb{A}) \times \mathrm{Lin}\,Q \to \mathrm{Lin}\,Q$ in (2.14) linearly, i.e., by putting

$$\Big(\sum_{\pi\in\mathrm{Aut}(\mathbb{A})} d_\pi \underline{\pi}\Big) \cdot v := \sum_{\pi\in\mathrm{Aut}(\mathbb{A})} d_\pi (\pi \cdot v)$$

for every $\sum_{\pi\in\mathrm{Aut}(\mathbb{A})} d_\pi \underline{\pi} \in \mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$ and $v \in \mathrm{Lin}\,Q$. It is easy to check by linearity that $- \cdot -$ satisfies the $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-module laws.

In particular, we have $c\underline{\mathrm{id}_{\mathbb{A}}} \cdot v = cv$ and $\underline{\pi} \cdot v = \pi \cdot v$ for all $c \in \mathbb{C}, \pi \in \mathrm{Aut}(\mathbb{A})$, and $v \in \mathrm{Lin}\,Q$. Using linearity, it is straightforward to see that a $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-submodule of $\mathrm{Lin}\,Q$ is precisely an equivariant subspace, whereas a $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-module homomorphism $f : \mathrm{Lin}\,Q \to \mathrm{Lin}\,Q'$ is precisely an equivariant linear map. ■

Now, given a weighted orbit-finite automaton $\mathcal{A} = (Q, \Sigma, I, \delta, F)$, we see that

$$\iota := \sum_{q\in Q} I(q)\underline{q}, \tag{2.17}$$

is a well-defined vector in $\mathrm{Lin}\,Q$ by the non-guessing condition. Similarly,

$$\begin{aligned} D_a : \mathrm{Lin}\,Q &\to \mathrm{Lin}\,Q \\ \underline{p} &\mapsto \sum_{q\in Q} \delta(p, a, q)\underline{q} \end{aligned} \tag{2.18}$$

is a well-defined linear map for any $a \in \Sigma$. Also,

$$\phi\Big(\sum_{q\in Q} c_q \underline{q}\Big) := \sum_{q\in Q} c_q F(q) \tag{2.19}$$

defines a linear functional $\phi : \operatorname{Lin} Q \to \mathbb{C}$.

An interesting but rather well-known property of the adjacency matrix A with entries in $\{0,1\}$ arising from a finite directed graph G is that the (v_0, v_n) entry in A^n counts the number of walks $(v_0, v_1, \ldots, v_n)$ in G. We give a generalisation for the weighted sum over labelled walks (2.12) that defines $L_{\mathcal{A}}$.

Proposition 2.20. *For any $w = a_1 \ldots a_n \in \Sigma^*$, we have*

$$L_{\mathcal{A}}(w) = (\phi \circ D_{a_n} \circ \cdots \circ D_{a_1})(\iota).$$

Proof. The assumptions can in fact be relaxed to allow automata whose initial weights are non-equivariant. We shall proceed by induction on the length of w for every automaton $\mathcal{A}$ at once. The base case is immediate: by (2.12), we have

$$L_{\mathcal{A}}(\varepsilon) = \sum_{q_0 \in Q} I(q_0) \cdot F(q_0) = \phi(\iota).$$

Now assume the result for $a_1 \ldots a_n$, and let $a_0 \in \Sigma$ be arbitrary. Given an automaton $\mathcal{A} = (Q, \Sigma, I, \delta, F)$, notice how

$$\begin{aligned}
L_{\mathcal{A}}(a_0 a_1 \ldots a_n) &= \sum_{(q_{-1}, q_0, \ldots, q_n) \in Q^{n+2}} I(q_{-1}) \prod_{0 \le i \le n} \delta(q_{i-1}, a_i, q_i) F(q_n) \\
&= \sum_{q_{-1} \in Q} \sum_{(q_0, \ldots, q_n) \in Q^{n+1}} \Big(I(q_{-1}) \delta(q_{-1}, a_0, q_0) \Big) \prod_{1 \le i \le n} \delta(q_{i-1}, a_i, q_i) F(q_n) \\
&= \sum_{(q_0, \ldots, q_n) \in Q^{n+1}} \Big(\sum_{q_{-1} \in Q} I(q_{-1}) \delta(q_{-1}, a_0, q_0) \Big) \prod_{1 \le i \le n} \delta(q_{i-1}, a_i, q_i) F(q_n) \\
&= L_{\mathcal{A}'}(a_1 \ldots a_n)
\end{aligned}$$

where $\mathcal{A}' \coloneqq (Q, \Sigma, I', \delta, F)$ with $I'(q) \coloneqq \sum_{q_{-1} \in Q} I(q_{-1}) \delta(q_{-1}, a_0, q)$. Note that $\mathcal{A}'$ is a well-defined automaton: as I, δ satisfy the non-guessing condition, so does I' because the set of states $\bigcup_{q_{-1} \in Q, I(q_{-1}) \neq 0} \{q \in Q \mid \delta(q_{-1}, a_0, q) \neq 0\}$ remains finite. By the inductive hypothesis, it follows that $L_{\mathcal{A}'}(a_1 \ldots a_n) = (\phi \circ D_{a_n} \circ \cdots \circ D_{a_1})(\iota')$ with $\iota' \coloneqq \sum_{q \in Q} I'(q) \underline{q} = \sum_{q_{-1} \in Q} I(q_{-1}) \sum_{q \in Q} \delta(q_{-1}, a_0, q) \underline{q} = D_{a_0}(\iota)$. □

2.3 Zeroness checking

We finally return to solving the zeroness problem for weighted orbit-finite automata. Given an automaton $\mathcal{A}$, we define ι, $(D_a)_{a\in\Sigma}$, and ϕ as in (2.17)–(2.19).

2.3.1 Configuration spaces

Definition 2.21. We call $(D_{a_m} \circ \cdots \circ D_{a_1})(\iota) \in \operatorname{Lin} Q$ the *configuration* of the word $a_1 \dots a_m \in \Sigma^*$. The *nth configuration space* V_n is the subspace of $\operatorname{Lin} Q$ spanned by

$$\{(D_{a_m} \circ \cdots \circ D_{a_1})(\iota) \mid a_1 \dots a_m \in \Sigma^m, 0 \leq m \leq n\}.$$

Proposition 2.22. *$L_{\mathcal{A}} : \Sigma^n \to \mathbb{C}$ is the zero function iff $\phi(V_m) = \{0\}$ for all $m \leq n$.*

Proof. Suppose that $L_{\mathcal{A}}(w) = 0$ for all $w \in \Sigma^n$. By Proposition 2.20, we see that ϕ is zero at each configuration. As ϕ is linear, we conclude that $\phi(V_m) = \{0\}$ for all $m \leq n$.

Conversely, suppose that $L_{\mathcal{A}}(w) \neq 0$ for some $a_1 \dots a_m \in \Sigma^*$ with $0 \leq m \leq n$. Then, again by Proposition 2.20, it is clear that ϕ is non-zero at the configuration of $a_1 \dots a_m$, and hence $\phi(V_m) \neq \{0\}$. □

We thus need to understand the V_n's.

Proposition 2.23. *The subspace $V_n \subseteq \operatorname{Lin} Q$ is equivariant for all n.*

Proof. Let $\pi \in \operatorname{Aut}(\mathbb{A})$. We have $\pi \cdot \iota = \sum_{q\in Q} I(q)\underline{\pi \cdot q} = \sum_{q\in Q} I(\pi \cdot q)\underline{\pi \cdot q} = \iota$ since I is equivariant and $\pi \cdot -$ is a bijection of Q. We also have

$$\pi \cdot D_a(\underline{p}) = \sum_{q\in Q} \delta(p, a, q)\underline{\pi \cdot q} = \sum_{q\in Q} \delta(\pi \cdot p, \pi \cdot a, \pi \cdot q)\underline{\pi \cdot q} = D_{\pi\cdot a}(\pi \cdot \underline{p})$$

for any $p \in Q$ by the equivariance of δ. It follows by the linearity of $\pi \cdot -$ that $\pi \cdot D_a(u) = D_{\pi\cdot a}(\pi \cdot u)$ for all $u \in \operatorname{Lin} Q$. Given a configuration $v = (D_{a_m} \circ \cdots \circ D_{a_1})(\iota)$ in V_n, we then have $\pi \cdot v = (D_{\pi\cdot a_m} \circ \cdots \circ D_{\pi\cdot a_1})(\pi \cdot \iota) = (D_{\pi\cdot a_m} \circ \cdots \circ D_{\pi\cdot a_1})(\iota)$ which is the configuration of the word $\pi \cdot a_1 \ldots a_m$ of length $m \leq n$. Again by the linearity of $\pi \cdot - : \operatorname{Lin} Q \to \operatorname{Lin} Q$, we conclude that $\pi \cdot w \in V_n$ for all $w \in V_n$. □

In light of Example 2.16, V_n is thus a $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-submodule of $\operatorname{Lin} Q$. It is clear from the definition that $V_n \subseteq V_{n+1}$ for all n, so we have an ascending sequence of $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-submodules $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$ in $\operatorname{Lin} Q$.

Proposition 2.24. *If $V_n = V_{n+1}$ for some n, then $V_n = V_{n+i}$ for all $i \geq 0$.*

Proof. Observe that

$$V_{n+1} = V_0 + \sum_{a \in \Sigma} D_a(V_n) \tag{2.25}$$

for all n. Indeed a word of length $m \leq n + 1$ is either the empty string ε, or $b_1 \ldots b_{m-1} a$ whose configuration is $D_a(v)$ where $v \in V_{m-1} \subseteq V_n$ is the configuration of the prefix $b_1 \ldots b_{m-1}$.

Now suppose that $V_{n+1} = V_n$ for some n. Note that $V_{n+i} = V_n$ implies that $V_{n+i+1} = V_0 + \sum_{a \in \Sigma} D_a(V_{n+i}) = V_0 + \sum_{a \in \Sigma} D_a(V_n) = V_{n+1} = V_n$, so the conclusion follows by induction. □

The crux is therefore the strictly ascending prefix of the sequence.

Definition 2.26. Let M be a $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-module. A *chain* is a sequence of $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-submodules

$$M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_l$$

in M; we say that the chain has *length l*. We define $\operatorname{length}(M)$ to be the largest possible length of any chain in M if such a maximum exists; otherwise we set $\operatorname{length}(M) = \infty$.

Remark (technical!). Equivalently and more commonly, $\operatorname{length}(M)$ is defined to be the length of *any* composition series of M — a chain which is maximal in the sense that no submodules can be added to extend it — thanks to the Jordan–Hölder Theorem. Curious readers are invited to consult [GW04, Chapter 4]. ■

Notice that if $\iota = 0$, then $\{0\} = V_0 = V_1 = \cdots$ stabilises right away by (2.25). Thus suppose that $0 \neq \iota \in V_0$. Provided that length(Lin Q) is finite, we must have

$$\{0\} \subsetneq V_0 \subsetneq \cdots \subsetneq V_s = V_{s+1} = \cdots \tag{2.27}$$

for some s with $s+1 \leq \text{length}(\operatorname{Lin} Q)$. By Proposition 2.22, if $L_{\mathcal{A}}$ is the zero function then $\phi(V_s) = \{0\}$. If conversely $L_{\mathcal{A}}(\Sigma^*) \neq \{0\}$ (which by Proposition 2.20 forces $\iota \neq 0$), $\phi(V_n)$ must be non-zero for some $n \leq s$; let n be minimal. Then $L_{\mathcal{A}}(w)$ must be non-zero for some word w with length $n \leq \text{length}(\operatorname{Lin} Q) - 1$.

Example 2.28. Consider a usual NFA $\mathcal{N}$ with states Q. Following Example 2.13, we see by above that if $L_{\mathcal{N}} \neq \emptyset$ then it contains a word w whose length is less than length(Lin Q). But Aut($\mathbb{A}$) acts trivially on Q, so a $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-submodule of Lin Q simply amounts to a subspace. It follows that $\text{length}(\operatorname{Lin} Q) = \dim(\operatorname{Lin} Q)$, which is just $|Q|$, the number of states.

We can arrive at the same conclusion with a much more rudimentary argument. Pick $a_1 \dots a_n \in L_{\mathcal{N}}$ with the minimal length, and suppose for a contradiction that $n \geq |Q|$. Then there is an accepting run

$$q_0 \xrightarrow{a_1} q_1 \cdots \xrightarrow{a_n} q_n$$

with $n + 1 > |Q|$ states, so necessarily $q_i = q_j$ for some $i < j$. But then

$$q_0 \xrightarrow{a_1} \cdots \xrightarrow{a_i} q_i \xrightarrow{a_{j+1}} q_{j+1} \cdots \xrightarrow{a_n} q_n$$

is also an accepting run in the NFA, so $a_1 \dots a_i a_{j+1} \dots a_n$ is a shorter word in $L_{\mathcal{N}}$ — which is absurd. Therefore the non-emptiness of $L_{\mathcal{N}}$ must be witnessed by a word whose length is strictly less than $|Q| = \text{length}(\operatorname{Lin} Q)$. ■

2.3.2 Schützenberger's algorithm

Example 2.29. For weighted finite automata, we also have $\text{length}(\operatorname{Lin} Q) = |Q|$. The resulting zeroness-checking algorithm that iteratively computes $V_0, V_1, \dots, V_{|Q|-1}$

until $\phi(V_n) \neq \{0\}$ or $V_n = V_{n+1}$ is often attributed to Marcel-Paul Schützenberger for his pioneering paper [Sch61].

Algorithm Zeroness of weighted automata

Input: Weighted finite automaton $\mathcal{A} = (Q, \Sigma, I, \delta, F)$

// Book-keeping

$\mathrm{V}_0 \leftarrow |Q|$-by-1 matrix whose $(p,1)$ entry is $I(p)$

for $a \in \Sigma$ **do**

  $\mathrm{D}_a \leftarrow |Q|$-by-$|Q|$ matrix whose (p,q) entry is $\delta(p,a,q)$

$\mathrm{F}_0 \leftarrow$ 1-by-$|Q|$ matrix whose $(1,q)$ entry is $F(q)$

// Iteratively compute a $|Q|$-by-$\dim(V_n)$ matrix V_n representing $V_n \subseteq \operatorname{Lin} Q$

$n \leftarrow 0$

repeat

  if FV_n has a non-zero entry **then**

    return "$L_{\mathcal{A}}$ `is NOT the zero function`"

  $\mathrm{V}_{n+1} \leftarrow \mathrm{V}_0$ and the matrices $\mathrm{D}_a\mathrm{V}_n, a \in \Sigma$ stacked horizontally ▷ by (2.25)

  Reduce V_{n+1} to column echelon form and remove zero columns

  $n \leftarrow n+1$

until V_{n-1} has the same number of columns as V_n

// Indeed $\phi(V_n) = \{0\}$ iff FV_n is the zero matrix,

// and $V_n \subseteq V_{n+1}$ is strict iff V_n has fewer columns than V_{n+1}.

return "$L_{\mathcal{A}}$ `IS the zero function`"

Note that the matrix operations can be performed in $O(|\Sigma| \cdot |Q|^3)$ time at each step. As there are at most $\operatorname{length}(\operatorname{Lin} Q) = |Q|$ steps, the overall algorithm runs in $O(|\Sigma| \cdot |Q|^4)$ time. ■

Perhaps surprisingly, the zeroness problem for weighted orbit-finite automata can be reduced to that of weighted finite automata: by Lemma V.2 of [BKM21] we need only check the zeroness of a finitised automaton[‡] with at most $n \cdot l^k$ states and l letters when $\Sigma = \mathbb{A}$, where

- n is the number of orbits in Q,
- $l = \operatorname{length}(\operatorname{Lin} Q)$, and
- k is the smallest k' such that every $q \in Q$ is supported by at most k' atoms.

All in all, we obtain an algorithm which runs in polynomial time with respect to $n \cdot l^k$, provided that l is finite.

But is $\operatorname{length}(\operatorname{Lin} Q)$ finite when the set of states Q is only orbit-finite? That is, do all chains of equivariant subspaces in the orbit-finite-dimensional state space $\operatorname{Lin} Q$ have finite lengths? A good starting point is $Q = \mathbb{A}^{(k)}$; in fact, it turns out that their lengths are all we need to know.

Lemma 2.30 (Generalised rank-nullity). *For $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-modules M, N,*

- $\operatorname{length}(M \oplus N) = \operatorname{length}(M) + \operatorname{length}(N)$*, and*
- $\operatorname{length}(M) = \operatorname{length}(\ker f) + \operatorname{length}(f(M)) \geq \operatorname{length}(f(M))$ *whenever $f : M \to N$ is a $\mathbb{C}[\operatorname{Aut} \mathbb{A}]$-module homomorphism.*

Remark. More generally, $\operatorname{length}(B) = \operatorname{length}(A) + \operatorname{length}(C)$ whenever $A \to B \to C$ is a "short exact sequence" — e.g., $M \hookrightarrow M \oplus N \twoheadrightarrow N$ and $\ker f \hookrightarrow M \twoheadrightarrow f(M)$. See [EH18, Proposition 3.17(b)] or [GW04, Proposition 4.12] for a proof. ■

Corollary 2.31. *Assume $\operatorname{Lin} \mathbb{A}^{(k)}$ has finite length for all $k \geq 0$. Then $\operatorname{Lin} Q$ has finite length whenever Q is orbit-finite.*

Proof. Let Q be orbit-finite. By Lemma 2.9, we have an equivariant surjection $f : \bigsqcup_{i=1}^{n} \mathbb{A}^{(k_i)} \to Q$. We can easily check that the induced linear map

$$\widehat{f} : \bigoplus_{i=1}^{n} \operatorname{Lin} \mathbb{A}^{(k_i)} \to \operatorname{Lin} Q$$
$$\underline{(a_1, \ldots, a_{k_i})} \mapsto \underline{f(a_1, \ldots, a_{k_i})}$$

[‡]Namely, the weighted finite automaton $(Q^\sigma, \sigma, I|_{Q^\sigma}, \delta_{Q^\sigma \times \sigma \times Q^\sigma}, F_{Q^\sigma})$ where $\sigma := \{1, \ldots, l\}$ and $Q^\sigma := \{q \in Q \mid q \text{ is supported by } \sigma\}$ are finite subsets of Σ and Q; more details can be found on `https://drive.google.com/file/d/14RIrUoZksd80oD2R1qeEWrbB9j18aeB1`.

is surjective and respects the $\mathrm{Aut}(\mathbb{A})$-actions. It follows by Lemma 2.30 that

$$\mathrm{length}(\mathrm{Lin}\, Q) \leq \mathrm{length}(\bigoplus_{i=1}^{n} \mathrm{Lin}\, \mathbb{A}^{(k_i)}) = \sum_{i=1}^{n} \mathrm{length}(\mathrm{Lin}\, \mathbb{A}^{(k_i)}),$$

which is finite by assumption. □

It remains to verify the assumption that $\mathrm{length}(\mathrm{Lin}\, \mathbb{A}^{(k)})$ is finite, which will be the focus of the two following chapters.

3 Equivariant subspaces

In orbit-finite-dimensional vector spaces,
Equivariant subspaces find their places,
Where symmetry reigns and invariance holds,
A beauty that mathematicians unfold.

ChatGPT

To compute length($\mathrm{Lin}\,\mathbb{A}^{(k)}$), perhaps the most straightforward way is to simply find all the equivariant subspaces in $\mathrm{Lin}\,\mathbb{A}^{(k)}$. We therefore attempt to classify all subsets of $\mathrm{Lin}\,\mathbb{A}^{(k)}$ that are closed under both linear combinations and permutations. In doing so, we will compile a bestiary of these $\mathbb{C}[\mathrm{Aut}\,\mathbb{A}]$-submodules for both $\mathbb{A} = (\mathbb{N}, =)$ and $\mathbb{A} = ([n], =)$.

3.1 $\mathrm{Lin}\,\mathbb{N}$ and $\mathrm{Lin}\,[n]$

Note that $[n] \subseteq \mathbb{N}$ gives rise to $\mathrm{Lin}\,[n] \subseteq \mathrm{Lin}\,\mathbb{N}$. Also, we view $\pi \in S_n = \mathrm{Aut}([n])$ as an element of S_{n+1} and $\mathrm{Aut}(\mathbb{N})$ by letting $\pi(i) := i$ for $i \notin [n]$, so that $S_n \subseteq S_{n+1} \subseteq \mathrm{Aut}(\mathbb{N})$ as groups.

Proposition 3.1. *Define* $\epsilon : \mathrm{Lin}\,\mathbb{N} \to \mathbb{C}$ *by* $\sum_i a_i \underline{i} \mapsto \sum_i a_i$*, and write* $\epsilon_n := \epsilon|_{\mathrm{Lin}\,[n]}$*. Then* $\ker(\epsilon) := \{v \in \mathrm{Lin}\,\mathbb{N} \mid \epsilon(v) = 0\} \subseteq \mathrm{Lin}\,\mathbb{N}$ *is an* $\mathrm{Aut}(\mathbb{N})$*-equivariant subspace, while* $\ker(\epsilon_n) \subseteq \mathrm{Lin}\,[n]$ *is an* $\mathrm{Aut}([n])$*-equivariant subspace.*

Proof. We can check that ϵ is a $\mathbb{C}[\mathrm{Aut}\,\mathbb{N}]$-module homomorphism, i.e., $\epsilon(\pi \cdot v + v') = \pi \cdot \epsilon(v) + \epsilon(v')$ for all $v, v' \in \mathrm{Lin}\,\mathbb{N}$ and $\pi \in \mathrm{Aut}(\mathbb{N})$; its $\ker(\epsilon)$ is then easily seen to be an equivariant subspace of $\operatorname{Lin}\mathbb{N}$. Similarly ϵ_n is a $\mathbb{C}[\operatorname{Aut}[n]]$-module homomorphism, so $\ker(\epsilon_n)$ is an equivariant subspace of $\operatorname{Lin}[n]$. □

Remark. $\{0\} \subsetneq \ker(\epsilon) \subsetneq \operatorname{Lin}\mathbb{N}$ and $\{0\} \subsetneq \ker(\epsilon_n) \subsetneq \operatorname{Lin}[n]$ for $n \geq 2$: consider the vectors $\underline{1} - \underline{2}$ and $\underline{1}$. ■

Lemma 3.2. $\ker(\epsilon)$ *and* $\ker(\epsilon_n), n \geq 2$ *respectively have bases* $\{\underline{1} - \underline{j} \mid 2 \leq j\}$ *and* $\{\underline{1} - \underline{j} \mid 2 \leq j \leq n\}$.

Proof. Clearly $\{\underline{1} - \underline{j} \mid 2 \leq j\} \subseteq \ker(\epsilon)$ is linearly independent: only $\underline{1} - \underline{j}$ contributes to the coefficient of $\underline{j}$. Moreover this set spans $\ker(\epsilon)$, as any $\sum_{i \in \mathbb{N}} c_i \underline{i} \in \ker(\epsilon)$ can be written as $\sum_{i \in \mathbb{N}} (-c_i)(\underline{1} - \underline{i})$ since $\sum_{i \in \mathbb{N}} c_i = 0$. Replacing $\mathbb{N}$ by $[n]$, the same argument shows that $\ker(\epsilon_n)$ has $\{\underline{1} - \underline{j} \mid 2 \leq j \leq n\}$ as a basis. □

3.1.1 $\operatorname{Lin}\mathbb{N}$

Proposition 3.3. $\{0\}$, $\ker(\epsilon)$, *and* $\operatorname{Lin}\mathbb{N}$ *are the only equivariant subspaces in* $\operatorname{Lin}\mathbb{N}$.

Proof. We paraphrase [BKM21, Example 5]. Let $V \subseteq \operatorname{Lin}\mathbb{N}$ be an equivariant subspace. Suppose that $V \neq \{0\}$, so we can find a non-zero vector $v = \sum_{i \in I} c_i \underline{i} \in V$ where we require $c_i \neq 0$ for each $i \in I$. Then $I \subseteq \mathbb{N}$ is finite but non-empty, and we can pick $i_0 \in \mathbb{N} \setminus I$ as well as $i_1 \in I$. By equivariance, V contains $v' := v - (i_0\ i_1) \cdot v = (0 - c_{i_1})\underline{i_0} + (c_{i_1} - 0)\underline{i_1} = c_{i_1}(\underline{i_1} - \underline{i_0})$. Now given $j \geq 2$, we can certainly find $\pi \in \operatorname{Aut}(\mathbb{N})$ with $\pi(i_1) = 1$ and $\pi(i_0) = j$. As $c_{i_1} \neq 0$ by assumption, we see that $c_{i_1}^{-1} \underline{\pi} \cdot v' = \underline{1} - \underline{j}$ lies in V. It follows by Lemma 3.2 that $\ker(\epsilon) \subseteq V$.

Now suppose that $\ker(\epsilon) \subsetneq V$, so there is $w = \sum_{j \in \mathbb{N}} d_i \underline{j} \in V \setminus \ker(\epsilon)$. Then $w + \sum_{j \in \mathbb{N}} d_j(\underline{1} - \underline{j}) = \sum_{j \in \mathbb{N}} d_j \underline{1} = \epsilon(w)\underline{1} \in V$. As $\epsilon(w) \neq 0$, by equivariance we have $\underline{n} \in V$ for all $n \in \mathbb{N}$. It follows that $V = \operatorname{Lin}\mathbb{N}$. □

Note that we were always able to choose a fresh atom $i_0 \in \mathbb{N} \setminus I$ above. This might not be possible when we work with $[n]$ instead of $\mathbb{N}$; indeed, the landscape of $\operatorname{Lin}[n]$ differs slightly from that of $\operatorname{Lin}\mathbb{N}$ as a new equivariant subspace now emerges.

3.1.2 Lin $[n]$

Proposition 3.4. *$\{0\}$, $\mathbb{C}(\underline{1}+\cdots+\underline{n})$, $\ker(\epsilon_n)$, and* Lin $[n]$ *are the only equivariant subspaces in* Lin $[n]$ *for $n \geq 2$. Moreover* Lin $[n] = \mathbb{C}(\underline{1}+\cdots+\underline{n}) \oplus \ker(\epsilon_n)$.

Proof. Firstly, observe that the 1-dimensional subspace $\mathbb{C}(\underline{1}+\cdots+\underline{n}) \subseteq \mathrm{Lin}\,[n]$ is equivariant: we have $\pi \cdot (\underline{1}+\cdots+\underline{n}) = \underline{1}+\cdots+\underline{n}$ for any $\pi \in \mathrm{Aut}([n])$.

Let $V \subseteq \mathrm{Lin}\,[n]$ be an equivariant subspace. Suppose that $V \subseteq \mathbb{C}(\underline{1}+\cdots+\underline{n})$. Then either

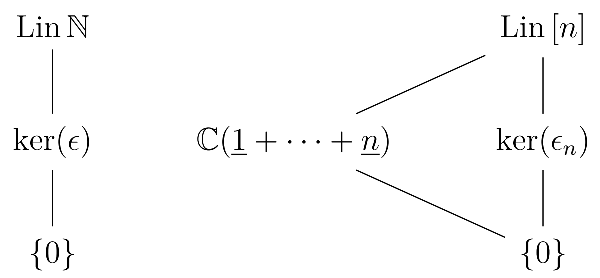
- $\dim V = 0$, in which case $V = \{0\}$, or
- $\dim V = 1$, in which case $V = \mathbb{C}(\underline{1}+\cdots+\underline{n})$.

Now suppose that $V \not\subseteq \mathbb{C}(\underline{1}+\cdots+\underline{n})$, so there is $v = \sum_{i\in[n]} c_i \underline{i} \in V$ with $c_{i_0} \neq c_{i_1}$ for some $i_0, i_1 \in I$. Then $v' := v - (i_0\ i_1)\cdot v = (c_{i_1} - c_{i_0})(\underline{i_1} - \underline{i_0}) \in V$. For $2 \leq j \leq n$, we can certainly find $\pi \in \mathrm{Aut}([n])$ with $\pi(i_1) = 1, \pi(i_0) = j$ so that $(c_{i_1} - c_{i_0})^{-1}\underline{\pi} \cdot v' = \underline{1} - \underline{j}$. It follows by Lemma 3.2 that $\ker(\epsilon_n) \subseteq V \subseteq \mathrm{Lin}\,[n]$. But $\dim(\ker \epsilon_n) = n-1$ whilst $\dim(\mathrm{Lin}\,[n]) = n$, so either

- $\dim V = n-1$, in which case $V = \ker(\epsilon_n)$, or
- $\dim V = n$, in which case $V = \mathrm{Lin}\,[n]$.

Finally we have $\epsilon(\underline{1}+\cdots+\underline{n}) = n \neq 0$, which shows that $\mathbb{C}(\underline{1}+\cdots+\underline{n})$ and $\ker(\epsilon_n)$ intersect trivially. By considering the dimensions, we conclude that $\mathrm{Lin}\,[n] = \mathbb{C}(\underline{1}+\cdots+\underline{n}) \oplus \ker(\epsilon_n)$. ☐

Corollary 3.5. *Below are the lattices of all equivariant subspaces in* Lin $\mathbb{N}$ *and* Lin $[n], n \geq 2$ *with respect to inclusion. Thus* $\mathrm{length}(\mathrm{Lin}\,\mathbb{N}) = 2 = \mathrm{length}(\mathrm{Lin}\,[n])$.



3.2 From $\operatorname{Lin}\mathbb{N}^{(k)}$ to $\operatorname{Lin}[n]^{(k)}$

Already when $k = 1$, we saw some connections (and disparities, no less) between $\operatorname{Lin}\mathbb{N}^{(k)}$ and $\operatorname{Lin}[n]^{(k)}$: notice that $\operatorname{Lin}\mathbb{N} = \mathbb{C}[\operatorname{Aut}\mathbb{N}] \odot \operatorname{Lin}[n]$ and that $\ker(\epsilon) = \mathbb{C}[\operatorname{Aut}\mathbb{N}] \odot \ker(\epsilon_n)$ for any $n \geq 2$, where the generating operator $\odot$ is defined as follows.

Definition 3.6. Let M be an R-module and let $X \subseteq M$ be arbitrary. We write $R \odot X \coloneqq \{\sum_{x\in X} r_x \cdot x \mid r_x \in R \text{ is zero for all but finitely many } x \in X\} \subseteq M$. (The more common notation is $R \cdot X$, but it conflicts with how we denote the orbit $\{r \cdot X \mid r \in R\}$ of X under R.)

Remark. As M is an R-set, given $x \in M$ we note that the orbit $R \cdot x$ coincides with $R \odot \{x\}$ and is easily seen to be an R-submodule of M. Given $X \subseteq M$, we see that $R \odot X = \sum_{x\in X} R \cdot x$ is an R-submodule of M too. ■

Example 3.7. By definition, the $\mathbb{C}$-vector space $\operatorname{Lin}[n]$ is a $\mathbb{C}$-module; $\mathbb{C}\odot X$ is then just the $\mathbb{C}$-linear span of $X \subseteq \operatorname{Lin}[n]$. For instance $\mathbb{C} \odot (\underline{1}+\cdots+\underline{n}) = \mathbb{C}(\underline{1}+\cdots+\underline{n})$, and $\mathbb{C} \odot \{\underline{1} - \underline{j} \mid 2 \leq j \leq n\} = \operatorname{Lin}[n]$ by Lemma 3.2. ■

3.2.1 Modules over $\mathbb{C}[\operatorname{Aut}\mathbb{N}]$, $\mathbb{C}[S_\infty]$, and $\mathbb{C}[S_n]$

Now fix $k \geq 1$. On the one hand, $\operatorname{Aut}(\mathbb{N})$ naturally acts on $\operatorname{Lin}\mathbb{N}^{(k)}$. On the other hand, as $\operatorname{Aut}([n]) = S_n$ acts naturally on $\operatorname{Lin}[n]^{(k)}$, it is

$$S_\infty \coloneqq \bigcup_{n\geq 1} S_n$$

that acts on $\bigcup_{n\geq 1} \operatorname{Lin}[n]^{(k)} = \operatorname{Lin}\mathbb{N}^{(k)}$. Note that $S_\infty \subsetneq \operatorname{Aut}(\mathbb{N})$: consider the permutation $\prod_{i\geq 1}(2k\ 2k+1)$ which moves infinitely many points. We can nonetheless reconcile this difference.

Proposition 3.8. *Let* $V \subseteq \operatorname{Lin}\mathbb{N}^{(k)}$. *Then* V *is a* $\mathbb{C}[S_\infty]$*-submodule iff* V *is a* $\mathbb{C}[\operatorname{Aut}\mathbb{N}]$*-submodule.*

Proof. If V is closed under the $\mathbb{C}[\operatorname{Aut}\mathbb{N}]$-actions, then clearly it is closed under the actions of $\mathbb{C}[S_\infty] \subseteq \mathbb{C}[\operatorname{Aut}\mathbb{N}]$.

Conversely, suppose that V is only closed under the $\mathbb{C}[S_\infty]$-actions. Let $v = \sum_{p\in\mathbb{N}^{(k)}} c_p \underline{p} \in V$, and let $\pi \in \operatorname{Aut}(\mathbb{N})$. Writing $\operatorname{supp}(p) \subseteq \mathbb{N}$ for the set of atoms occurring in $p \in \mathbb{N}^{(k)}$, we see that $\operatorname{supp}(v) \coloneqq \bigcup_{p\in\mathbb{N}^{(k)}, c_p\neq 0} \operatorname{supp}(p) \subseteq \mathbb{N}$ is finite. Since $\pi|_{\operatorname{supp}(v)} : \operatorname{supp}(v) \to \pi\cdot\operatorname{supp}(v)$ is still bijective, it extends to a bijection π' of $[n]$ for $n \geq \max\bigl(\operatorname{supp}(v) \cup \pi\cdot\operatorname{supp}(v)\bigr)$. But $\pi' \in S_n \subseteq S_\infty$, so V contains $\pi'\cdot v = \pi\cdot v$. □

Therefore $\operatorname{length}(\operatorname{Lin}\mathbb{N}^{(k)})$ does not depend on whether we view $\operatorname{Lin}\mathbb{N}^{(k)}$ as a $\mathbb{C}[S_\infty]$-module or as a $\mathbb{C}[\operatorname{Aut}\mathbb{N}]$-module. Moreover, we have the following bound.

3.2.2 $\sup_{n\gg 0} \operatorname{length}(\operatorname{Lin}[n]^{(k)})$ bounds $\operatorname{length}(\operatorname{Lin}\mathbb{N}^{(k)})$ from above

Theorem 3.9. *Assume* $l \in \mathbb{N}$ *satisfies* $\operatorname{length}(\operatorname{Lin}[n]^{(k)}) \leq l$ *for sufficiently large* n. *Then* $\operatorname{length}(\operatorname{Lin}\mathbb{N}^{(k)}) \leq l$.

Proof. Say $\operatorname{length}(\operatorname{Lin}[n]^{(k)}) \leq l$ whenever $n \geq N$. Suppose to the contrary that $\operatorname{length}(\operatorname{Lin}\mathbb{N}^{(k)}) > l$, so we can find a chain $\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_{l+1}$ of length $l+1$ in $\operatorname{Lin}\mathbb{N}^{(k)}$. By induction, there exist $v_1, \ldots, v_{l+1} \in \operatorname{Lin}\mathbb{N}^{(k)}$ such that

$$v_i \in V_i \setminus \mathbb{C}[S_\infty] \odot \{v_1, \ldots, v_{i-1}\} = V_i \setminus (\mathbb{C}[S_\infty]\cdot v_1 + \cdots + \mathbb{C}[S_\infty]\cdot v_{i-1})$$

for all i: we can pick any $v_1 \in V_1 \setminus \{0\}$; having picked $v_1, \ldots, v_{i-1}$, notice that $\mathbb{C}[S_\infty]\cdot v_1 + \cdots + \mathbb{C}[S_\infty]\cdot v_{i-1} \subseteq V_1 + \cdots + V_{i-1} \subseteq V_{i-1} \subsetneq V_i$. Now $v_i \in \operatorname{Lin}\mathbb{N}^{(k)} = \bigcup_n \operatorname{Lin}[n]^{(k)}$, so each v_i lies in $\operatorname{Lin}[n_i]^{(k)}$ for some n_i. Then $v_i \in \operatorname{Lin}[N']^{(k)}$ for all i if we put $N' \coloneqq \max\{n_1, \ldots, n_{l+1}, N\}$, so that

$$W_i \coloneqq \mathbb{C}[S_{N'}] \odot \{v_1, \ldots, v_i\} = \mathbb{C}[S_{N'}]\cdot v_1 + \cdots + \mathbb{C}[S_{N'}]\cdot v_i$$

is a $\mathbb{C}[S_{N'}]$-submodule of $\operatorname{Lin}[N']^{(k)}$. Clearly $W_{i-1} \subseteq W_i$ whenever $1 \leq i \leq l+1$, but by assumption v_i does not lie in $C[S_\infty] \odot \{v_1, \ldots, v_{i-1}\}$, which contains W_{i-1}. Therefore $v_i \in W_i \setminus W_{i-1}$, and we obtain a chain $\{0\} = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{l+1}$ of length $l+1$ in $\mathrm{Lin}\,[N']^{(k)}$. But this is impossible since $\mathrm{length}(\mathrm{Lin}\,[N']^{(k)}) \leq l$. □

Remark. For $k=1$, both the premise and the conclusion follow from Corollary 3.5: we have $\mathrm{length}(\mathrm{Lin}\,[n]) \leq 2$ for all $n \geq 2$, and indeed $\mathrm{length}(\mathrm{Lin}\,\mathbb{N}) \leq 2$. ■

Corollary 3.10. *Under the same hypothesis, any equivariant subspace $V \subseteq \mathrm{Lin}\,\mathbb{N}^{(k)}$ is generated by at most l elements. In particular, $V = \mathbb{C}[S_\infty] \odot W$ where W is an equivariant subspace of some $\mathrm{Lin}\,[n]^{(k)}$ with $n \geq N$.*

Proof. Suppose to the contrary that the $\mathbb{C}[\mathrm{Aut}\,\mathbb{N}]$-submodule $V \subseteq \mathrm{Lin}\,\mathbb{N}^{(k)}$ cannot be written as $\mathbb{C}[S_\infty] \odot \{v_1, \ldots, v_m\}$ whenever $m \leq l$. Then we can find $v_1, \ldots, v_{l+1}$ such that $v_{i+1} \in V \backslash \mathbb{C}[S_\infty] \odot \{v_1, \ldots, v_i\}$ for $0 \leq i \leq l$. Now, by putting $V_i \coloneqq \mathbb{C}[S_\infty] \odot \{v_1, \ldots v_i\}$, we see that $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{l+1}$ is a chain of length $l+1$ in $V \subseteq \mathrm{Lin}\,\mathbb{N}^{(k)}$. By Theorem 3.9 this cannot occur, so we must have $V = \mathbb{C}[S_\infty] \odot \{v_1, \ldots, v_m\}$ for some $v_1, \ldots, v_m \in V$ with $m \leq l$.

Choosing a large $n \geq N$, we can accommodate $v_1, \ldots, v_m$ all inside $\mathrm{Lin}\,[n]^{(k)}$. Then $W \coloneqq \mathbb{C}[S_n] \odot \{v_1, \ldots, v_m\} \subseteq \mathrm{Lin}\,[n]^{(k)}$ is an equivariant subspace, and it is immediate that $V = \mathbb{C}[S_\infty] \odot W$ as desired. □

It remains to understand the equivariant subspaces of $\mathrm{Lin}\,[n]^{(k)}$ for all large n, which seems to be an easier task. For one, any $\mathbb{C}[S_n]$-submodule $W \subseteq \mathrm{Lin}\,[n]^{(k)}$ is finite-dimensional and thus automatically finitely generated. We will make use of the following criteria to compare the resulting $\mathbb{C}[S_\infty]$-modules of the form $\mathbb{C}[S_\infty] \odot W$.

Lemma 3.11. *Let $X, Y \subseteq \mathrm{Lin}\,\mathbb{N}^{(k)}$ be finite. The following are equivalent:*

(*a*) $\mathbb{C}[S_\infty] \odot X \subseteq \mathbb{C}[S_\infty] \odot Y$*;*

(*b*) *there is n with* $\mathbb{C}[S_n] \odot X \subseteq \mathbb{C}[S_n] \odot Y$*;*

(*c*) *there is n with* $\mathbb{C}[S_{n+i}] \odot X \subseteq \mathbb{C}[S_{n+i}] \odot Y$ *for all* $i \geq 0$.

It follows that $\mathbb{C}[S_\infty] \odot X = \mathbb{C}[S_\infty] \odot Y$ iff $\mathbb{C}[S_n] \odot X = \mathbb{C}[S_n] \odot Y$ for some n.

Proof. It is routine to check that $S \odot (R \odot Z) = S \odot Z$ whenever $R \subseteq S$ is a subring, and that $S \odot Z_X \subseteq S \odot Z_Y$ whenever $Z_X \subseteq Z_Y$. But $\mathbb{C}[S_n] \subseteq \mathbb{C}[S_{n+i}] \subseteq \mathbb{C}[S_\infty]$ as rings, so $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ follow immediately.

Now assume (a). Given $x \in X$, we then have $x = \sum_{y \in Y} \phi_{x,y} \cdot y$ where each $\phi_{x,y} \in \mathbb{C}[S_\infty]$ is of the form $\sum_{\pi \in S_\infty} c_{x,y,\pi} \underline{\pi}$ with all but finitely many $c_{x,y,\pi} \in \mathbb{C}$ being zero. As X and Y are finite, so is $P \coloneqq \bigcup_{x \in X, y \in Y} \{\pi \in S_\infty \mid c_{x,y,\pi} \neq 0\}$. By taking n large enough, we have $P \subseteq S_n$ so that each $\phi_{x,y}$ lies in $\mathbb{C}[S_n]$. Therefore $X \subseteq \mathbb{C}[S_n] \odot Y$, which implies that $\mathbb{C}[S_n] \odot X \subseteq \mathbb{C}[S_n] \odot (\mathbb{C}[S_n] \odot Y) = \mathbb{C}[S_n] \odot Y$.

Using double inclusion, the final assertion follows straightforwardly. □

3.3 Lin $[n]^{(2)}$

To find all the equivariant subspaces of $\operatorname{Lin} \mathbb{N}^{(k)}$, by Corollary 3.10 we need only determine the equivariant subspaces of $\operatorname{Lin} [n]^{(k)}$ for all large n. We shall presently do so for $k = 2$; thus let $n \geq 2$ so that $[n]^{(2)} \neq \emptyset$.

Here, it is helpful to view a vector $\sum_{a \neq b} c_{(a,b)} \underline{(a,b)} \in \operatorname{Lin} \mathbb{N}^{(2)}$ as a weighted, finite, directed simple graph on the vertices $\{a \in \mathbb{N} \mid c_{(a,b)}, c_{(b,a)} \neq 0\}$: simply assign the weight $c_{(a,b)} \in \mathbb{C}$ to each edge (a, b). This helps explain the nomenclature of the linear maps $\mathrm{op} : \operatorname{Lin} \mathbb{N}^{(2)} \to \operatorname{Lin} \mathbb{N}^{(2)}$, $\mathrm{in} : \operatorname{Lin} \mathbb{N}^{(2)} \to \operatorname{Lin} \mathbb{N}$, and $\mathrm{out} : \operatorname{Lin} \mathbb{N}^{(2)} \to \operatorname{Lin} \mathbb{N}$ defined on the standard basis elements by

$$\begin{aligned} \mathrm{op} &: \underline{(a,b)} \mapsto \underline{(b,a)}, \\ \mathrm{in} &: \underline{(a,b)} \mapsto \underline{b}, \\ \mathrm{out} &: \underline{(a,b)} \mapsto \underline{a}. \end{aligned}$$

These are clearly equivariant, and moreover satisfy

$$\begin{aligned} \mathrm{op}\Big(\sum_{a \neq b} c_{(a,b)} \underline{(a,b)}\Big) &= \sum_{a \neq b} c_{(b,a)} \underline{(a,b)}, \\ \mathrm{in}\Big(\sum_{a \neq b} c_{(a,b)} \underline{(a,b)}\Big) &= \sum_{b} \Big(\sum_{a \neq b} c_{(a,b)}\Big) \underline{b}, \\ \mathrm{out}\Big(\sum_{a \neq b} c_{(a,b)} \underline{(a,b)}\Big) &= \sum_{a} \Big(\sum_{b \neq a} c_{(a,b)}\Big) \underline{a}. \end{aligned}$$

We shall leverage these maps (or more pedantically, their restrictions to $\operatorname{Lin} [n]^{(2)}$, which will be left implicit hereafter) to find equivariant subspaces in $\mathrm{Lin}\,[n]^{(2)}$.

Proposition 3.12. $\mathrm{Lin}\,[n]^{(2)} = \ker(\mathrm{id} + \mathrm{op}) \oplus \ker(\mathrm{id} - \mathrm{op})$, *where* $\ker(\mathrm{id} \pm \mathrm{op})$ *are equivariant subspaces with an* $\binom{n}{2}$*-element basis* $\{\underline{(i,j)} \mp \underline{(j,i)} \mid 1 \leq i < j \leq n\}$.

Remark. $\ker(\mathrm{id} \pm \mathrm{op})$ respectively consist of the graphs with antisymmetric and symmetric weight functions with respect to edge reversal. ■

Proof. Given any vector $v \in \mathrm{Lin}\,[n]^{(2)}$, we can write

$$v = \frac{1}{2}(v - \mathrm{op}(v)) + \frac{1}{2}(v + \mathrm{op}(v))$$

where $(\mathrm{id} \pm \mathrm{op})(v \mp \mathrm{op}(v)) = (v \mp \mathrm{op}(v)) \pm (\mathrm{op}(v) \mp v) = 0$ since $\mathrm{op} \circ \mathrm{op} = \mathrm{id}$. Thus $\mathrm{Lin}\,[n]^{(2)} = \ker(\mathrm{id} + \mathrm{op}) + \ker(\mathrm{id} - \mathrm{op})$, where the sum is furthermore direct as $v \in \ker(\mathrm{id} + \mathrm{op}) \cap \ker(\mathrm{id} - \mathrm{op})$ implies that $v = \frac{1}{2}0 + \frac{1}{2}0 = 0$ by the above equation.

Also, as id and op are both $\mathbb{C}[S_n]$-module homomorphisms, so is $\mathrm{id} + \mathrm{op}$; thus $\ker(\mathrm{id} + \mathrm{op}) \subseteq \mathrm{Lin}\,[n]^{(2)}$ is a $\mathbb{C}[S_n]$-submodule. Now consider

$$\mathfrak{B}_+ := \{\underline{(i,j)} - \underline{(j,i)} \mid 1 \leq i < j \leq n\}.$$

It is clear that $\underline{(i,j)} - \underline{(j,i)}$ lies in $\ker(\mathrm{id} + \mathrm{op})$ and is the only vector in $\mathfrak{B}_+$ that contributes to the coefficient of $\underline{(i,j)}$. So $\mathfrak{B}_+ \subseteq \ker(\mathrm{id} + \mathrm{op})$ is linearly independent, and we have $\dim(\ker(\mathrm{id} + \mathrm{op})) \geq |\mathfrak{B}_+| = \binom{n}{2}$. But, by defining $\mathfrak{B}_-$ analogously, we have $\dim(\ker(\mathrm{id} - \mathrm{op})) \geq |\mathfrak{B}_-| = \binom{n}{2}$ by the same argument. Since

$$n(n-1) = \dim(\mathrm{Lin}\,[n]^{(2)}) = \dim(\ker(\mathrm{id} + \mathrm{op})) + \dim(\ker(\mathrm{id} - \mathrm{op}))$$

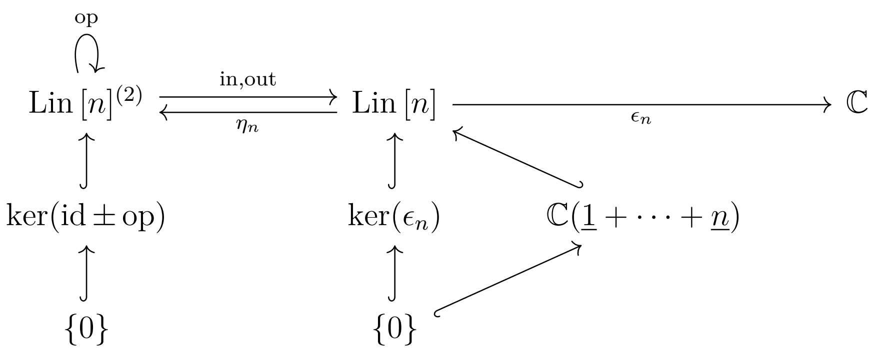
by the direct sum decomposition above, we see that $\dim(\ker(\mathrm{id} \pm \mathrm{op})) = \binom{n}{2} = |\mathfrak{B}_\pm|$ precisely. It follows that $\mathfrak{B}_\pm$ give a basis for $\ker(\mathrm{id} \pm \mathrm{op})$ respectively. □

3.3.1 $\mathbb{1}_n$ and V_n

We would like to decompose $\ker(\mathrm{id} \pm \mathrm{op})$ further. One idea is to reuse the earlier classification of equivariant subspaces in $\mathrm{Lin}\,[n]$ and see if any isomorphic copy appears in the kernels. We are thus led to consider $e_j := \sum_{i \neq j} \underline{(i,j)} \in \operatorname{Lin}[n]^{(2)}$ for $1 \leq j \leq n$ which satisfies $\pi \cdot e_j = e_{\pi \cdot j}$ for all $\pi \in S_n$, so that

$$\eta_n : \underline{i} \mapsto e_i$$

extends to an equivariant linear map $\eta_n : \operatorname{Lin}[n] \to \operatorname{Lin}[n]^{(2)}$. As $\mathbb{C}[S_n]$-module homomorphisms are closed under compositions and pointwise linear combinations, we have gathered quite a few $\mathbb{C}[S_n]$-modules and module homomorphisms by now; the (non-commutative) diagram below gives a summary of our current repertoire.



(3.13)

Proposition 3.14. *For $n \geq 3$, let*

$$\mathbb{1}_n := \eta_n(\mathbb{C}(\underline{1} + \cdots + \underline{n})) \tag{3.15}$$

and

$$V_n := \eta_n(\ker \epsilon_n). \tag{3.16}$$

Suppose that $\lambda, \mu \in \mathbb{C}$ are not both zero. Then $\mathbb{1}_n$ and $(\lambda \operatorname{id} + \mu \operatorname{op})(V_n)$

- *are both equivariant subspaces of* $\operatorname{Lin}[n]^{(2)}$*,*
- *satisfy* $\operatorname{length}(\mathbb{1}_n) = 1 = \operatorname{length}((\lambda \operatorname{id} + \mu \operatorname{op})(V_n))$*, and*
- *have bases $\{e_1 + \cdots + e_n\}$ and $\{(\lambda \operatorname{id} + \mu \operatorname{op})(e_1 - e_j) \mid 2 \leq j \leq n\}$ respectively.*

Proof. Let ϕ denote the restriction of $(\lambda \operatorname{id} + \mu \operatorname{op}) \circ \eta_n$ to $\ker(\epsilon_n)$. Then clearly $\phi : \ker(\epsilon_n) \to \operatorname{Lin}[n]^{(2)}$ is a $\mathbb{C}[S_n]$-module homomorphism, and we can easily check that its image $\phi(\ker \epsilon_n) = (\lambda \operatorname{id} + \mu \operatorname{op})(V_n)$ is a $\mathbb{C}[S_n]$-submodule of $\operatorname{Lin}[n]^{(2)}$. Now

$$\phi(\underline{1} - \underline{2}) = (\lambda \operatorname{id} + \mu \operatorname{op})(\sum_{i \neq 1} \underline{(i,1)} - \sum_{j \neq 2} \underline{(j,2)})$$

must be non-zero: observe that the coefficients of $\underline{(3,1)}$ and $\underline{(2,3)}$ are λ and $-\mu$ respectively. Thus $\ker(\phi)$ is an equivariant subspace of $\ker(\epsilon_n)$ but $\underline{1} - \underline{2} \notin \ker(\phi)$, so by Corollary 3.5 we must have $\ker(\phi) = \{0\}$. It follows that ϕ is a $\mathbb{C}[S_n]$-module isomorphism between $\ker(\epsilon_n)$ and $\phi(\ker \epsilon_n) = (\lambda \,\mathrm{id} + \mu \,\mathrm{op})(V_n)$; by Lemma 2.30

$$\mathrm{length}((\lambda \,\mathrm{id} + \mu \,\mathrm{op})(V_n)) = \mathrm{length}(\ker \epsilon_n) = 1.$$

Also, ϕ is *a fortiori* an injective linear map; as $\{\underline{1} - \underline{j} \mid 2 \leq j \leq n\}$ is a basis of $\ker(\epsilon_n)$, $\{\phi(\underline{1} - \underline{j}) = (\lambda \,\mathrm{id} + \mu \,\mathrm{op})(e_1 - e_j) \mid 2 \leq j \leq n\}$ forms a basis for $\phi(\ker \epsilon_n)$.

By applying the same argument to $\eta_n|_{\mathbb{C}(\underline{1}+\cdots+\underline{n})}$, we see that $\mathbb{1}_n = \mathbb{C}(e_1 + \cdots + e_n)$ is an equivariant subspace of length 1 as well. □

3.3.2 U_n

Observe that $\mathbb{1}_n$ and $(\mathrm{id} + \mathrm{op})(V_n)$ are both contained in $\ker(\mathrm{id} - \mathrm{op})$: indeed $\mathrm{op}(e_1 + \cdots + e_n) = \mathrm{op}(\sum_{i \neq j} \underline{(i,j)}) = \sum_{i \neq j} \underline{(j,i)} = e_1 + \cdots + e_n$ for the former, whereas $(\mathrm{id} - \mathrm{op}) \circ (\mathrm{id} + \mathrm{op}) = 0$ for the latter.

Proposition 3.17. *Let* $n \geq 4$. *We have* $\ker(\mathrm{id} - \mathrm{op}) = \mathbb{1}_n \oplus (\mathrm{id} + \mathrm{op})(V_n) \oplus U_n$, *where*

$$U_n := \mathbb{C} \odot \{(\mathrm{id} + \mathrm{op})\left(\begin{matrix} \underline{(a,b)} - \underline{(b,c)} \\ + \underline{(c,d)} - \underline{(d,a)} \end{matrix}\right) \mid (a,b,c,d) \in \mathrm{Lin}\,[n]^{(4)}\} \tag{3.18}$$

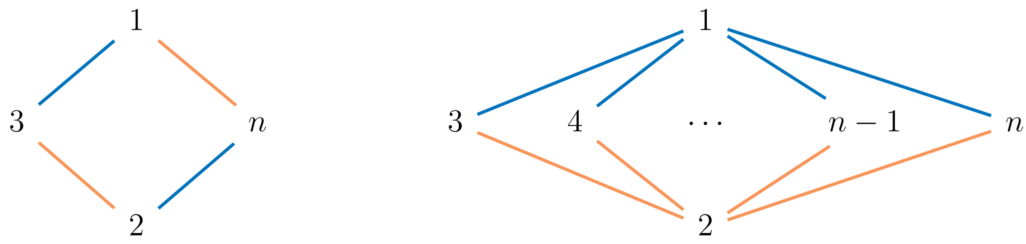
is a $\frac{1}{2}n(n-3)$*-dimensional equivariant subspace of* $\mathrm{Lin}\,[n]^{(2)}$.

Proof. Start by writing $u_{a,b,c,d}$ for $(\mathrm{id} + \mathrm{op})(\underline{(a,b)} - \underline{(b,c)} + \underline{(c,d)} - \underline{(d,a)})$, and consider

$$\mathfrak{U} := \{u_{1,2,c,d} \mid 3 \leq c < d \leq n\} \cup \{u_{1,3,2,d} \mid 4 \leq d \leq n\}$$

which consists of $\binom{n-2}{2} + (n-3) = \frac{1}{2}n(n-3)$ elements.

Recall that vectors in $\ker(\mathrm{id} - \mathrm{op})$ correspond to directed graphs with a symmetric weight function. We can thus visualise them as undirected weighted graphs; by way of illustration, $u_{1,3,2,n}$ and $(\mathrm{id} + \mathrm{op})(e_1 - e_2)$ are depicted below — a blue edge has a positive weight of +1, whilst an orange edge has a negative weight of −1.



Claim: *U_n is equivariant.*

Any vector $u \in U_n$ is a linear combination of the $u_{a,b,c,d}$'s. As $\mathrm{id}+\mathrm{op}$ is equivariant, for any $\pi \in S_n$ we have $\pi \cdot u_{a,b,c,d} = u_{\pi(a),\pi(b),\pi(c),\pi(d)} \in U_n$, so by linearity $\pi \cdot u$ lies in U_n too. ∎

Claim: *$U_n = \mathbb{C} \odot \mathfrak{U}$, so $\dim(U_n) \leq |\mathfrak{U}|$.*

We first show that $u_{a,b,c,d}$ lies in $\mathbb{C} \odot \mathfrak{U}$ whenever $\{a,b,c,d\} \cap \{1,2\} = \{1,2\}$. As $u_{a,b,c,d} = -u_{b,c,d,a}$, by cyclically permuting the atoms if necessary we may assume that $a = 1$; as $u_{1,b,c,d} = -u_{1,d,c,b}$, we may assume that $b < d$ and that $d \neq 2$. Now

- $u_{1,3,2,d} \in \mathfrak{U}$ for $4 \leq d \leq n$, and
- $u_{1,b,2,d} = -u_{1,3,2,b} + u_{1,3,2,d} \in \mathbb{C} \odot \mathfrak{U}$ for all $4 \leq b \neq d \leq n$;
- $u_{1,2,c,d} \in \mathfrak{U}$ for $3 \leq c < d \leq n$, and
- $u_{1,2,d,c} = u_{1,2,c,d} - u_{1,c,2,d} \in \mathbb{C} \odot \mathfrak{U}$.

It follows that $u_{1,b,c,d} = -u_{1,2,c,b} + u_{1,2,c,d} \in \mathbb{C} \odot \mathfrak{U}$ whenever $b,c,d \in [n] \setminus \{1,2\}$ are distinct, which covers the case $\{a,b,c,d\} \cap \{1,2\} = \{1\}$. The remaining case $\{a,b,c,d\} \cap \{1,2\} \subseteq \{2\}$ is now immediate: $u_{a,b,c,d} = -u_{1,b,a,d} + u_{1,b,c,d} \in \mathbb{C} \odot \mathfrak{U}$ whenever $a,b,c,d \in [n] \setminus \{1\}$ are distinct. This establishes the inclusion $\mathbb{C} \odot \mathfrak{U} \supseteq \mathbb{C} \odot \{u_{a,b,c,d} \mid (a,b,c,d) \in [n]^{(4)}\}$; the other inclusion is obvious. ∎

Claim: *$U_n + (\mathrm{id}+\mathrm{op})(V_n)$ is $(\binom{n}{2} - 1)$-dimensional.*

Recall from Proposition 3.14 that

$$\mathfrak{V} := \{(\mathrm{id}+\mathrm{op})(e_1 - e_j) \mid 2 \leq j \leq n\}$$

forms a basis for $(\mathrm{id}+\mathrm{op})(V_n)$, and thus $U_n+(\mathrm{id}+\mathrm{op})(V_n) = \mathbb{C}\odot\mathfrak{U}+\mathbb{C}\odot\mathfrak{V}$. Writing

$$\mathfrak{D} := \{(\mathrm{id}+\mathrm{op})(\underline{(1,2)} - \underline{(i,j)}) \mid 1 \leq i < j \leq n, (i,j) \neq (1,2)\},$$

we will first show that $\mathbb{C}\odot\mathfrak{U}+\mathbb{C}\odot\mathfrak{V} = \mathbb{C}\odot\mathfrak{D}$. For any $(a_1,a_2),(b_1,b_2) \in [n]^{(2)}$, note that

$$\begin{aligned}(\mathrm{id}+\mathrm{op})(\underline{(a_1,a_2)} - \underline{(b_1,b_2)}) = & -(\mathrm{id}+\mathrm{op})(\underline{(1,2)} - \underline{(\min_i a_i, \max_i a_i)}) \\ & + (\mathrm{id}+\mathrm{op})(\underline{(1,2)} - \underline{(\min_i b_i, \max_i b_i)}) \in \mathbb{C}\odot\mathfrak{V}.\end{aligned}$$

It is then clear that $\mathfrak{U} \subseteq \mathbb{C}\odot\mathfrak{D}$. Also, given $2 \leq j \leq n$ observe that

$$\begin{aligned}(\mathrm{id}+\mathrm{op})(e_1 - e_j) &= (\mathrm{id}+\mathrm{op})(\sum_{a\neq 1}\underline{(a,1)} - \sum_{b\neq j}\underline{(b,j)}) \\ &= \sum_{d\neq 1,j}(\mathrm{id}+\mathrm{op})(\underline{(d,1)} - \underline{(d,j)})\end{aligned}$$

as the $a = j$ term cancels with the $b = 1$ term. Thus $\mathfrak{V} \subseteq \mathbb{C}\odot\mathfrak{D}$ as well, which establishes the inclusion $\mathbb{C}\odot\mathfrak{U}+\mathbb{C}\odot\mathfrak{V} \subseteq \mathbb{C}\odot\mathfrak{D}$.

On the other hand, as $u_{1,3,2,d} = (\mathrm{id}+\mathrm{op})(\underline{(3,1)} - \underline{(3,2)} + \underline{(d,2)} - \underline{(d,1)})$, we see that $\mathbb{C}\odot\mathfrak{U}+\mathbb{C}\odot\mathfrak{V}$ contains

$$(\mathrm{id}+\mathrm{op})(e_1 - e_2) + \sum_{4\leq d\leq n} u_{1,3,2,d} = (n-2)(\mathrm{id}+\mathrm{op})(\underline{(3,1)} - \underline{(3,2)}) =: v.$$

But $U_n + V_n$ is equivariant since both U_n and V_n are equivariant, so

$$\begin{aligned}\frac{1}{n-2}(\underline{\mathrm{id}} + \underline{(1\ 3\ 2\ 4)}) \cdot v &= (\mathrm{id}+\mathrm{op})(\underline{(3,1)} - \underline{(3,2)} + \underline{(2,3)} - \underline{(2,4)}) \\ &= (\mathrm{id}+\mathrm{op})(\underline{(3,1)} - \underline{(2,4)}) =: v'\end{aligned}$$

also lies in $\mathbb{C}\odot\mathfrak{U}+\mathbb{C}\odot\mathfrak{V}$. Now each $(\mathrm{id}+\mathrm{op})(\underline{(1,2)} - \underline{(i,j)}) \in \mathfrak{D}$ is easily seen to be a permutation of v or v' depending on whether $\{i,j\}\cap\{1,2\}$ is an empty set or a singleton. By invoking equivariance again, we conclude that $\mathbb{C}\odot\mathfrak{U}+\mathbb{C}\odot\mathfrak{V} \supseteq \mathbb{C}\odot\mathfrak{D}$.

Finally, note that $\mathfrak{D}$ is linearly independent: only $(\mathrm{id}+\mathrm{op})(\underline{(1,2)} - \underline{(i,j)}) \in \mathfrak{D}$ contributes to the coefficient of $\underline{(i,j)}$ for $1 \leq i < j \leq n, (i,j) \neq (1,2)$. Therefore

$\mathbb{C} \odot \mathfrak{U} + \mathbb{C} \odot \mathfrak{V} = \mathbb{C} \odot \mathfrak{D}$ has dimension $\binom{n}{2} - 1$. ■

Claim: $\dim(U_n) = \frac{1}{2}n(n-3)$.

By above we have $\dim(\mathbb{C} \odot \mathfrak{U}) + \dim(\mathbb{C} \odot \mathfrak{V}) - \dim(\mathbb{C} \odot \mathfrak{U} \cap \mathbb{C} \odot \mathfrak{V}) = \dim(\mathbb{C} \odot \mathfrak{D})$, so $\dim(\mathbb{C} \odot \mathfrak{U}) \geq \dim(\mathbb{C} \odot \mathfrak{D}) - \dim(\mathbb{C} \odot \mathfrak{V}) = (\frac{1}{2}n(n-1) - 1) - (n-1) = \frac{1}{2}n(n-3)$ with equality iff $\mathbb{C} \odot \mathfrak{U} \cap \mathbb{C} \odot \mathfrak{V} = \{0\}$. But $\dim(\mathbb{C} \odot \mathfrak{U}) \leq |\mathfrak{U}| = \frac{1}{2}n(n-3)$; thus $U_n = \mathbb{C} \odot \mathfrak{U}$ has dimension $\frac{1}{2}n(n-3)$ precisely, and $\mathbb{C} \odot \mathfrak{D} = (\mathbb{C} \odot \mathfrak{U}) \oplus (\mathbb{C} \odot \mathfrak{V})$. ■

Claim: $(\mathbb{C} \odot \mathfrak{D}) \oplus \mathbb{1}_n = \ker(\mathrm{id} - \mathrm{op})$.

It is clear that $\mathbb{C} \odot \mathfrak{D} + \mathbb{1}_n \subseteq \ker(\mathrm{id} - \mathrm{op})$ is an equivariant subspace. Recall that $\mathbb{1}_n$ contains $e_1 + \cdots + e_n = \sum_{1 \leq i \neq j \leq n} \underline{(i,j)} = \sum_{1 \leq i < j \leq n} (\mathrm{id} + \mathrm{op})(\underline{(i,j)})$, so $\mathbb{C} \odot \mathfrak{D} + \mathbb{1}_n$ contains

$$(e_1 + \cdots + e_n) + \sum_{v \in \mathfrak{D}} v = \binom{n}{2} (\mathrm{id} + \mathrm{op})(\underline{(1,2)})$$

where $\binom{n}{2} \neq 0$ as we assumed that $n \geq 4$. By equivariance we see that $\mathbb{C} \odot \mathfrak{D} + \mathbb{1}_n$ contains $\{(\mathrm{id} + \mathrm{op})(\underline{(i,j)}) \mid 1 \leq i < j \leq n\}$, which by Proposition 3.12 forms a basis for $\ker(\mathrm{id} - \mathrm{op})$. Therefore $\mathbb{C} \odot \mathfrak{D} + \mathbb{1}_n = \ker(\mathrm{id} - \mathrm{op})$, and furthermore $\dim(\mathbb{C} \odot \mathfrak{D} \cap \mathbb{1}_n) = (\binom{n}{2} - 1) + 1 - \binom{n}{2} = 0$. ■

Thus $\ker(\mathrm{id} - \mathrm{op}) = (\mathbb{C} \odot \mathfrak{D}) \oplus \mathbb{1}_n = U_n \oplus (\mathrm{id} + \mathrm{op})(V_n) \oplus \mathbb{1}_n$, which completes the lengthy proof of Proposition 3.17. □

3.3.3 W_n

We tackle $\ker(\mathrm{id} + \mathrm{op})$ analogously.

Proposition 3.19. *Let $n \geq 3$. We have $\ker(\mathrm{id} + \mathrm{op}) = (\mathrm{id} - \mathrm{op})(V_n) \oplus W_n$, where*

$$W_n := \mathbb{C} \odot \{(\mathrm{id} - \mathrm{op})(\underline{(a,b)} + \underline{(b,c)} + \underline{(c,a)}) \mid (a,b,c) \in [n]^{(3)}\} \tag{3.20}$$

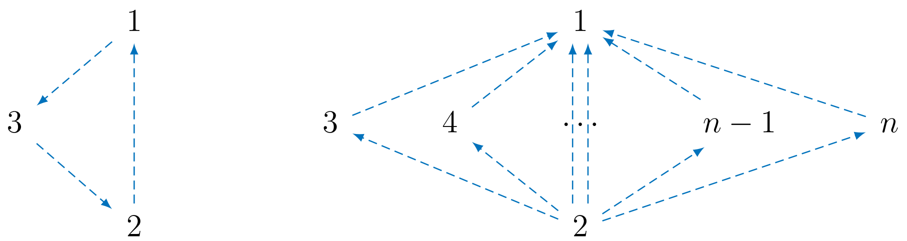
is an $\binom{n-1}{2}$-dimensional equivariant subspace of Lin $[n]^{(2)}, n \geq 3$.

Proof. Again start by writing $w_{a,b,c}$ for $(\mathrm{id}-\mathrm{op})(\underline{(a,b)}+\underline{(b,c)}+\underline{(c,a)}$, and consider

$$\mathfrak{W} := \{w_{1,i,j} \mid 2 \leq i < j \leq n\}$$

which consists of $\binom{n-1}{2}$ elements.

Recall that vectors in $\ker(\mathrm{id}+\mathrm{op})$ correspond to directed graphs with an anti-symmetric weight function; that is, if $\underline{(i,j)}$ has weight c, then $\underline{(j,i)}$ necessarily has weight $-c$. Thus we need only keep track of the edges $\underline{(i,j)}$ with a positive weight. For instance, $w_{1,3,2}$ and $(\mathrm{id}-\mathrm{op})(e_1-e_2)$ are depicted as follows — a dashed edge from i to j represents the vector $\underline{(i,j)}-\underline{(j,i)}$.

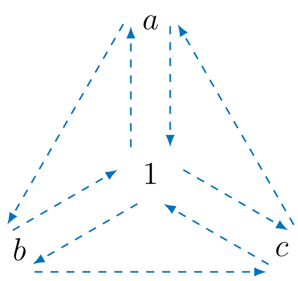


Claim: *W_n is equivariant.*

Any $w \in W_n$ is a linear combination of the $w_{a,b,c}$'s. For any $\pi \in S_n$ we have $\pi \cdot w_{a,b,c} = w_{\pi(a),\pi(b),\pi(c)} \in W_n$, so by linearity $\pi \cdot w$ lies in W_n too. ■

Claim: *$W_n = \mathbb{C} \odot \mathfrak{W}$, so $\dim(W_n) \leq |\mathfrak{W}|$.*

Clearly $\mathbb{C} \odot \mathfrak{W} \subseteq W_n$; it remains to show that $\mathbb{C} \odot \mathfrak{W}$ contains $w_{a,b,c}$ given any $(a,b,c) \in [n]^{(3)}$. Observe that $w_{a,b,c} = w_{1,a,b} + w_{1,b,c} + w_{1,c,a}$ if $a,b,c \in [n] \setminus \{1\}$, which is aptly demonstrated by summing the boundaries of a tetrahedron without its bottom face as illustrated below.



Now either $a < b$ and $w_{1,a,b}$ is in $\mathfrak{W}$ directly, or $a > b$ and $w_{1,a,b} = -w_{1,b,a} \in \mathbb{C} \odot \mathfrak{W}$; hence $w_{a,b,c} \in \mathbb{C} \odot \mathfrak{W}$. In the remaining case that $1 \in \{a, b, c\}$, we may assume that $a = 1$ since $w_{a,b,c} = w_{b,c,a} = w_{c,b,a}$; the argument right above then shows that $w_{1,b,c}$ lies in $\mathbb{C} \odot \mathfrak{W}$. ■

Claim: $\ker(\mathrm{id} + \mathrm{op}) = (\mathrm{id} - \mathrm{op})(V_n) + W_n$.

As both $(\mathrm{id} - \mathrm{op})(V_n)$ and W_n are equivariant subspaces of $\ker(\mathrm{id} + \mathrm{op})$, so is their sum $(\mathrm{id} - \mathrm{op})(V_n) + W_n$.

We now show the reverse inclusion $\ker(\mathrm{id} + \mathrm{op}) \subseteq (\mathrm{id} - \mathrm{op})(V_n) + W_n$. Recall that $(\mathrm{id} - \mathrm{op})(V_n)$ contains

$$\begin{aligned}(\mathrm{id} - \mathrm{op})(e_1 - e_2) &= (\mathrm{id} - \mathrm{op})\Big(\sum_{a\neq 1} \underline{(a,1)} - \sum_{b \neq 2} \underline{(b,2)}\Big) \\ &= (\mathrm{id} - \mathrm{op})\Big(2\underline{(2,1)} + \sum_{c\neq 1,2} \underline{(c,1)} + \underline{(2,c)}\Big).\end{aligned}$$

As $w_{2,1,c} = (\mathrm{id} - \mathrm{op})(\underline{(2,1)} - \underline{(c,1)} - \underline{(2,c)})$, we see that $(\mathrm{id} - \mathrm{op})(V_n) + W_n$ contains

$$(\mathrm{id} - \mathrm{op})(e_1 - e_2) + \sum_{3 \leq c \leq n} w_{2,1,c} = n(\mathrm{id} - \mathrm{op})(\underline{2,1}).$$

By equivariance $(\mathrm{id} - \mathrm{op})(V_n) + W_n$ contains $\{(\mathrm{id} - \mathrm{op})(\underline{i,j}) \mid 1 \leq i < j \leq n\}$, which by Proposition 3.12 forms a basis for $\ker(\mathrm{id} + \mathrm{op})$. Thus $\ker(\mathrm{id} + \mathrm{op}) = (\mathrm{id} - \mathrm{op})(V_n) + W_n$. ■

Finally, by passing to dimensions we obtain $\dim(W_n) \geq \binom{n}{2} - (n-1) = \binom{n-1}{2}$ with equality iff $(\mathrm{id} - \mathrm{op})(V_n) \cap W_n = \{0\}$. But we showed above that $\dim(W_n) \leq |\mathfrak{W}| = \binom{n-1}{2}$, so $\dim(W_n) = \binom{n-1}{2}$ precisely and it follows that $\ker(\mathrm{id} + \mathrm{op}) = (\mathrm{id} - \mathrm{op})(V_n) \oplus W_n$. □

We have, for any $n \geq 4$, arrived at

$$\begin{aligned}\mathrm{Lin}\,[n]^{(2)} &= \ker(\mathrm{id} - \mathrm{op}) \oplus \ker(\mathrm{id} + \mathrm{op}) \\ &= \mathbb{1}_n \oplus (\mathrm{id} + \mathrm{op})(V_n) \oplus U_n \oplus (\mathrm{id} - \mathrm{op})(V_n) \oplus W_n, \end{aligned} \tag{3.21}$$

which by Lemma 2.30 shows that $\text{length}(\text{Lin}\,[n]^{(2)}) = 3 + \text{length}(U_n) + \text{length}(W_n)$. We still need to further decompose U_n and W_n or, ideally, prove that no proper equivariant subspaces are lurking around. It should be evident by now that reasoning from first principles can become quite arduous. Instead, we will harness the full power of character theory and discover that the heavy lifting has already been done.

3.4 Character theory

$\mathbb{C}[S_n]$-modules enjoy some miraculous properties; we will enumerate a few of them following the exposition of [Ser77].

3.4.1 Some results

In Proposition 3.17, we painstakingly exhibited an equivariant complement U_n to $\mathbb{1}_n \oplus (\text{id} + \text{op})(V_n)$ in $\ker(\text{id} - \text{op})$. We will now see that, in fact, such a U_n is guaranteed to exist.

Lemma 3.22 (Maschke's Theorem). *Let $U \subseteq V$ be finite-dimensional $\mathbb{C}[S_n]$-modules. Then there exists another $\mathbb{C}[S_n]$-submodule $U' \subseteq N$ such that $U \oplus U' = V$.*

Proof. By picking a basis for U and extending it to a basis of V, we can find a subspace $W \subseteq V$ such that $U \oplus W = V$. This linear complement W will certainly not be closed under the S_n-actions in general. Nonetheless, we can employ an "averaging" trick: writing $\varphi : V = U \oplus W \to U$ to be the projection map sending $u + w \mapsto u$, we define a new map $\overline{\varphi} : V \to V$ by

$$\overline{\varphi}(v) \coloneqq \frac{1}{|S_n|} \sum_{\pi \in S_n} \pi \cdot \varphi(\pi^{-1} \cdot v).$$

It is easy to check that $\overline{\varphi}$ is linear; moreover, given any $\tau \in S_n$ we have $\overline{\varphi}(\tau \cdot v) = \frac{1}{|S_n|} \sum_{\pi \in S_n} \pi \cdot \varphi((\tau^{-1} \circ \pi)^{-1} \cdot v) = \tau \cdot \frac{1}{|S_n|} \sum_{\pi \in S_n} (\tau^{-1} \circ \pi) \cdot \varphi((\tau^{-1} \circ \pi)^{-1} \cdot v) = \tau \cdot \overline{\varphi}(v)$ since $\pi \mapsto \tau^{-1} \circ \pi$ gives a bijection of S_n, showing that $\overline{\varphi}$ is equivariant. In particular, $\ker(\overline{\varphi}) \subseteq V$ is an equivariant subspace.

Now, for any $u \in U$, we have $\overline{\varphi}(u) = \frac{1}{|S_n|} \sum_{\pi \in S_n} \pi \cdot (\pi^{-1} \cdot u) = u$ because each $\pi^{-1} \cdot u$ still lies in U. Therefore $\overline{\varphi}(U) = U$, and by the Rank-Nullity Theorem we get $\dim(\ker \overline{\varphi}) + \dim(U) = \dim(V)$. But $u \in \ker(\overline{\varphi}) \cap U$ implies that $0 = \overline{\varphi}(u) = u$, so by considering the dimensions we conclude that $V = \ker(\overline{\varphi}) \oplus U$. Thus $\ker(\overline{\varphi})$ is the desired equivariant complement to U in V. □

Remark. Despite the constructive nature of the proof, the calculations are only practical for small, individual values of n. Also, the theory does not apply to the $\mathbb{C}[S_\infty]$-module $\operatorname{Lin} \mathbb{N}$: $\ker(\epsilon) \subseteq \operatorname{Lin} \mathbb{N}$ does not admit an equivariant complement, since $\ker(\epsilon)$ itself is the only proper equivariant subspace of $\operatorname{Lin} \mathbb{N}$ by Corollary 3.5. ■

The existence alone of equivariant complements carries important theoretical consequences concerning the structure of finite-dimensional $\mathbb{C}[S_n]$-modules.

Definition 3.23. An R-module M is said to be *simple* if $\operatorname{length}(M) = 1$. Equivalently, M is simple if it is non-zero and contains no proper R-submodules.

Corollary 3.24. *Let V be a finite-dimensional $\mathbb{C}[S_n]$-module. Then there exist simple $\mathbb{C}[S_n]$-submodules $U_1, \ldots, U_n \subseteq V$ such that $\bigoplus_i U_i = V$.*

Proof. We proceed by induction on $d = \dim(V)$. The base cases are straightforward: when $d = 0$, $V = \{0\}$ is the empty direct sum; when $d = 1$, V must be simple as $\{0\} \subsetneq V$ is the only chain for dimension reasons.

Now suppose that $d > 1$. Let $U \subseteq V$ be a non-zero $\mathbb{C}[S_n]$-submodule of the least dimension; note that U must be simple. If $U = V$, then the decomposition is trivial. Otherwise, by Maschke's Theorem, we can write $V = U \oplus U'$ for some $\mathbb{C}[S_n]$-submodule $U' \subseteq V$ where $\dim(U') = \dim(V) - \dim(U) < \dim(V)$. By the inductive hypothesis, we obtain $U' = \bigoplus_j U'_j$ where each U'_j is simple. Thus $V = U \oplus (\bigoplus_j U'_j)$, which completes the induction. □

As we will see, the existence of a decomposition into simples is but one of the algebraic miracles. But first, more theory.

Definition 3.25. Let V be a finite-dimensional $\mathbb{C}[S_n]$-module with $\mathfrak{B} = \{b_1, \dots, b_d\}$ as a basis. Given a linear map $\varphi : V \to V$, we write ${}_{\mathfrak{B}}(\varphi)_{\mathfrak{B}}$ for the d-by-d matrix whose (i, j) entry is the coefficient $c_{i,j}$ of b_i in $\varphi(b_j)$. Then the *character* $\chi_V : S_n \to \mathbb{C}$ of V is the map sending $\pi \in S_n$ to the trace $\mathrm{tr}({}_{\mathfrak{B}}(\pi \cdot -)_{\mathfrak{B}}) = \sum_{i=1}^{d} c_{i,i}$.

Remark. χ_V does not depend on the basis $\mathfrak{B}$ chosen: if $\mathfrak{B}'$ is another basis of V, then ${}_{\mathfrak{B}'}(\pi \cdot -)_{\mathfrak{B}'} = {}_{\mathfrak{B}'}(\mathrm{id})_{\mathfrak{B}}\ {}_{\mathfrak{B}}(\pi \cdot -)_{\mathfrak{B}}\ {}_{\mathfrak{B}}(\mathrm{id})_{\mathfrak{B}'}$; since $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$ and ${}_{\mathfrak{B}}(\mathrm{id})_{\mathfrak{B}'}\ {}_{\mathfrak{B}'}(\mathrm{id})_{\mathfrak{B}}$ is the identity matrix, we see that $\mathrm{tr}({}_{\mathfrak{B}'}(\pi \cdot -)_{\mathfrak{B}'}) = \mathrm{tr}({}_{\mathfrak{B}}(\pi \cdot -)_{\mathfrak{B}})$.

The same argument also shows that isomorphic $\mathbb{C}[S_n]$-modules share the same characters. Indeed if $\varphi : V' \to V$ is an equivariant linear bijection and $\mathfrak{B}, \mathfrak{B}'$ are any bases for V, V', then we have ${}_{\mathfrak{B}'}(\pi \cdot -)_{\mathfrak{B}'} = {}_{\mathfrak{B}'}(\varphi)_{\mathfrak{B}}\ {}_{\mathfrak{B}}(\pi \cdot -)_{\mathfrak{B}}\ {}_{\mathfrak{B}}(\varphi^{-1})_{\mathfrak{B}'}$. ■

At first glance, we lose a lot of information by reducing each linear map $\pi \cdot -$ to a mere scalar $\chi_V(\pi) \in \mathbb{C}$. As it turns out, this is not the case at all.

Lemma 3.26. *Given $\mathbb{C}[S_n]$-modules V and V', consider*

$$\langle \chi_V, \chi_{V'} \rangle \coloneqq \frac{1}{|S_n|} \sum_{\pi \in S_n} \overline{\chi_V(\pi)} \chi_V(\pi)$$

where $\overline{c}$ denotes the complex conjugate of $c \in \mathbb{C}$. We have the following.

(*a*) *If U and U' are simple, then*

$$\langle \chi_U, \chi_{U'} \rangle = \begin{cases} 0 & \text{if } U \not\simeq U', \\ 1 & \text{if } U \simeq U'. \end{cases}$$

Moreover

$$\frac{\dim U}{|S_n|} \sum_{\pi \in S_n} \overline{\chi_U(\pi)} \pi \cdot u' = \begin{cases} 0 & \text{if } U \not\simeq U', \\ u' & \text{if } U \simeq U' \end{cases}$$

for all $u' \in U'$.

(*b*) $\chi_{V \oplus V'} = \chi_V + \chi_{V'}$.

(*c*) $\chi_V = \chi_{V'}$ *iff* $V \simeq V'$.

Remark. The proof for (a) is standard but slightly lengthy; (b) is easy, and (c) then follows quite straightforwardly by applying Corollary 3.24 of Maschke's Theorem. We refer avid readers to [Ser77, §2]. ■

3.4.2 Decomposing $\mathrm{Lin}\,[n]^{(2)}$

Let us return to $\mathrm{Lin}\,[n]^{(2)}$. Since $[n]^2 = \{(i,i) \mid i \in [n]\} \sqcup [n]^{(2)}$ as S_n-sets, we have $\mathrm{Lin}\,[n]^2 \simeq \mathrm{Lin}\,[n] \oplus \mathrm{Lin}\,[n]^{(2)}$ as $\mathbb{C}[S_n]$-modules, so $\chi_{\mathrm{Lin}\,[n]^{(2)}} = \chi_{\mathrm{Lin}\,[n]^2} - \chi_{\mathrm{Lin}\,[n]}$ as functions $S_n \to \mathbb{C}$. Now fix $\pi \in S_n$. Observe that, given an S_n-set X, the matrix $\underline{X}(\pi \cdot -)_{\underline{X}}$ has an entry of 1 at $(\underline{\pi \cdot x}, \underline{x})$ for each $x \in X$ and zero everywhere else; writing $\mathrm{Fix}_\pi(X) \coloneqq \{x \in X \mid \pi \cdot x = x\}$, we then see that the trace $\chi_{\mathrm{Lin}\,X}(\pi) = |\mathrm{Fix}_\pi(X)|$ is a non-negative integer. In particular, as $\pi \cdot (a_1, \ldots, a_k) = (a_1, \ldots, a_k)$ iff each $a_i \in \mathrm{Fix}_\pi([n])$, we have $\mathrm{Fix}_\pi([n]^k) = \mathrm{Fix}_\pi([n])^k$. Therefore $\chi_{\mathrm{Lin}\,[n]^{(2)}}(\pi) = |\mathrm{Fix}_\pi([n]^2)| - |\mathrm{Fix}_\pi([n])| = s^2 - s$ with $s \coloneqq |\mathrm{Fix}_\pi([n])|$, and

$$\begin{aligned}\langle \chi_{\mathrm{Lin}\,[n]^{(2)}}, \chi_{\mathrm{Lin}\,[n]^{(2)}} \rangle &= \frac{1}{|S_n|} \sum_{\pi \in S_n} (s^2 - s)^2 \\ &= \frac{1}{|S_n|} \sum_{\pi \in S_n} \left|\mathrm{Fix}_\pi([n]^4)\right| - 2\left|\mathrm{Fix}_\pi([n]^3)\right| + \left|\mathrm{Fix}_\pi([n]^2)\right|.\end{aligned}$$

But $\frac{1}{|S_n|} \sum_{\pi \in S_n} |\mathrm{Fix}_\pi(X)|$ counts the number of orbits in X by Burnside's Lemma. Using (2.8), we compute that $[n]^4$, $[n]^3$, and $[n]^2$ respectively have $1 + 7 + 6 + 1 = 15$, $1 + 3 + 1 = 5$, and $1 + 1 = 2$ orbits provided that $n \geq 4$. It follows that $\langle \mathrm{Lin}\,[n]^{(2)}, \mathrm{Lin}\,[n]^{(2)} \rangle = 15 - 2 \cdot 5 + 2 = 7$.

On the other hand, by Maschke's Theorem we may decompose U_n and W_n into direct sums of simple $\mathbb{C}[S_n]$-modules. After grouping the simple modules by their isomorphism types, from (3.21) we obtain

$$\mathrm{Lin}\,[n]^{(2)} = \bigoplus_i \bigoplus_{j=1}^{m_i} U_{i,j};$$

more elaborately, we require the simple modules $U_{i,j}$ to satisfy $U_{i,j} \simeq U_{i,j'}$ for all i as well as $U_{i,j} \not\simeq U_{i',j'}$ whenever $i \neq i'$. We may assume that $U_{1,1} = \mathbb{1}_n$ and $U_{2,1} = (\mathrm{id} + \mathrm{op})(V_n), U_{2,2} = (\mathrm{id} - \mathrm{op})(V_n)$: these are simple by Proposition 3.14

with $(\mathrm{id}+\mathrm{op})(V_n) \simeq \ker(\epsilon_n) \simeq (\mathrm{id}-\mathrm{op})(V_n)$; additionally, $\mathbb{1}_n \not\simeq \ker(\epsilon_n)$ because $\dim(\mathbb{1}_n) = 1 \neq n-1 = \dim(\ker \epsilon_n)$ — a $\mathbb{C}[S_n]$-module isomorphism is *a fortiori* a linear bijection and thus must preserve dimensions. Hence $m_1 \geq 1$ and $m_2 \geq 2$. Now $\chi_{\mathrm{Lin}\,[n]^{(2)}} = \sum_i \sum_{j=1}^{m_i} \chi_{U_{i,j}} = \sum_i m_i \chi_{U_{i,1}}$, so we obtain

$$\langle \chi_{\mathrm{Lin}\,[n]^{(2)}}, \chi_{\mathrm{Lin}\,[n]^{(2)}} \rangle = \sum_i \sum_{i'} m_i m_{i'} \langle \chi_{U_{i,1}}, \chi_{U_{i',1}} \rangle = \sum_i {m_i}^2 \langle \chi_{U_{i,1}}, \chi_{U_{i,1}} \rangle = \sum_i {m_i}^2$$

by applying both cases of Lemma 3.26(a) consecutively. But $7 = 1^2 + \cdots + 1^2$ and $7 = 2^2 + 1^2 + 1^2 + 1^2$ are the only ways to write 7 as a sum of squares, so by our assumption on m_1, m_2 we are forced to conclude that $m_2 = 2$ and $m_1, m_3, m_4 = 1$. As each $U_{i,j}$ has length 1, it follows that

$$\mathrm{length}(\mathrm{Lin}\,[n]^2) = m_1 + m_2 + m_3 + m_4 = 5 \tag{3.27}$$

for all $n \geq 4$. Moreover, recall that by construction the unidentified subspaces $U_{3,1}, U_{4,1}$ are disjointly the simple constituents of U_n and of W_n. As $U_n, W_n \neq 0$, without loss of generality we must have $U_{3,1} = U_n, U_{4,1} = W_n$. In particular U_n, W_n are already simple, so (3.21) is already a decomposition into simple constituents.

3.4.3 Uniqueness of the decomposition

Remarkably, the $\mathbb{C}[S_n]$-submodules of $\mathrm{Lin}\,[n]^{(2)}$ are almost uniquely determined by the decomposition (3.21). To this end, suppose $M \subseteq \mathrm{Lin}\,[n]^{(2)}$ is an equivariant subspace. By Maschke's Theorem, there exists another equivariant subspace M' with $M \oplus M' = \mathrm{Lin}\,[n]^{(2)}$, and we can make the further decomposition $M \oplus M' = (S_1 \oplus \cdots \oplus S_m) \oplus (S_{m+1} \oplus \cdots \oplus S_l)$ into simple constituents; note that $0 \leq m \leq l = 5$ for length reasons. Firstly, each S_i must be isomorphic to one of $\mathbb{1}_n$, V_n, U_n, or W_n: otherwise by letting $\sum_{\pi \in S_n} \overline{\chi_{S_i} \pi}$ act on $\bigoplus_i S_i = \mathbb{1}_n \oplus ((\mathrm{id} \pm \mathrm{op})(V_n)) \oplus U_n \oplus W_n$ we obtain $S_i = \{0\}$ by Lemma 3.26(a), so $\mathrm{length}(S_i) = 0 \neq 1$ which contradicts the assumption that S_i is simple. Using the same technique with $\mathbb{1}_n$, U_n, W_n, and V_n in place of S_i, we see that

- $\mathbb{1}_n = S_{i_1}$, $U_n = S_{i_2}$, and $W_n = S_{i_3}$ where i_1, i_2, i_3 are distinct;
- $V_n \simeq S_{i_4}, S_{i_5}$ is the remaining isomorphism type, and
- $(\mathrm{id}+\mathrm{op})(V_n) \oplus (\mathrm{id}-\mathrm{op})(V_n) = S_{i_4} \oplus S_{i_5}$.

We now determine S_{i_4}.

Lemma 3.28. $(\mathrm{id}+\mathrm{op})(V_n) \oplus (\mathrm{id}-\mathrm{op})(V_n) = (\lambda\,\mathrm{id} + \mu\,\mathrm{op})(V_n) + (\lambda'\,\mathrm{id} + \mu'\,\mathrm{op})$ *iff* $(\lambda, \mu), (\lambda', \mu') \in \mathbb{C}^2$ *are linearly independent. Moreover, if either condition holds, then* $(\lambda\,\mathrm{id} + \mu\,\mathrm{op})(V_n) \cap (\lambda'\,\mathrm{id} + \mu'\,\mathrm{op})(V_n) = \{0\}$.

Proof. Suppose $(\lambda, \mu), (\lambda', \mu')$ are linearly dependent. Then $c(\lambda, \mu) + d(\lambda', \mu') = (0, 0)$ for some $c, d \in \mathbb{C}$ not both zero; without loss of generality, say $c \neq 0$. Now $(\lambda', \mu') = \frac{d}{c}(\lambda, \mu)$, so $(\lambda'\,\mathrm{id} + \mu'\,\mathrm{op})(V_n) = \frac{d}{c}(\lambda\,\mathrm{id} + \mu\,\mathrm{op})(V_n) \subseteq (\lambda\,\mathrm{id} + \mu\,\mathrm{op})(V_n)$. Hence $(\lambda\,\mathrm{id} + \mu\,\mathrm{op})(V_n) + (\lambda'\,\mathrm{id} + \mu'\,\mathrm{op})(V_n) = (\lambda\,\mathrm{id} + \mu\,\mathrm{op})(V_n)$ is either $\{0\}$ in the case that $\lambda = 0 = \mu$, or has dimension $n-1$ by Proposition 3.14; however, $(\mathrm{id}+\mathrm{op})(V_n) \oplus (\mathrm{id}-\mathrm{op})(V_n)$ has dimension $2(n-1)$.

Suppose now that $(\lambda, \mu), (\lambda', \mu')$ are linearly independent. Then λ, μ cannot be both zero, and neither can λ', μ'; also

$$\mathbb{C} \odot \{(\lambda, \mu), (\lambda', \mu')\} = \mathbb{C}^2 = \mathbb{C} \odot \{(1, 1), (1, -1)\}.$$

But by Proposition 3.14, $\{(\lambda\,\mathrm{id}+\mu\,\mathrm{op})(e_1-e_j), (\lambda'\,\mathrm{id}+\mu'\,\mathrm{op})(e_1-e_j) \mid 2 \leq j \leq n\}$ and $\{(\mathrm{id}+\mathrm{op})(e_1 - e_j), (\mathrm{id}-\mathrm{op})(e_1 - e_j) \mid 2 \leq j \leq n\}$ respectively form bases for both sides of the purported equation; by above, these two span the same vector space. Moreover $(\lambda\,\mathrm{id}+\mu\,\mathrm{op})(V_n) \cap (\lambda'\,\mathrm{id}+\mu'\,\mathrm{op})$ has dimension $2(n+1)-(n+1)-(n+1)$, so the sum is direct. $\square$

Thus $S_{i_4} \subseteq (\mathrm{id}+\mathrm{op})(V_n) \oplus (\mathrm{id}-\mathrm{op})(V_n) = V_n \oplus \mathrm{op}(V_n)$, giving us projection maps

$$\varphi : S_{i_4} \to V_n, \quad \varphi' : S_{i_4} \to \mathrm{op}(V_n)$$

that satisfy $\mathrm{id}_{S_{i_4}} = \varphi + \varphi'$. If $\varphi = 0$, then $\{0\} \subsetneq S_{i_4} \subseteq \mathrm{op}(V_n)$ which implies that $S_{i_4} = \mathrm{op}(V_n)$ by simplicity; analogously, if $\varphi' = 0$ then $S_{i_4} = V_n$. The only remaining case is when $\varphi, \varphi' \neq 0$.

As $V_n \simeq S_{i_4}$, pick an isomorphism $\psi : V_n \to S_{i_4}$; we then get a $\mathbb{C}[S_n]$-module endomorphism

$$\varphi \circ \psi : V_n \to V_n$$

which is certainly a linear endomorphism of a finite-dimensional vector space. Because we work over $\mathbb{C}$, its characteristic polynomial must admit a root $\lambda \in \mathbb{C}$. We can thus find an eigenvector $v \neq 0$ associated with λ which satisfies $(\varphi \circ \psi - \lambda \operatorname{id})(v) = 0$, so $\{0\} \subsetneq \ker(\varphi \circ \psi - \lambda \operatorname{id}) \subseteq V_n$. Since $\varphi \circ \psi - \lambda \operatorname{id}$ is moreover a $\mathbb{C}[S_n]$-module homomorphism, by simplicity we must have $\ker(\varphi \circ \psi - \lambda \operatorname{id}) = V_n$. In other words, the identity $\varphi \circ \psi = \lambda \operatorname{id}$ holds on V_n, which shows that $\varphi = \lambda \psi^{-1}$ as ψ is invertible; note also that $\lambda \neq 0$ since we assumed that $\varphi \neq 0$. This simple but powerful result is known as Schur's Lemma.

But we can apply the same Lemma to $\operatorname{op} \circ \varphi' \circ \psi$ and obtain $\operatorname{op} \circ \varphi' = \mu \psi^{-1}$ for some $\mu \neq 0$. It follows that $\mu\varphi = \lambda\mu\psi^{-1} = \lambda(\operatorname{op} \circ \varphi')$ and that

$$\mu \operatorname{id}_{S_{i_4}} = \mu\varphi + \mu\varphi' = (\lambda \operatorname{op} + \mu \operatorname{id}_{\operatorname{op}(V_n)}) \circ \varphi',$$

which shows that

$$\{0\} \subsetneq S_{i_4} = (\mu \operatorname{id}_{S_{i_4}})(S_{i_4}) \subseteq (\lambda \operatorname{op} + \mu \operatorname{id})(\operatorname{op}(V_n)) = (\lambda \operatorname{id} + \mu \operatorname{op})(V_n).$$

But this last is simple by Proposition 3.14; hence $S_{i_4} = (\lambda \operatorname{id} + \mu \operatorname{op})(V_n)$ precisely. We also see that the cases $\varphi = 0$ and $\varphi' = 0$ can be subsumed by putting $(\lambda, \mu) = (0, 1), (1, 0)$ respectively. Of course, the same applies to S_{i_5}, giving $S_{i_5} = (\lambda' + \mu')(V_n)$ where λ', μ' are not both zero. As $S_{i_4} \oplus S_{i_5} = (\operatorname{id} + \operatorname{op})(V_n) \oplus (\operatorname{id} - \operatorname{op})(V_n)$, by Lemma 3.28 we see that (λ, μ) and (λ', μ') must be linearly independent.

Finally, recall that $M \subseteq \operatorname{Lin}[n]^{(2)}$ is the direct sum of a subset of $\{S_{i_1}, \ldots, S_{i_5}\}$. As we have determined what each S_{i_k} must look like — to recapitulate,

- $(S_{i_1}, S_{i_2}, S_{i_3}) = (\mathbb{1}_n, U_n, W_n)$, whilst
- $(S_{i_4}, S_{i_5}) = ((\lambda \operatorname{id} + \mu \operatorname{op})(V_n), (\lambda' \operatorname{id} + \mu' \operatorname{op})(V_n))$

where $(\lambda, \mu), (\lambda', \mu') \in \mathbb{C}^2 \setminus \{(0,0)\}$ is linearly independent — we also know what M must look like. Additionally, if we have $S_{i_4} \oplus S_{i_5} \subseteq M$, then by Lemma 3.28 we might as well set $(\lambda, \mu) = (1,1)$ and $(\lambda', \mu') = (1,-1)$. Conversely, the direct sum M of any subset of such $\{S_{i_1}, \ldots, S_{i_5}\}$ is certainly an equivariant subspace of $\mathrm{Lin}\,[n]^{(2)}$. We summarise our findings as follows.

Theorem 3.29. *For $n \geq 4$, the equivariant subspaces of $\mathrm{Lin}\,[n]^{(2)}$ are in bijection with*

$$\mathfrak{X}_n \coloneqq \left\{ \mathfrak{a} \cup \mathfrak{b} \;\middle|\; \begin{array}{r} \mathfrak{a} \subseteq \{t_n, u, w\}, \mathfrak{b} \in \big\{\emptyset\big\} \cup \big\{(\lambda\,\mathrm{id} + \mu\,\mathrm{op})(v_n) \mid [\lambda : \mu] \in \mathbb{C}P^1\big\} \\ \cup \big\{\{(\mathrm{id} + \mathrm{op})(v_n), (\mathrm{id} - \mathrm{op})(v_n)\}\big\} \end{array} \right\}$$

where

- $t_n \coloneqq e_1 + \cdots + e_n$ *(where recall that $e_j = \sum_{i \in [n] \setminus \{j\}} \underline{(i,j)}$ depends on n),*
- $u \coloneqq (\mathrm{id} + \mathrm{op})(\underline{(1,2)} - \underline{(2,3)} + \underline{(3,4)} - \underline{(4,1)})$,
- $w \coloneqq (\mathrm{id} - \mathrm{op})(\underline{(1,2)} + \underline{(2,3)} + \underline{(3,1)})$,
- $v_n \coloneqq e_1 - e_2$

and $\mathbb{C}P^1$ denotes the complex projective line (or equivalently, the Riemann sphere. More elaborately $\mathbb{C}P^1$ is the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ under $\sim$, where $(\lambda, \mu) \sim (c\lambda, c\mu)$ whenever $c \in \mathbb{C}$ is non-zero; customarily we write $[\lambda : \mu]$ for the equivalence class of (λ, μ). See, e.g., [Gol99, §1.1.1 and §1.3.2] for more details.)

Proof. The correspondence is just

$$\Psi : X \subseteq \mathfrak{X}_n \mapsto \mathbb{C}[S_n] \odot X = \sum_{x \in X} \mathbb{C}[S_n] \cdot x \subseteq \mathrm{Lin}\,[n]^{(2)}.$$

Indeed, notice how t_n, u, w, and $(\lambda\,\mathrm{id} + \mu\,\mathrm{op})(v_n)$ respectively generate $\mathbb{1}_n$, U_n, W_n, and $(\lambda\,\mathrm{id} + \mu\,\mathrm{op})(V_n)$. Note also that Ψ is well-defined: given $c \in \mathbb{C} \setminus \{0\}$, the identity $[\lambda : \mu] = [c\lambda : c\mu]$ in $\mathbb{C}P^1$ is respected by Ψ as $\mathbb{C}[S_n] \cdot (\lambda\,\mathrm{id} + \mu\,\mathrm{op})(v_n) = \mathbb{C}[S_n] \cdot (c\lambda\,\mathrm{id} + c\mu\,\mathrm{op})(v_n)$. Given a $\mathbb{C}[S_n]$-submodule $M \subseteq \mathrm{Lin}\,[n]^{(2)}$, by decomposing it into simple constituents we can thus write $M = \Psi(X)$ for some $X \subseteq \mathfrak{X}_n$ as we have discussed at length; hence Ψ is surjective.

As for injectivity, we see that characters allow us to easily distinguish between the $\Psi(\mathfrak{a} \cup \mathfrak{b})$'s that arise from distinct choices of $\mathfrak{a} \subseteq \{t_n, u, w\}$ and distinct cardinalities of $\mathfrak{b} \in \left\{\emptyset\right\} \cup \left\{(\lambda \,\mathrm{id} + \mu \,\mathrm{op})(v_n) \mid [\lambda : \mu] \in \mathbb{C}P^1\right\} \cup \left\{\{(\mathrm{id} + \mathrm{op})(v_n), (\mathrm{id} - \mathrm{op})(v_n)\}\right\}$. Hence it is enough to show that $\Psi(\mathfrak{b}) \neq \Psi(\mathfrak{b}')$ whenever $\mathfrak{b} \neq \mathfrak{b}'$ are both singletons. But $[\lambda : \mu] \neq [\lambda' : \mu']$ says precisely that any representatives (λ, μ) and (λ', μ') are not collinear and are thus linearly independent. It follows by Lemma 3.28 that $\mathbb{C}[S_n] \cdot (\lambda \,\mathrm{id} + \mu \,\mathrm{op})(v_n)$ and $\mathbb{C}[S_n] \cdot (\lambda' \,\mathrm{id} + \mu' \,\mathrm{op})(v_n)$ intersect trivially, so these two cannot be equal. □

3.5 $\mathrm{Lin}\,\mathbb{N}^{(2)}$

Fulfilling the vision of understanding equivariant subspaces of $\mathrm{Lin}(\mathbb{N}^{(k)})$ through the equivariant subspaces of $\mathrm{Lin}([n]^{(k)})$, we immediately obtain $\mathrm{length}(\mathrm{Lin}\,\mathbb{N}^{(2)}) \leq 5$ by combining (3.27) and Theorem 3.9. But the Structure Theorem 3.29 for $\mathrm{Lin}\,[n]^{(2)}$ together with Corollary 3.10 empowers us to discern much more about the structure of $\mathrm{Lin}\,\mathbb{N}^{(2)}$.

Corollary 3.30. *Any equivariant subspace in* $\mathrm{Lin}\,\mathbb{N}^{(2)}$ *is of the form* $\mathbb{C}[S_\infty] \odot X$*, where* X *is a subset of* $\mathfrak{X}_n$ *for some* $n \geq 4$.

That is, the $\mathbb{C}[S_\infty] \odot X$'s constitute a complete yet possibly highly redundant collection of the equivariant subspaces in $\mathrm{Lin}\,\mathbb{N}^{(2)}$. To unveil the lattice of such equivariant subspaces ordered by containment, given $X, X' \in \bigcup_{n \geq 4} \mathcal{P}(\mathfrak{X}_n)$ we need to compare $\mathbb{C}[S_\infty] \odot X$ and $\mathbb{C}[S_\infty] \odot X'$ for inclusion and equality. Conveniently, in view of Lemma 3.11, it suffices to compare $\mathbb{C}[S_{n'}] \odot X$ with $\mathbb{C}[S_{n'}] \odot X'$ for all sufficiently large n'. Also, as $\mathbb{C}[S_{n'}] \odot X = \sum_{x \in X} \mathbb{C}[S_{n'}] \cdot x$, we need only compute $\mathbb{C}[S_{n'}] \cdot x$ for $x \in \{(\lambda \,\mathrm{id} + \mu \,\mathrm{op})(v_n) \mid [\lambda : \mu] \in \mathbb{C}P^1, n \geq 4\} \cup \{t_n \mid n \geq 4\} \cup \{u, w\}$. This already makes a handful of generators; in fact, we will define two more to ease the computations — put $y \coloneqq \underline{(3,1)} - \underline{(3,2)}$ and $z \coloneqq \underline{(3,1)} - \underline{(3,2)} - \underline{(4,1)} + \underline{(4,2)}$. A useful catalogue is available in Figure A.

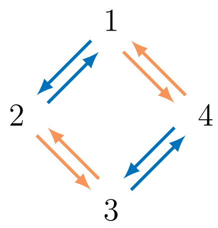
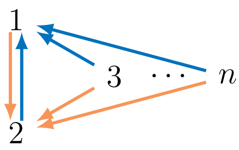
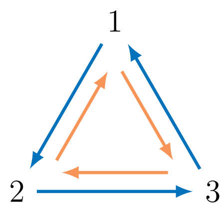
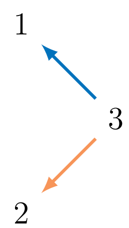
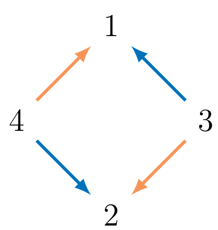
Generator x		$\mathbb{C}[S_n] \cdot x$, if seen
$t_n = \sum_{(i,j) \in [n]^{(2)}} \underline{(i,j)}$		$\mathbb{1}_n$
$u = (\mathrm{id} + \mathrm{op}) \begin{pmatrix} \underline{(1,2)} \\ - \underline{(2,3)} \\ + \underline{(3,4)} \\ - \underline{(4,1)} \end{pmatrix}$		U_n
$v_n = \begin{matrix} \sum_{i \in [n] \setminus \{1\}} \underline{(i,1)} \\ - \sum_{j \in [n] \setminus \{2\}} \underline{(j,2)} \end{matrix}$		V_n
$w = (\mathrm{id} - \mathrm{op}) \begin{pmatrix} \underline{(1,2)} \\ + \underline{(2,3)} \\ + \underline{(3,1)} \end{pmatrix}$		W_n
$y = \underline{(3,1)} - \underline{(3,2)}$		
$z = \begin{matrix} \underline{(3,1)} - \underline{(3,2)} \\ - \underline{(4,1)} + \underline{(4,2)} \end{matrix}$		

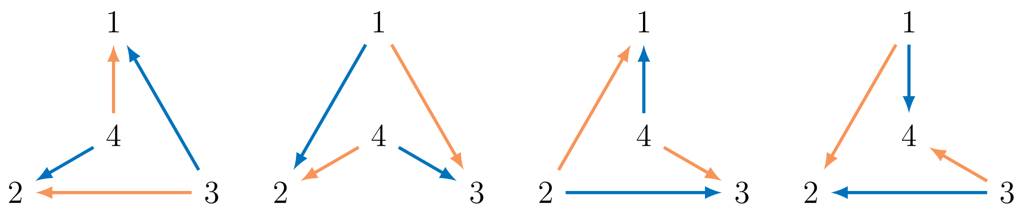
Figure A: Named generators. Blue/orange edges respectively have weights $+/-1$.

3.5.1 v_n

We begin by studying the spaces generated by v_n; hereafter we assume that $n \geq 4$. Even though we showed that $\mathbb{C}[S_n] \cdot v_n = V_n$ is simple in Proposition 3.14, we will see that $\mathbb{C}[S_{n+1}] \cdot v_n$ already fails to be simple as it contains $\mathbb{C}[S_{n+1}] \cdot u = U_{n+1}$ from (3.18) and $\mathbb{C}[S_{n+1}] \cdot w = W_{n+1}$ from (3.20) as submodules.

Lemma 3.31. $\mathbb{C}[S_n] \odot \{u, w\} = \mathbb{C}[S_n] \cdot z$.

Proof. Consider z, $(1\ 2\ 3) \cdot z$, $(1\ 3\ 2) \cdot z$, and $(1\ 2\ 4) \cdot z$ depicted in order below.



Observe that $z + (1\ 2\ 3) \cdot z + (1\ 3\ 2) \cdot z = (\mathrm{id} - \mathrm{op})(\underline{(1,2)} + \underline{(2,3)} + \underline{(3,1)}) = w$, whereas $(1\ 3\ 2) \cdot z + (1\ 2\ 4) \cdot z = (\mathrm{id} + \mathrm{op})(-\underline{(1,2)} + \underline{(2,3)} - \underline{(3,4)} + \underline{(4,1)}) = -u$; hence $u, w \in \mathbb{C}[S_n] \cdot z$. In the other direction, notice that

$$\begin{aligned} u_{1,3,2,4} + v_{1,2,3} + v_{1,4,2} = &\ (\mathrm{id} + \mathrm{op})(\underline{(3,1)} - \underline{(3,2)} + \underline{(4,2)} - \underline{(4,1)}) \\ &+ (\mathrm{id} - \mathrm{op})(\underline{(3,1)} - \underline{(3,2)} + \underline{(1,2)}) \\ &+ (\mathrm{id} - \mathrm{op})(- \underline{(1,2)} + \underline{(4,2)} - \underline{(4,1)}) \\ = &\ 2(\underline{(3,1)} - \underline{(3,2)} + \underline{(4,2)} - \underline{(4,1)}) = 2z. \end{aligned}$$

It then follows from Propositions 3.17 and 3.19 that $\mathbb{C}[S_n] \odot \{u, w\}$ contains z. □

Corollary 3.32. $u, w \in \mathbb{C}[S_{n+1}] \cdot v_n$.

Proof. As $v_n = \sum_{1 \leq i \leq n, i \neq 1} \underline{(i,1)} - \sum_{1 \leq j \leq n, j \neq 2} \underline{(j,2)}$, we have $(n\ \ n+1) \cdot v_n = \sum_{1 \leq i \lneq n, i \neq 1} \underline{(i,1)} + \underline{(n+1,1)} - \sum_{1 \leq j \lneq n, j \neq 2} \underline{(j,2)} - \underline{(n+1,2)}$. Now $\mathbb{C}[S_{n+i}] \cdot v_n$ contains their difference, which simplifies to $\underline{(n,1)} - \underline{(n,2)} - \underline{(n+1,1)} + \underline{(n+1,2)}$. But this becomes z after we apply the renaming $(n\ 3) \circ (n+1\ \ 4)$. □

Lemma 3.33. *$\mathbb{C}[S_n] \odot (\alpha\,\mathrm{id} + \beta\,\mathrm{op})(X) \subseteq \mathbb{C}[S_n] \odot (\alpha\,\mathrm{id} + \beta\,\mathrm{op})(X')$ for any $\alpha, \beta \in \mathbb{C}$ whenever $\mathbb{C}[S_n] \odot X \subseteq \mathbb{C}[S_n] \odot X'$.*

Proof. It suffices to show that $(\alpha\,\mathrm{id} + \beta\,\mathrm{op})(x) \in \mathbb{C}[S_n] \odot (\alpha\,\mathrm{id} + \beta\,\mathrm{op})(X')$ holds for any $x \in X$; thus let $x \in X$. As $x \in \mathbb{C}[S_n] \odot X \subseteq \mathbb{C}[S_n] \odot X'$, we can write

$$x = \sum_{x' \in X'} \phi_{x'} \cdot x'$$

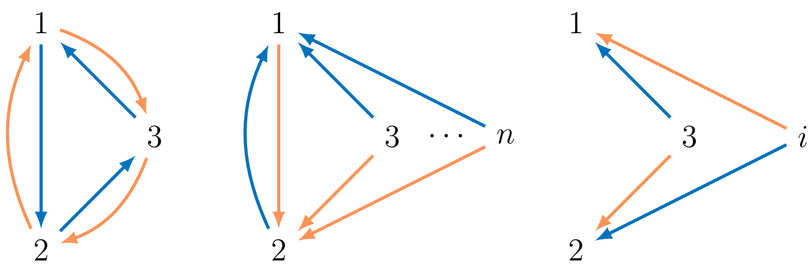
where each $\phi_{x'}$ is in $\mathbb{C}[S_n]$. By applying the $\mathbb{C}[S_n]$-module homomorphism $\alpha\,\mathrm{id} + \beta\,\mathrm{op}$, we obtain

$$(\alpha\,\mathrm{id} + \beta\,\mathrm{op})(x) = (\alpha\,\mathrm{id} + \beta\,\mathrm{op})\Big(\sum_{x' \in X'} \phi_{x'} \cdot x'\Big) = \sum_{x' \in X'} \phi_{x'} \cdot (\alpha\,\mathrm{id} + \beta\,\mathrm{op})(x')$$

which certainly lies in $\mathbb{C}[S_n] \odot (\alpha\,\mathrm{id} + \beta\,\mathrm{op})(X')$, as desired. □

Lemma 3.34. $\mathbb{C}[S_n] \odot \{v_n, u, w\} = \mathbb{C}[S_n] \cdot \big((n-1)\,\mathrm{id} - \mathrm{op}\big)(y)$.

Proof. It is hard not to see that $z = y - (3\ 4) \cdot y$ lies in $\mathbb{C}[S_n] \cdot y$. By Lemma 3.31, we then have $u, w \in \mathbb{C}[S_n] \cdot z \subseteq \mathbb{C}[S_n] \cdot y$; in turn, $\big((n-1)\,\mathrm{id} - \mathrm{op}\big)(u) = (n-2)u$ and $\big((n-1)\,\mathrm{id} - \mathrm{op}\big)(w) = nw$ lie in $\mathbb{C}[S_n] \cdot \big((n-1)\,\mathrm{id} - \mathrm{op}\big)(y)$ by Lemma 3.33. Therefore $u, w \in \mathbb{C}[S_n] \cdot \big((n-1)\,\mathrm{id} - \mathrm{op}\big)(y)$ as well. Now consider w, v_n, and $(4\ i) \cdot z$ for $4 \le i \le n$ (where we let $(4\ 4) = \mathrm{id}$ for convenience) depicted below.



Observe that these share common edges, and indeed we have

$$w + v_n + \sum_{4 \le i \le n} (4\ i) \cdot z = \big((n-1)\,\mathrm{id} - \mathrm{op}\big)\Big(\overbrace{\underline{(3,1)} - \underline{(3,2)}}^{y}\Big) \tag{3.35}$$

after many cancellations. By rearranging the equation, we get an expression of v_n as a $\mathbb{C}[S_n]$-linear combination of w, z, and $\big((n-1)\,\mathrm{id} - \mathrm{op}\big)(y)$. But we showed above that $\mathbb{C}[S_n] \cdot \big((n-1)\operatorname{id} - \operatorname{op}\big)(y)$ contains u, w and hence z as well by Lemma 3.31; we conclude that $v_n, u, w \in \mathbb{C}[S_n] \cdot \big((n-1)\operatorname{id} - \operatorname{op}\big)(y)$.

On the other hand, (3.35) already shows that $\big((n-1)\operatorname{id} - \operatorname{op}\big)(y)$ belongs to $\mathbb{C}[S_n] \odot \{v_n, u, w\}$: the submodule $\mathbb{C}[S_n] \odot \{v_n, u, w\}$ certainly contains w and v_n, but by Lemma 3.31 it also contains z since it contains u, w. □

In other words, each v_n is associated with a 'twisted' copy of y. These twists parametrised by n seem inevitable, so instead we will learn to work with them.

Lemma 3.36. *The maps* $(\{\alpha\operatorname{id} + \beta\operatorname{op} \mid \alpha, \beta \in \mathbb{C}\}; \circ; \operatorname{id})$ *form a commutative monoid, whose invertible elements* $(\{\alpha\operatorname{id} + \beta\operatorname{op} \mid \alpha, \beta \in \mathbb{C}, \alpha \neq \pm\beta\}; \circ; \operatorname{id})$ *form an abelian subgroup that in particular contains*

$$\sigma_n^{\pm} := (n-1)\operatorname{id} \pm \operatorname{op}$$

for $n > 2$. *Additionally, given any* $\rho = \alpha\operatorname{id} + \beta\operatorname{op}$ *we have*

(*a*) $\rho \circ (\operatorname{id} + \operatorname{op}) = c^{+}(\operatorname{id} + \operatorname{op})$ *where* $c^{+} := \alpha + \beta$ *is non-zero whenever* $\alpha \neq -\beta$*;*

(*b*) $\rho \circ (\operatorname{id} - \operatorname{op}) = c^{-}(\operatorname{id} - \operatorname{op})$ *where* $c^{-} := \alpha - \beta$ *is non-zero whenever* $\alpha \neq \beta$*.*

Remark. One can readily identify this monoid with the multiplicative structure of the group ring $\mathbb{C}[S_2]$, but we will pursue a more elementary line of reasoning. ■

Proof. Note that composition of $\mathbb{C}[S_n]$-module homomorphisms is associative and $\mathbb{C}$-bilinear. Thus

$$(\alpha\operatorname{id} + \beta\operatorname{op}) \circ (\alpha'\operatorname{id} + \beta'\operatorname{op}) = (\alpha\alpha' + \beta\beta')\operatorname{id} + (\alpha\beta' + \beta\alpha')\operatorname{op},$$

which furthermore establishes that $\{\alpha\operatorname{id} + \beta\operatorname{op} \mid \alpha, \beta \in \mathbb{C}\}$ is closed under $\circ$, that $\circ$ is commutative, and that id is a unit. The assertions (a) and (b) also follow straightforwardly. Observe moreover that $(\alpha\operatorname{id} + \beta\operatorname{op}) \circ (\alpha\operatorname{id} - \beta\operatorname{op}) = (\alpha^2 - \beta^2)\operatorname{id}$. If $\alpha \neq \pm\beta$, then $\frac{1}{\alpha^2 - \beta^2}(\alpha\operatorname{id} - \beta\operatorname{op})$ furnishes the inverse to $\alpha\operatorname{id} + \beta\operatorname{op}$; in particular, σ_n^{+} and σ_n^{-} are mutually inverse up to a non-zero scalar $(n-1)^2 - 1$ for $n \geq 3$. Conversely, if $\alpha = \pm\beta$, then $(\operatorname{id} + \operatorname{op}) \circ (\operatorname{id} - \operatorname{op}) = 0$ shows that $\alpha\operatorname{id} + \beta\operatorname{op}$ cannot be invertible because id $\neq 0$. Hence the invertible maps are precisely $\{\alpha\,\mathrm{id} + \beta\,\mathrm{op} \mid \alpha \neq \pm\beta\}$. But the invertible elements of a (commutative) monoid evidently form an (abelian) subgroup: if ρ and τ are invertible, then so is their composition $\rho \circ \tau$ — consider $\tau^{-1} \circ \rho^{-1}$. ☐

Corollary 3.37. *For all $n \geq 4$, we have*

(*a*) $\mathbb{C}[S_n] \odot \{\sigma_n^+(v_n), u, w\} = \mathbb{C}[S_n] \cdot y$,

(*b*) $\mathbb{C}[S_{n+i}] \odot \{\sigma_n^+(v_n), u, w\} = \mathbb{C}[S_{n+i}] \odot \{\sigma_{n+i}^+(v_{n+i}), u, w\}$ *for all* $i \geq 1$*, and*

(*c*) $\mathbb{C}[S_{n+1}] \cdot \sigma_n^+(v_n) = \mathbb{C}[S_{n+1}] \odot \{\sigma_n^+(v_n), u, w\}$.

Proof. Recall from Lemma 3.34 that $\mathbb{C}[S_n] \odot \{v_n, u, w\} = \mathbb{C}[S_n] \cdot \sigma_n^-(y)$. By applying Lemma 3.33 with σ_n^+, we get

$$\mathbb{C}[S_n] \odot \{\sigma_n^+(v_n), u, w\} = \mathbb{C}[S_n] \cdot y$$

since $\sigma_n^+(u) = nu$, $\sigma_n^+(w) = (n-2)w$, and $(\sigma_n^+ \circ \sigma_n^-)(y) = ((n-1)^2 - 1)y$ are non-zero scalar multiples of u, w, and y respectively for $n \geq 3$. This establishes (a). But $\mathbb{C}[S_{n+i}] \odot \{\sigma_{n+i}^+(v_{n+i}), u, w\} = \mathbb{C}[S_{n+i}] \cdot y$ since n is arbitrary, and also $\mathbb{C}[S_{n+i}] \cdot y = \mathbb{C}[S_{n+i}] \odot \{\sigma_n^+(v_n), u, w\}$ by Lemma 3.11, thus proving (b). For (c), manifestly $\mathbb{C}[S_{n+1}] \cdot \sigma_n^+(v_n) \subseteq \mathbb{C}[S_{n+1}] \odot \{\sigma_n^+(v_n), u, w\}$; the reverse inclusion $\mathbb{C}[S_{n+1}] \cdot \sigma_n^+(v_n) \supseteq \mathbb{C}[S_{n+1}] \odot \{\sigma_n^+(v_n), u, w\}$ follows immediately by Corollary 3.32 and Lemma 3.33. ☐

Remark. By (b), over $\mathbb{C}[S_{n+i}]$ the module generated by $\{\sigma_n^+(v_n), u, w\}$ decomposes into the simple submodules generated respectively by $\sigma_{n+i}^+(v_{n+i})$, u, and w as per Theorem 3.29; this facilitates the comparison against other modules at the level $n+i$. But (a) tells us that $\{\sigma_n^+(v_n), u, w\}$, $n \geq 4$ all generate the same module over the limit $\mathbb{C}[S_\infty]$ — namely, $\mathbb{C}[S_\infty] \cdot y$. Moreover, we see from (c) that $\{\sigma_n^+(v_n)\}$ and $\{\sigma_n^+(v_n), u, w\}$ in $\mathfrak{X}_n$ both generate this $\mathbb{C}[S_\infty]$-module. We will capitalise on these observations to obtain a structure theorem for the $\mathbb{C}[S_\infty]$-submodules of $\operatorname{Lin}\mathbb{N}^{(2)}$ once we complete such calculations for other sets of generators in $\mathfrak{X}_n$. ■

Corollary 3.38. *Let $n \geq 4$ and $i \geq 1$. Given $\lambda_n, \mu_n \in \mathbb{C}$ not both zero, there exist $\lambda, \mu \in \mathbb{C}$ not both zero such that $\tau = \lambda \operatorname{id} + \mu \operatorname{op}$ satisfies $\tau \circ \sigma_n^+ = \lambda_n \operatorname{id} + \mu_n \operatorname{op}$. Moreover $[\lambda : \mu] = [\pm 1 : 1]$ iff $[\lambda_n : \mu_n] = [\pm 1 : 1]$, and we have*

$$(a)\quad \left.\begin{array}{ll} \mathbb{C}[S_n] \odot \{(\operatorname{id} + \operatorname{op})(v_n), u\} & \text{if } [\lambda : \mu] = [1 : 1], \\ \mathbb{C}[S_n] \odot \{(\operatorname{id} - \operatorname{op})(v_n), w\} & \text{if } [\lambda : \mu] = [-1 : 1], \\ \mathbb{C}[S_n] \odot \{(\tau \circ \sigma_n^+)(v_n), u, w\} & \text{otherwise} \end{array}\right\} = \mathbb{C}[S_n] \cdot \tau(y),$$

(b)
- $\mathbb{C}[S_{n+i}] \odot \{(\operatorname{id} + \operatorname{op})(v_n), u\} = \mathbb{C}[S_{n+i}] \odot \{(\operatorname{id} + \operatorname{op})(v_{n+i}), u\}$ *whenever* $[\lambda : \mu] = [1 : 1]$,
- $\mathbb{C}[S_{n+i}] \odot \{(\operatorname{id} - \operatorname{op})(v_n), w\} = \mathbb{C}[S_{n+i}] \odot \{(\operatorname{id} - \operatorname{op})(v_{n+i}), w\}$ *whenever* $[\lambda : \mu] = [-1 : 1]$,
- $C[S_{n+i}] \odot \{(\tau \circ \sigma_n^+)(v_n), u, w\} = C[S_{n+i}] \odot \{(\tau \circ \sigma_{n+i}^+)(v_{n+i}), u, w\}$ *whenever* $[\lambda : \mu] \neq [\pm 1 : 1]$, *and*

$$(c)\quad \mathbb{C}[S_{n+1}] \cdot (\tau \circ \sigma_n^+)(v_n) = \begin{cases} \mathbb{C}[S_{n+1}] \odot \{(\operatorname{id} + \operatorname{op})(v_n), u\} & \text{if } [\lambda : \mu] = [1 : 1], \\ \mathbb{C}[S_{n+1}] \odot \{(\operatorname{id} - \operatorname{op})(v_n), w\} & \text{if } [\lambda : \mu] = [-1 : 1], \\ \mathbb{C}[S_{n+1}] \odot \{(\tau \circ \sigma_n^+)(v_n), u, w\} & \text{otherwise.} \end{cases}$$

Proof. We can simply take $\tau = (\lambda_n \operatorname{id} + \mu_n \operatorname{op}) \circ \frac{1}{(n-1)^2 - 1} \sigma_n^-$. If $[\lambda_n : \mu_n] \neq [\pm 1 : 1]$, then by Lemma 3.36 $\lambda_n \operatorname{id} + \mu_n \operatorname{op}$ is invertible; since so is σ_n^-, their composite τ is also invertible and thus $[\lambda : \mu] \neq [\pm 1 : 1]$. We see by (a) and (b) of the same lemma that $[\lambda_n : \mu_n] = [\pm 1 : 1]$ implies $[\lambda : \mu] = [\pm 1 : 1]$ respectively. Because these 3 cases are disjoint, we conclude that $[\lambda_n : \mu_n] = [\pm 1 : 1]$ iff $[\lambda : \mu] = [\pm 1 : 1]$ as claimed.

Now, we apply Lemma 3.33 with τ to Corollary 3.37. Noticing that $\tau(u) = 0$ iff $[\lambda : \mu] = [-1 : 1]$ whilst $\tau(w) = 0$ iff $[\lambda : \mu] = [1 : 1]$, the assertions (a)–(c) follow straightforwardly. □

3.5.2 t_n

Fortunately, the spaces generated by t_n can now be easily described.

Proposition 3.39. *Let $n \geq 4$ and $i \geq 2$. Then* (*a*) $\mathbb{C}[S_n] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u\} = \mathbb{C}[S_n] \cdot (\mathrm{id} + \mathrm{op})(\underline{(1,2)})$,

(*b*) $\mathbb{C}[S_{n+i}] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u\} = \mathbb{C}[S_{n+i}] \odot \{t_{n+i}, (\mathrm{id} + \mathrm{op})(v_{n+i}), u\}$, *and*

(*c*) $\mathbb{C}[S_{n+2}] \cdot t_n = \mathbb{C}[S_{n+2}] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u\}$.

Proof. For (a), recall from Propositions 3.12 and 3.17 that

$$\begin{aligned}\mathbb{C}[S_n] \cdot (\mathrm{id} + \mathrm{op})(\underline{(1,2)}) &= \ker((\mathrm{id} - \mathrm{op})|_{\mathrm{Lin}\,[n]^{(2)}}) \\ &= \mathbb{1}_n + (\mathrm{id} + \mathrm{op})(V_n) + U_n = \mathbb{C}[S_n] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u\}.\end{aligned}$$

As $n \geq 4$ can be taken arbitrary, (b) follows easily by Lemma 3.11. It remains to show that $(\mathrm{id} + \mathrm{op})(v_n), u \in \mathbb{C}[S_{n+2}] \cdot t_n$.

As $t_n = \sum_{1 \leq i \neq j \leq n} \underline{(i,j)}$, we have

$$\begin{aligned}&(2\ \ n+1) \cdot t_n - (1\ \ n+1) \cdot t_n \\ &= \sum_{i \neq j \in [n+1] \setminus \{2\}} \underline{(i,j)} - \sum_{i \neq j \in [n+1] \setminus \{1\}} \underline{(i,j)} \\ &= \sum_{k \in [n+1] \setminus \{1,2\}} (\underline{(1,k)} + \underline{(k,1)}) - \sum_{k \in [n+1] \setminus \{1,2\}} (\underline{(2,k)} + \underline{(k,2)}) \\ &= (\mathrm{id} + \mathrm{op})(\sum_{k' \in [n+1] \setminus \{1\}} \underline{(1,k')} - \sum_{k'' \in [n+1] \setminus \{2\}} \underline{(2,k'')}) \\ &= (\mathrm{id} + \mathrm{op})(v_{n+1})\end{aligned}$$

where the second to the last equality follows since the terms associated with $k' = 2$ and with $k'' = 1$ have the same image under the symmetrising map $\mathrm{id} + \mathrm{op}$; hence $(\mathrm{id} + \mathrm{op})(v_{n+1}) \in \mathbb{C}[S_{n+1}] \cdot t_n$, which gives $\mathbb{C}[S_{n+2}] \cdot (\mathrm{id} + \mathrm{op})(v_{n+1}) \subseteq \mathbb{C}[S_{n+2}] \cdot t_n$. But we are done: Corollary 3.38 tells us that

$$\begin{aligned}&\mathbb{C}[S_{n+2}] \cdot (\mathrm{id} + \mathrm{op})(v_{n+1}) \\ &\overset{\text{(c)}}{=} \mathbb{C}[S_{n+2}] \odot \{(\mathrm{id} + \mathrm{op})(v_{n+1}), u\} \\ &\overset{\text{(b)}}{=} \mathbb{C}[S_{n+2}] \odot \{(\mathrm{id} + \mathrm{op})(v_{n+2}), u\} \\ &\overset{\text{(b)}}{=} \mathbb{C}[S_{n+2}] \odot \{(\mathrm{id} + \mathrm{op})(v_n), u\}.\end{aligned}$$

□

3.5.3 The structure theorem

Theorem 3.40. *Figure B gives all the equivariant subspaces of* $\operatorname{Lin}\mathbb{N}^2$. *In particular*

(*a*) $\operatorname{length}(\operatorname{Lin}\mathbb{N}^{(2)}) = 5 = \operatorname{length}(\operatorname{Lin}[n]^{(2)})$ *for* $n \geq 4$, *and*

(*b*) *there is an infinite family, indexed by* $\mathbb{C}P^1 \setminus \{[1:1],[-1:1]\}$, *of distinct but isomorphic equivariant subspaces in* $\operatorname{Lin}\mathbb{N}^{(2)}$.

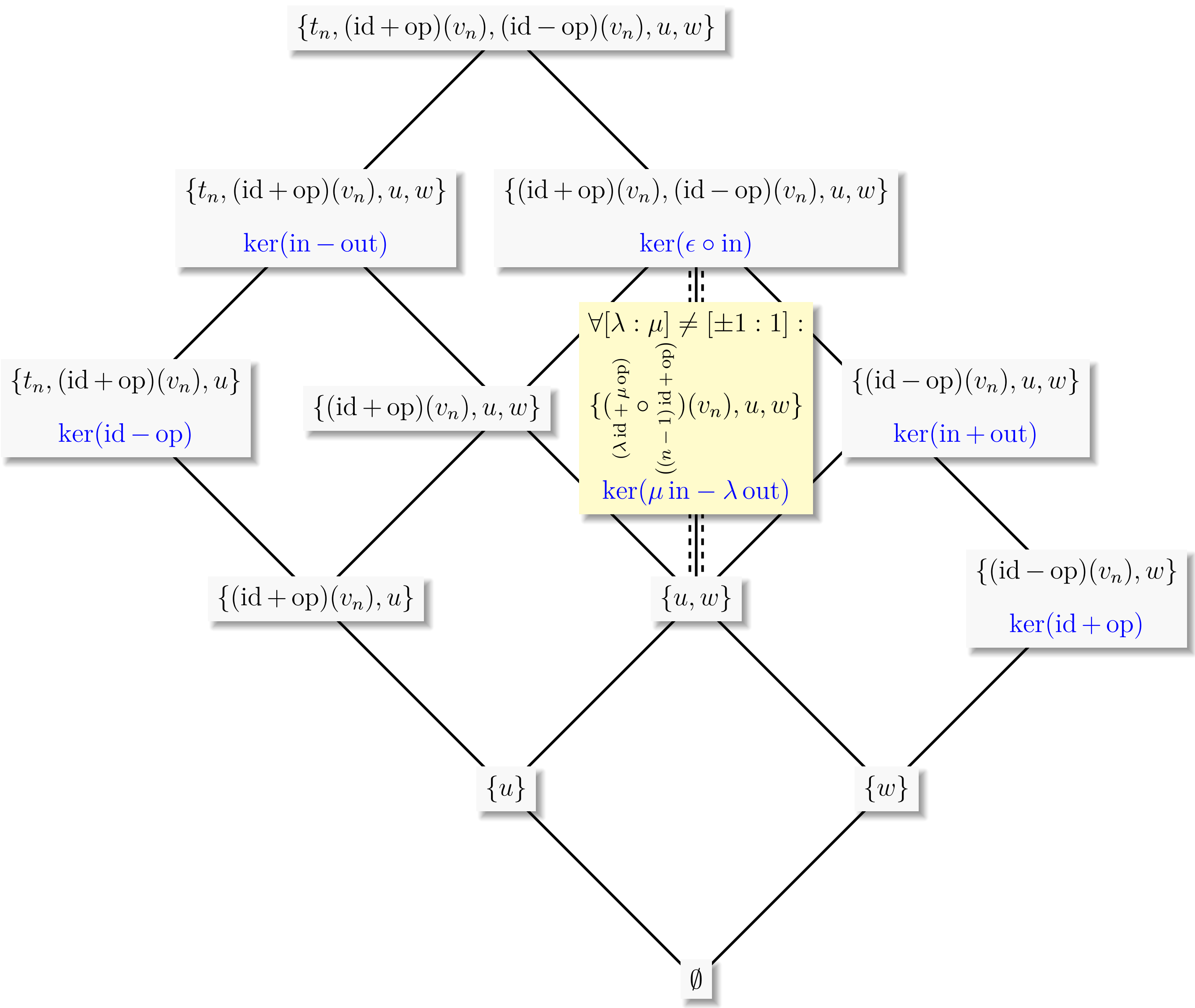
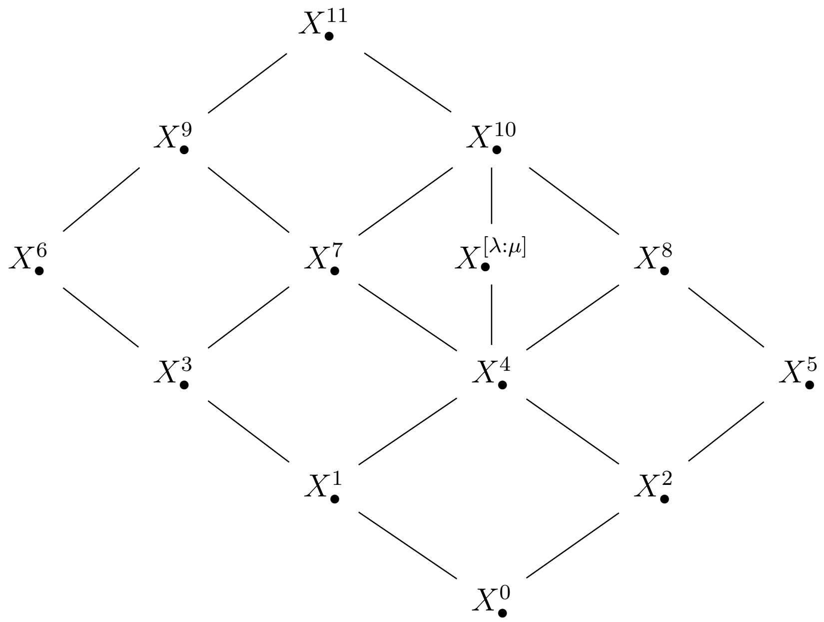


Figure B: Lattice of all equivariant subspaces in $\operatorname{Lin}\mathbb{N}^{(2)}$ with respect to inclusion, labelled by their respective generators, where each $n \geq 4$ can be arbitrarily large. Equivalent characterisations of certain spaces are given in blue.

Proof. Let $\mathfrak{A} := \{0, \dots, 11\} \sqcup (\mathbb{C}P^1 \setminus \{[1:1], [-1:1]\})$, and consider the poset $\mathcal{P} := (\{X_\bullet^\alpha \mid \alpha \in \mathfrak{A}\}; \leq)$ given by the Hasse diagram



where $X_\bullet^\alpha$ denotes the family, indexed by $n \geq 4$, of generating sets given by the label at the corresponding position in Figure B. For instance, the leftmost family $X_\bullet^6$ consists of $X_n^6 := \{(\mathrm{id} + \mathrm{op})(v_n), u\}$ for each $n \geq 4$. Note also that $X_\bullet^{[\lambda:\mu]}$ and $X_\bullet^{[\lambda':\mu']}$ are by definition incomparable whenever $[\lambda : \mu], [\lambda' : \mu'] \in \mathbb{C}P^1$ are distinct. Aside from the $X_\bullet^{[\lambda:\mu]}$'s, we tally 12 distinct families of generators.

Further, for any families $X_\bullet^\alpha, X_\bullet^\beta$ and any $n \geq 4$ we have

(i) $\mathbb{C}[S_n] \odot X_n^\alpha \subseteq \mathbb{C}[S_n] \odot X_n^\beta$ iff $X_\bullet^\alpha \leq X_\bullet^\beta$, and

(ii) $\mathbb{C}[S_{n+i}] \odot X_n^\alpha = \mathbb{C}[S_{n+i}] \odot X_{n+i}^\alpha$ for all $i \geq 2$.

Indeed, for (i) notice the poset $\mathcal{P}_n := (\{\mathbb{C}[S_n] \cdot X_n^\alpha \mid \alpha \in \mathfrak{A}\}; \subseteq)$ admits the same Hasse diagram as $\mathcal{P}$ above owing to Theorem 3.29; (ii) is a routine check using Corollary 3.38(b) and Proposition 3.39(b). But given $n, n' \geq 4$, (ii) moreover entails

$$\mathbb{C}[S_{n+n'+2}] \odot X_n^\alpha = \mathbb{C}[S_{n+n'+2}] \odot X_{n+n'+2}^\alpha = \mathbb{C}[S_{n+n'+2}] \odot X_{n'}^\alpha$$

and hence $\mathbb{C}[S_\infty] \odot X_n^\alpha = \mathbb{C}[S_\infty] \odot X_{n'}^\alpha$. Denote this unique equivariant subspace that every $X_n^\alpha, n \geq 4$ generates by $(X_\bullet^\alpha)$. Then (i) above together with Lemma 3.11

says precisely that

$$\mathcal{P} = (\{X_\bullet^\alpha \mid \alpha \in \mathfrak{A}\}; \leq) \to (\text{equivariant subspaces of } \operatorname{Lin} \mathbb{N}^{(2)}; \subseteq)$$
$$X_\bullet^\alpha \mapsto (X_\bullet^\alpha)$$

is an order embedding. To show that we have an order isomorphism, it suffices to establish surjectivity. By Corollary 3.30, any equivariant subspace of $\operatorname{Lin} \mathbb{N}^{(2)}$ is of the form $\mathbb{C}[S_\infty] \odot X$ where $X \subseteq \mathfrak{X}_n$ with $n \geq 4$. Observe that X is already one of the X_n^α's unless it contains

- $\{(\mathrm{id} + \mathrm{op})(v_n)\}$ but not $\{u\}$,
- $\{(\mathrm{id} - \mathrm{op})(v_n)\}$ but not $\{w\}$,
- $\{((\lambda\,\mathrm{id} + \mu\,\mathrm{op}) \circ \sigma_n^+)(v_n)\}$ where $[\lambda' : \mu'] \neq [\pm 1 : 1]$ but not $\{u, w\}$, or
- $\{t_n\}$ but not $\{(\mathrm{id} + \mathrm{op})(v_n), u\}$.

But Corollary 3.38(c) and Proposition 3.39(c) tell us that these degeneracies can be corrected without changing the $\mathbb{C}[S_\infty]$-module generated; that is, $\mathbb{C}[S_\infty] \cdot X$ agrees with $(X_\bullet^\alpha)$ for some $\alpha \in \mathfrak{A}$ as desired.

We conclude that Figure B indeed gives the full poset of equivariant subspaces in $\operatorname{Lin} \mathbb{N}^{(2)}$ with respect to inclusion. (This is additionally a lattice as claimed: given $\mathbb{C}[S_\infty]$-modules M and M', it is clear that their joins and meets are respectively given by $M + M'$ and $M \cap M'$.) The assertions (a) and (b) then follow immediately, with $\lambda\,\mathrm{id} + \mu\,\mathrm{op}$ affording the $\mathbb{C}[S_\infty]$-module isomorphism $(X_\bullet^{[1:0]}) \to (X_\bullet^{[\lambda:\mu]})$. □

We finish the section by supplying the alternative characterisations in Figure B.

Proposition 3.41. *For any $n \geq 4$, we have*

(a) $\mathbb{C}[S_\infty] \odot \{(\mathrm{id} - \mathrm{op})(v_n), w\} = \ker(\mathrm{id} + \mathrm{op})$;

(b) $\mathbb{C}[S_\infty] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u\} = \ker(\mathrm{id} - \mathrm{op})$;

(c) $\mathbb{C}[S_\infty] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u, w\} = \ker(\mathrm{in} - \mathrm{out})$;

(d) $\mathbb{C}[S_\infty] \odot \{((\lambda\,\mathrm{id} + \mu\,\mathrm{op}) \circ \sigma_n^+)(v_n), u, w\} = \ker(\mu\,\mathrm{in} - \lambda\,\mathrm{out})$ *for* $[\lambda : \mu] \neq [1 : 1]$;

(e) $\mathbb{C}[S_\infty] \odot \{(\mathrm{id} + \mathrm{op})(v_n), (\mathrm{id} - \mathrm{op})(v_n), u, w\} = \ker(\epsilon \circ \mathrm{in})$.

Proof. Recall the diagram with the restrictions of op, in, out, and ϵ to $\operatorname{Lin}[n]^{(2)}$ in (3.13). It is easy to verify that each generator on the left falls in the corresponding kernel on the right. Indeed, observe from Figure A that $\operatorname{in}(u) = 0 = \operatorname{out}(u)$, $\operatorname{in}(w) = 0 = \operatorname{out}(w)$, and $\operatorname{in}(v_n) = (n-1)\underline{1} - (n-1)\underline{2}$ whilst $\operatorname{out}(v_n) = -\underline{1} + \underline{2}$; notice also that

$$\begin{aligned}
&\big((\mu \operatorname{in} - \lambda \operatorname{out}) \circ (\lambda \operatorname{id} + \mu \operatorname{op}) \circ \rho_n^+\big)(v_n) \\
&= \big((-\lambda^2 + \mu^2) \operatorname{out} \circ \rho_n^+\big)(v_n) \\
&= (-\lambda^2 + \mu^2)\big((n-1) \operatorname{out} + \operatorname{in}\big)(v_n) \\
&= 0
\end{aligned}$$

and that

$$\begin{aligned}
\big(\epsilon \circ \operatorname{in} \circ (\operatorname{id} \pm \operatorname{op})\big)(v_n) &= (\epsilon \circ \operatorname{in})(v_n) \pm (\epsilon \circ \operatorname{out})(v_n) \\
&= \big((n-1) - (n-1)\big) \pm \big(-1 + 1\big) = 0
\end{aligned}$$

because of the basic identity $\operatorname{in} \circ \operatorname{op} = \operatorname{out}$. We now demonstrate the reverse containments by considering the lengths.

To start with, we see that $\mathbb{C}[S_\infty] \odot \{(\operatorname{id} - \operatorname{op})(v_n), w\} \subseteq \ker(\operatorname{id} + \operatorname{op})$ implies by dint of Theorem 3.40 that $\operatorname{length}\big(\ker(\operatorname{id} + \operatorname{op})\big) \geq 2$ with equality iff the two $\mathbb{C}[S_\infty]$-modules are equal. Similarly, we have $\operatorname{length}\big(\ker(\operatorname{id} - \operatorname{op})\big) \geq 3$ with equality iff $\mathbb{C}[S_\infty] \odot \{t_n, (\operatorname{id} + \operatorname{op})(v_n), u\} = \ker(\operatorname{id} - \operatorname{op})$. But the beginning of the proof to Proposition 3.12 also shows $\operatorname{Lin}\mathbb{N}^{(2)} = \ker(\operatorname{id} + \operatorname{op}) \oplus \ker(\operatorname{id} - \operatorname{op})$, so by Lemma 2.30 it follows that $5 = \operatorname{length}(\operatorname{Lin}\mathbb{N}^{(2)}) = \operatorname{length}\big(\ker(\operatorname{id} + \operatorname{op})\big) + \operatorname{length}\big(\ker(\operatorname{id} - \operatorname{op})\big)$. Thus $\operatorname{length}\big(\ker(\operatorname{id} + \operatorname{op})\big) = 2$ and $\operatorname{length}\big(\ker(\operatorname{id} - \operatorname{op})\big) = 3$ precisely, which by above establishes (a) and (b).

As for (c), note that $(\operatorname{in} - \operatorname{out})(\operatorname{Lin}\mathbb{N}^{(2)}) = \ker(\epsilon)$: indeed

$$(\epsilon \circ \operatorname{in})\Big(\sum_{a \neq b} c_{(a,b)} \underline{(a,b)}\Big) = \sum_{a \neq b} c_{(a,b)} = (\epsilon \circ \operatorname{out})\Big(\sum_{a \neq b} c_{(a,b)} \underline{(a,b)}\Big),$$

so $(\mathrm{in} - \mathrm{out})(\operatorname{Lin} \mathbb{N}^{(2)})$ is an equivariant subspace of $\ker(\epsilon)$ that is moreover non-zero as it contains $(\mathrm{in} - \mathrm{out})(\underline{(1,2)}) = \underline{2} - \underline{1}$. It follows by Lemma 2.30 and Corollary 3.5 that

$$\begin{aligned} 5 = \operatorname{length}(\operatorname{Lin} \mathbb{N}^{(2)}) &= \operatorname{length}\big((\mathrm{in} - \mathrm{out})(\operatorname{Lin} \mathbb{N}^{(2)})\big) + \operatorname{length}\big(\ker(\mathrm{in} - \mathrm{out})\big) \\ &= \operatorname{length}(\ker \epsilon) + \operatorname{length}\big(\ker(\mathrm{in} - \mathrm{out})\big) \\ &= 1 + \operatorname{length}\big(\ker(\mathrm{in} - \mathrm{out})\big). \end{aligned}$$

We then have

$$\operatorname{length}\big(\ker(\mathrm{in} - \mathrm{out})\big) = 4 = \operatorname{length}\big(\mathbb{C}[S_\infty] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u, w\}\big)$$

in view of Theorem 3.40, which forces the containment $\ker(\mathrm{in} - \mathrm{out}) \supseteq \mathbb{C}[S_\infty] \odot \{t_n, (\mathrm{id} + \mathrm{op})(v_n), u, w\}$ that we established earlier to be an equality. The assertions (d) and (e) follow analogously from the observations that $(\mu\, \mathrm{in} - \lambda\, \mathrm{out})(\operatorname{Lin} \mathbb{N}^{(2)}) = \operatorname{Lin} \mathbb{N}$ given $[\lambda : \mu] \neq [1 : 1]$ and that $(\epsilon \circ \mathrm{in})(\operatorname{Lin} \mathbb{N}^{(2)}) = \mathbb{C}$. □

Remark. These characterisations readily supply us with efficient membership tests: for example, checking whether some $v \in \operatorname{Lin} \mathbb{N}^{(2)}$ falls in $\mathbb{C}[S_\infty] \odot \{(\mathrm{id} - \mathrm{op})(v_n), w\}$ by (a) amounts to checking whether $(\mathrm{id} + \mathrm{op})(v) = 0$. Moreover, we can exploit the lattice structure of Figure B to describe other subspaces as meets: we now see that

- $\mathbb{C}[S_\infty] \odot \{u, w\} = \ker(\mathrm{in}) \cap \ker(\mathrm{out})$,
- $\mathbb{C}[S_\infty] \cdot u = \ker(\mathrm{id} - \mathrm{op}) \cap \ker(\mathrm{out})$, and
- $\mathbb{C}[S_\infty] \cdot w = \ker(\mathrm{id} + \mathrm{op}) \cap \ker(\mathrm{in})$. ■

4 Lengths

We now leave the constructive world of $k \leq 2$, switch gears and rely on more abstract machinery to work out upper bounds on $\operatorname{length}(\operatorname{Lin} \mathbb{N}^{(k)})$ for arbitrary k. The culmination is Corollary 4.4, which improves the following result from Bojańczyk et al. by at least a factor of $k!$.

Proposition ([BKM21, Lemma IV.9]). *Given any $k \geq 0$,*

$$\operatorname{length}(\operatorname{Lin} \mathbb{N}^{(k)}) \leq k!(k+1)!$$

and is in particular finite.

4.1 $\operatorname{length}(\operatorname{Lin} [n]^{(k)})$

We begin with a brief survey of the representation theory of finite symmetric groups following [Sag01, §2–3].

Partitions. A *partition of* n is a weakly decreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$; we write $\lambda \vdash n$ and $|\lambda| \coloneqq n$. A *Young tableau of shape* λ is a bijection between $[n]$ and the n boxes arranged in l left-justified rows with λ_i boxes in row i for $1 \leq i \leq l$; the tableau is moreover *standard* if the rows and columns are all increasing sequences. For example, given

$\lambda = (2, 1, 1)$, the tableau

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

is standard, whereas

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

is not. The number of standard tableaux of shape λ is denoted by f^λ.

We also record two ways to produce new partitions from old. Given $m \geq |\lambda| + \lambda_1$ (where we put $\lambda_1 := 0$ if $\lambda = () \vdash 0$), we obtain a partition

$$\lambda[m] := (m - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_l,)$$

of m. Conversely, for $|\lambda| > 0$ we may remove the first entry of λ, which gives the partition

$$\lambda^* := (\lambda_2, \ldots, \lambda_l)$$

of $|\lambda| - \lambda_1$.

Specht modules. Each partition $\lambda \vdash n$ is associated with a simple $\mathbb{C}[S_n]$-module V_λ (or S^λ in some notations) of dimension f^λ; known as the *Specht modules*, the collection

$$\{V_\lambda \mid \lambda \vdash n\}$$

gives a complete, irredundant list of all simple $\mathbb{C}[S_n]$-modules up to isomorphism [Sag01, Theorems 2.6.5 and 2.4.6]. In fact, we have already met

- $\mathbb{1}_n \simeq V_{(n)}$,
- $V_n \simeq V_{(n-1,1)}$,
- $U_n \simeq V_{(n-2,2)}$,
- $W_n \simeq V_{(n-2,1,1)}$

for all $n \geq 4$. Indeed, the first isomorphism is trivial whilst the second is standard (pun intended) — see the first example after Theorem 4.3 along with Exercise 4.6* and the hints at the back of the book [FH04]. Once one realises that Lin $[n] \simeq \mathbb{C}^n \simeq V_{(n)} \oplus V_{(n-1,1)}$ and that Lin $[n] \oplus$ Lin $[n]^{(2)} \simeq$ Lin $[n]^2 \simeq (\mathbb{C}^n)^{\otimes 2} \simeq V_{(n)} \oplus V_{(n-1,1)} \oplus V_{(n-1,1)} \oplus V_{(n-1,1)}^{\otimes 2}$, the remaining two isomorphisms follow directly from [FH04, Exercise 4.19*]. Alternatively, one can follow [Ham89, (7-167)–(7-177)] for an intuitive, graphical rule to decompose $V_{(n-1,1)}^{\otimes 2}$.

Notice that (n), $(n-1,1)$, $(n-2,2)$, $(n-2,1,1)$ are respectively given by $\lambda[n]$ if we put $\lambda = ()$, (1), (2), $(1,1)$.

Schur–Weyl dualities. Consider Lin $[n]^k$, where we now allow repeating entries in the basis. Consider also the standard basis $\{\mathbf{e}_i \mid 1 \leq i \leq n\}$ of $\mathbb{C}^n$, where $\mathbf{e}_i$ is the column vector with 1 in the ith entry and 0 everywhere else; the kth tensor power $(\mathbb{C}^n)^{\otimes k}$ then has $\mathfrak{B} \coloneqq \{\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$ as a basis. We can readily identify Lin $[n]^k$ with $(\mathbb{C}^n)^{\otimes k}$ via $\underline{(i_1, \ldots, i_k)} \mapsto \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}$. But an invertible n-by-n matrix $A \in \mathrm{GL}(n)$ with complex entries has a natural linear action $A \cdot (\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k}) = A\mathbf{e}_{i_1} \otimes \cdots \otimes A\mathbf{e}_{i_k}$ on $(\mathbb{C}^n)^{\otimes k}$. In fact, thus far we have been concerned with the actions of

$$S_n \subseteq \mathrm{GL}(n)$$

where we view $\pi \in S_n$ as the n-by-n matrix with 1's in the $(i, \pi \cdot i)$ entries and 0 elsewhere. It turns out that, if we take G to be one of these two matrix groups, the collection of linear maps $\phi : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes k}$ whose matrix form ${}_{\mathfrak{B}}[\phi]_{\mathfrak{B}}$ satisfies ${}_{\mathfrak{B}}[\phi]_{\mathfrak{B}}\, A = A\, {}_{\mathfrak{B}}[\phi]_{\mathfrak{B}}$ for all $A \in G$ are precisely the actions of

$$\mathbf{P}_k(n) \supseteq \mathbb{C}[S_k]$$

provided that $n \geq 2k$. Here $\mathbf{P}_k(n)$ is the *partition algebra*, whilst $\mathbb{C}[S_k]$ is the group ring of S_k where $\pi \in S_k$ acts via $\pi \cdot \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_k} \coloneqq \mathbf{e}_{i_{\pi^{-1} \cdot 1}} \otimes \cdots \otimes \mathbf{e}_{i_{\pi^{-1} \cdot k}}$ (so that $\mathbf{e}_{i_j}$ becomes the $(\pi \cdot j)$th component). The duality between $\mathrm{GL}(n)$ and $\mathbb{C}[S_k]$ is a classical result of representation theory named after the pioneers Issai Schur and

Hermann Weyl, whereas the analogue concerning S_n and $\mathbf{P}_k(n)$ is a fairly recent discovery in the early 1990s [HR05]. We moreover have

$$(\mathbb{C}^n)^{\otimes k} \simeq \bigoplus_{\lambda \vdash n, 0 \le |\lambda^*| \le k} \mathbf{P}_k^\lambda \otimes V_\lambda \tag{4.1}$$

as $\big(\mathbf{P}_k(n) \times \mathbb{C}[S_n]\big)$-modules [HJ20, (2.14)–(2.15)], where the action on the right hand side satisfies $(d, \pi) \cdot (u \otimes v) = (d \cdot u) \otimes (\pi \cdot v)$. Now $\mathbb{C}[S_n]$ is certainly a subring of $\mathbf{P}_k(n) \times \mathbb{C}[S_n]$ via the inclusion $\pi \mapsto (1, \pi)$, through which $\mathbf{P}_k^\lambda \otimes V_\lambda$ restricts to a $\mathbb{C}[S_n]$-module with action $\pi \cdot (u \otimes v) = u \otimes (\pi \cdot v)$. If we pick a basis $\{\mathbf{p}_1, \dots, \mathbf{p}_{\dim \mathbf{P}_k^\lambda}\}$ for $\mathbf{P}_k^\lambda$, it is then clear that $\mathbf{p}_i \otimes v \mapsto (0, \dots, 0, \underbrace{v}_{i\text{th entry}}, 0, \dots, 0)$ gives an isomorphism

$$\mathbf{P}_k^\lambda \otimes V_\lambda \simeq \bigoplus_{1 \le i \le \dim \mathbf{P}_k^\lambda} V_\lambda$$

of $\mathbb{C}[S_n]$-modules. Since each V_λ has length 1, returning to (4.1) and taking lengths gives

$$\begin{aligned} \operatorname{length}(\mathbb{C}^n)^{\otimes k} &= \sum_{\lambda \vdash n, 0 \le |\lambda^*| \le k} \dim \mathbf{P}_k^\lambda \qquad (4.2) \\ &= \sum_{\lambda \vdash n, 0 \le |\lambda^*| \le k} \sum_{t=0}^{k} S(k,t) \binom{t}{|\lambda^*|} f^{\lambda^*} \end{aligned}$$

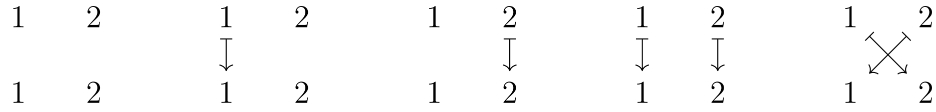
by (4.4c) of [HJ20].

It is worth mentioning that an earlier work [GC05, Proposition 2] seems — based on numerical evidence for small $|\lambda^*|$ — to also yield (4.2), but we have not been able to iron out the combinatorial details.

Theorem 4.3. $\operatorname{length}(\operatorname{Lin}[n]^{(k)}) = a_k$ *whenever* $n \ge 2k$, *where the sequence* $(a_k)_{k \ge 0}$ *is defined recursively by*

$$\begin{cases} a_0 = 1, \\ a_1 = 2, a_2 = 5, \\ a_{k+1} = 2a_k + k a_{k-1} \quad \textit{if } k \ge 2. \end{cases}$$

Proof. We show first that $(a_k)_{k\geq 1}$ counts the number of *partial involutions of* $[k]$, i.e., functions $\pi : X \to X$ with $X \subseteq [k]$ such that $\pi \circ \pi = \mathrm{id}_X$. Here it is important to explicitly write out the 1-cycles to distinguish between fixed points and elements outside the domain. Clearly the empty function $\emptyset$ and the identity (1) are the only partial involutions of [1],[§] whilst as illustrated below $\emptyset$, (1), (2), (1)(2), and (1 2) give all the partial involutions of [2].[¶]



But given $k \geq 2$, a partial involution of $[k+1]$ either does not contain $k+1$ in its domain, has $k+1$ as a fixed point, or maps $k+1$ to some $x \in [k]$ and x back to $k+1$; furthermore, it restricts to a unique partial involution of $[k]$, $[k]$, and $[k] \setminus \{x\}$ respectively. In total we end up with $a_k + a_k + ka_{k-1}$ partial involutions of $[k+1]$, which completes the inductive proof.

We proceed by another induction to show length(Lin $[n]^{(k)}$) $= a_k$ for $n \geq 2k \geq 2$. The base cases $k = 1$ and $k = 2$ are immediate by Corollary 3.5 and (3.27). For the inductive case, we will build upon the two following properties about the number b_m of (total) involutions of $[m]$:

(i) $b_m = \sum_{\mu \vdash m} f^\mu$ for all $m \geq 0$, and

(ii) $\sum_{m=0}^{t} \binom{t}{m} b_m = a_t$ for all $t \geq 0$.

The former is a well-known consequence of the celebrated RSK correspondence [Sag01, Exercise 3.7(a)], whilst the latter follows straightforwardly from the definition of partial involutions. Notice also that for $0 \leq m \leq k$ there is a bijection

$$\{\lambda \vdash n, |\lambda^*| = m\} \rightleftarrows \{\mu \vdash m\}$$

$$\lambda \mapsto \lambda^*$$

$$\mu[n] \leftarrow\!\shortmid \mu$$

[§]Ceci n'est pas une référence bibliographique.
[¶]*Idem.*

where $\mu[n]$ is a *bona fide* partition since $n \geq 2k \geq 2m \geq |\mu| + \mu_1$. Continuing with (4.2), we then have

$$\begin{aligned}
\operatorname{length}(\mathbb{C}^n)^{\otimes k} &= \sum_{m=0}^{k} \sum_{\substack{\lambda \vdash n,\\ |\lambda^*|=m}} \sum_{t=0}^{k} S(k,t) \binom{t}{m} f^{\lambda^*} \\
&= \sum_{m=0}^{k} \sum_{\mu \vdash m} \sum_{t=0}^{k} S(k,t) \binom{t}{m} f^{\mu} \\
&\overset{\text{(i)}}{=} \sum_{m=0}^{k} \sum_{t=0}^{k} S(k,t) \binom{t}{m} b_m \\
&= \sum_{t=0}^{k} S(k,t) (\sum_{m=0}^{t} \binom{t}{m} b_m + \sum_{m=t+1}^{k} 0 b_m) \\
&\overset{\text{(ii)}}{=} \sum_{t=0}^{k} S(k,t) a_t.
\end{aligned}$$

On the other hand, by (2.8) and Lemma 2.30 we see that

$$\operatorname{length}(\mathbb{C}^n)^{\otimes k} = \sum_{t=0}^{k} S(k,t) \operatorname{length}(\operatorname{Lin}[n]^{(t)}).$$

As $S(k,0) = 0$ and $S(k,k) = 1$, we conclude by the inductive hypothesis that $a_k = \operatorname{Lin}[n]^{(k)}$ too.

Our analysis is unnecessarily complicated by singling out the case $k = 0$. But $\operatorname{Lin}[n]^{(0)} = \mathbb{C}\underline{()}$ has length $1 = a_0$ for a dull reason: it has no proper subspaces, let alone equivariant ones. □

Remark. For $n \geq 2k$, the lengths of $(\mathbb{C}^n)^{\otimes k}$ and of $\operatorname{Lin}[n]^{(k)}$ are given by the [OEIS] sequences `A002872`(k) and `A005425`(k) respectively. ■

4.2 $\operatorname{length}(\operatorname{Lin}\mathbb{N}^{(k)})$

Corollary 4.4. $\operatorname{length}(\operatorname{Lin}\mathbb{N}^{(k)}) \leq \operatorname{length}(\operatorname{Lin}[2k]^{(k)}) = a_k$ *for any* $k \geq 0$. *Also* $2^k \leq a_k \leq (k+1)!$, *where the inequalities are furthermore strict for* $k \geq 2$.

Proof. In light of Theorem 4.3 above, the first part is an immediate consequence of

Theorem 3.9. The second part is an easy induction using the definition of $(a_k)_{k\geq 0}$ noting that $2^2 < a_2 < (2+1)!$. □

Conjecture 4.5. length(Lin $\mathbb{N}^{(k)}$) = length(Lin $[2k]^{(k)}$) $= a_k$ *for any* $k \geq 0$.

Remark. The equality appears to hold, at least to the authors of [SS15], for very simple reasons in view of their results — insomuch that they assert its validity in (1.3.4) and (8.7) at the beginning and end of their work without explicit proof.

Embarrassingly, even after friendly correspondence with the original authors and MathOverflow contributor Christopher Ryba, we have only been able to sketch a proof that hinges upon a homomorphism between the Grothendieck rings of $\mathbb{C}[S_\infty]$ and $\mathbb{C}[S_{2k}]$ induced by derived specialisation functors. The machinery is out of the scope of this paper, and we direct keen readers to `https://mathstrek.blog/2015/01/20/exact-sequences-and-the-grothendieck-group/`. Instead we content ourselves with Corollary 3.5 and Theorem 3.40 that settle the cases $k \in \{0, 1, 2\}$, but leave the result for $k > 2$ as an open problem. ■

5 Future work

> Take a minute to consider your achievement.
>
> ---
>
> John Cutter, *The Prestige* (2006)

Suppose that $\mathcal{A}$ is a weighted orbit-finite automaton with alphabet $\mathbb{N}$ and states

$$Q = \bigsqcup_{c_0=1}^{n_0} \mathbb{N}^{(0)} \sqcup \bigsqcup_{c_1=1}^{n_1} \mathbb{N}^{(1)} \sqcup \cdots \sqcup \bigsqcup_{c_k=1}^{n_k} \mathbb{N}^{(k)}.$$

As we discussed in Section 2.3.2, using Schützenberger's algorithm we can check whether $L_{\mathcal{A}}$ is the zero function in polynomial time with respect to

$$n \cdot \operatorname{length}(\operatorname{Lin} Q)^k \leq (\sum_{i=0}^{k} n_i) \cdot (\sum_{i=0}^{k} n_i a_i)^k$$

by Lemma 2.30 and Corollary 4.4; the decidability of the zeroness problem in the orbit-finite setting alone is of importance to the nominal theory of computation. But we are left with two questions, one algorithmic and one algebraic.

Attainability by automata. Recall from (2.27) that, provided $L_{\mathcal{A}}$ is not the zero function,

$$n_{\mathcal{A}} \coloneqq \min\{|w| \mid w \in \Sigma^*, L_{\mathcal{A}}(w) \neq 0\}$$

satisfies $n_{\mathcal{A}}+1 \leq \operatorname{Lin} Q \leq \sum_{i=0}^{k} n_i a_i$ and that Schützenberger's algorithm terminates upon reaching the $n_{\mathcal{A}}$th configuration space. Therefore Conjecture 4.5 will have limited automata-theoretic significance if the upper bound $\sum_{i=0}^{k} n_i a_i$ cannot in fact be asymptotically (with respect to n_k) attained by $n_{\mathcal{A}}$.

How do we devise an automaton $\mathcal{A}$ with a large $n_{\mathcal{A}}$? For $k = 0$, this is trivial:

we can arrange n_0 copies of the empty register $\mathbb{N}^{(0)}$ in a chain, thus achieving

$$n_{\mathcal{A}} = n_0 - 1 = \mathrm{Lin}(\bigsqcup_{c_0=1}^{n_0} \mathbb{N}^{(0)}) - 1$$

which is the maximal possible for any $n_0 \geq 1$. For $k = 1$ and $n_0 = 1$, we describe an automaton in Appendix A which we conjecture to achieve

$$n_{\mathcal{A}} = 1 + 2n_1 - 2 = \mathrm{length}(\bigsqcup_{c_0=1}^{1} \mathbb{N}^{(0)} \sqcup \bigsqcup_{c_1=1}^{n_1} \mathbb{N}^{(1)}) - 2$$

for any $n_1 \geq 2$; its conception is grounded on the chain $\{0\} \subsetneq \ker \epsilon \subsetneq \mathrm{Lin}\,[n]$ from Corollary 3.5 and the knowledge of the corresponding generators. For $k = 2$ a contrived automaton may still possibly be constructed from a chain in Figure B, but for $k > 2$ there is no clear way to proceed. This leads us to the algebraic question.

Explicit chain of length a_k. Can we exhibit a chain of length a_k in $\mathrm{Lin}\,[n]^{(k)}$ and, assuming Conjecture 4.5, in $\mathrm{Lin}\,\mathbb{N}^{(k)}$ for arbitrarily large k?

The problem is that the isomorphism (4.1) does not lend itself to an explicit decomposition of $\mathrm{Lin}\,[n]^{(k)}$ into simple constituents that is uniform across large n's, and our ad hoc analysis for $k \leq 2$ from Chapter 3 does not generalise naturally. We mention an orthogonal, but possibly fruitful, line of research using induced representations: for $n \geq k$, we can make the identifications $\mathrm{Ind}_{S_{n-k} \times (S_1)^k}^{S_n} \mathbb{C} \simeq \mathrm{Lin}\,[n]^{(k)}$ and $\mathrm{Ind}_{\{\pi \in S_\infty : \pi|_{[k]} = \mathrm{id}_{[k]}\} \times (S_1)^k}^{S_\infty} \mathbb{C} \simeq \mathrm{Lin}\,\mathbb{N}^{(k)}$ as explained in [BHH17, (5.7)] and [TV07, §3.3.2]; [NTV18] builds upon these.

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A Example in Haskell

```haskell
-- A weighted orbit-finite automaton with n_0 = 1, n_1 = _n states
-- which, we conjecture (and verified for _n <= 7),
-- only produces zero outputs for words with < 2*_n - 1 letters
-- but a non-zero output for "b" ++ replicate (2*_n - 2) 'a'.
import Data.Char (chr, ord)
import Data.List (findIndex, groupBy, sortBy)
import Data.Maybe (fromJust)

type Atom = Char
type Weight = Int
-- | Parameter n \geq 2
_n :: Int
_n = 6

-- Components of the automaton \mathcal{A}_n
-- | Set of states Q_n
data State = P | Q Int Atom deriving (Eq, Ord, Show)
transition :: (State, Weight) -> Atom -> [(State, Weight)]
transition (P, w) a = [ (Q 1 a, w) ]
transition (Q n a, w) b -- The gadgets can be organised in 3 phases:
  -- initialisation
  | n == 1 =
      [ (Q 2 a, w),
        (Q 2 b, -w) -- so that config is in \ker(\epsilon)
        -- Note that this is equivariant and non-guessing:
        -- \delta(Q 1 a,  a, Q 2 a ) = 1 - 1 = 0,
        -- \delta(Q 1 a, b', Q 2 a ) = 1, and
        -- \delta(Q 1 a, b', Q 2 b') = -1 for a \neq b'.
      ]
  -- propagation
  | 1 < n && n < _n =
      -- foward
      (Q (n + 1) a, w)
        -- backward
        : [(Q (n - 1) a, w) | n > 2]
  -- contamination
  | n == _n =
      [(Q n b, w) | a /= b]
        ++ [(Q (n - 1) a, w) | n > 2]
  | otherwise = []
```

```haskell
initialWeights :: [(State, Weight)]
initialWeights = [(P, 1)]

finalWeights :: (State, Weight) -> Weight
finalWeights (Q 2 _, w) = w
finalWeights _ = 0

-- | Simplify weighted sum of states.
-- >>> simplify [(P, 1), (P, -1)]
-- []
simplify :: [(State, Weight)] -> [(State, Weight)]
simplify =
  filter ((/= 0) . snd)
    . map (\xs -> (fst $ head xs, sum $ map snd xs))
    . groupBy (\x y -> fst x == fst y)
    . sortBy (\x y -> compare (fst x) (fst y))

-- | Compute the configuration after reading in a word.
-- >>> config ""
-- [(P,1)]
-- >>> config "a"
-- [(Q 1 'a',1)]
-- >>> config "aa"
-- []
-- >>> config "ab"
-- [ (Q 2 'a', 1),
--   (Q 2 'b',-1) ]
--
-- Now suppose _n = 6.
-- >>> config "abX"
-- [              (Q 3 'a', 1),
--                (Q 3 'b',-1)  ]
-- >>> config "abXX"
-- [ (Q 2 'a', 1),              (Q 4 'a', 1),
--   (Q 2 'b',-1),              (Q 4 'b',-1) ]
-- >>> config "abXXX"
-- [              (Q 3 'a', 2),              (Q 5 'a', 1),
--                (Q 3 'b',-2),              (Q 5 'b',-1) ]
-- >>> config "abXXXX"
-- [ (Q 2 'a', 2),              (Q 4 'a', 3),              (Q 6 'a', 1),
--   (Q 2 'b',-2),              (Q 4 'b',-3),              (Q 6 'b',-1) ]
-- >>> config "abXXXXa"
-- [              (Q 3 'a', 5),              (Q 5 'a', 4),
--                (Q 3 'b',-5),              (Q 5 'b',-4),(Q 6 'b',-1) ]
-- >>> config "abXXXXX"
-- [              (Q 3 'a', 5),              (Q 5 'a', 4),
--                (Q 3 'b',-5),              (Q 5 'b',-4) ]
-- >>> config "abXXXXb"
-- [              (Q 3 'a', 5),              (Q 5 'a', 4),(Q 6 'a', 1),
--                (Q 3 'b',-5),              (Q 5 'b',-4), ]
-- >>> config "abXXXXbX"
-- [ (Q 2 'a', 5),              (Q 4 'a', 9),(Q 5 'a', 1),(Q 6 'a', 4),
--   (Q 2 'b',-5),              (Q 4 'b',-9),              (Q 6 'b',-4),
--                                                         (Q 6 'X', 1) ]
-- and etc.
config :: [Atom] -> [(State, Weight)]
```

```haskell
config =
  foldl
    ( \xs a -> simplify $ concatMap (`transition` a) xs
    )
    initialWeights

-- | The weighted language L_{\mathcal{A}_n}
-- >>> language ("ab" ++ replicate (_n-2) 'X' ++
-- ...              "a" ++ replicate (_n-2) 'X')
-- -1
-- >>> language ("ab" ++ replicate (_n-2) 'X' ++
-- ...              "c" ++ replicate (_n-2) 'X')
-- 0
-- >>> language ("ab" ++ replicate (_n-2) 'X' ++
-- ...              "b" ++ replicate (_n-2) 'X')
-- 1
language :: [Atom] -> Weight
language = sum . map finalWeights . config

-- | Enumerate all partitions of a list;
-- courtesy of https://stackoverflow.com/a/46596325.
-- >>> partitions [1, 2]
-- [ [[1, 2]],  [[1], [2]] ]
partitions :: [a] -> [[[a]]]
partitions [] = [[]]
partitions (x : xs) = expand x $ partitions xs
  where
    expand :: a -> [[[a]]] -> [[[a]]]
    expand x = concatMap (extend x)

    extend :: a -> [[a]] -> [[[a]]]
    extend x [] = [[[x]]]
    extend x (y : ys) = ((x : y) : ys) : map (y :) (extend x ys)

-- >>> partition2word [[1], [2, 3]]
-- "abb"
partition2word :: [[Int]] -> [Atom]
partition2word p = map (chr . (+ ord 'a')) indices
  where
    n = length . concat $ p
    indices = [fromJust $ findIndex (i `elem`) p | i <- [1 .. n]]

-- | Guess a shortest word that prododuces a non-zero output.
-- As L_{\mathcal{A}_n} is equivariant, we need only check the output
-- on each orbit of length-k words for k \geq 0.
findWord :: [Atom]
findWord = head . filter ((/= 0) . language) $ ws
  where
    ws =
      map partition2word $
        concatMap partitions [[1 .. k] | k <- [0 ..]]

main :: IO ()
main = do
  putStrLn $ "n = " ++ show _n
  let w = findWord
```

```
putStrLn $
  "least accepting length = "
    ++ show (length w)
    ++ " (word: "
    ++ w
    ++ ")"
putStrLn $ "Compare: 2n-1 = " ++ show (2 * _n - 1)
```